

Some New Subclasses of Nonsingular H-matrices

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Abstract—In this paper, we obtain some new subclasses of nonsingular H-matrices by using α diagonally dominant matrix.

Keywords—H-matrix, diagonal dominance, α diagonally dominant matrix.

I. INTRODUCTION

LET $A = (a_{ij})_{n \times n} \in C^{n \times n}$, $M(A) = (m_{ij})$, where

$$m_{ij} = \begin{cases} |a_{ii}|, & i = j, \\ -|a_{ij}|, & i \neq j, \end{cases} \quad i = 1, 2, \dots, n.$$

Then we call $M(A)$ is the comparison matrix of A . Suppose A is an n by n matrix over the field of real numbers. If A can be expressed in the form $A = \sigma I - B$ where B is a nonnegative matrix and $\sigma > \rho(B)$ the spectral radius of B , then A is called a nonsingular M-matrix. This class of matrices has been much studied [1].

If $M(A)$ is nonsingular M-matrix, then A is called a nonsingular H-matrix. If

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|, \quad i = 1, 2, \dots, n,$$

then we say A is strictly diagonally dominant. If there exist positive number x_1, x_2, \dots, x_n such that

$$x_i |a_{ii}| > \sum_{j \neq i} x_j |a_{ij}|, \quad i = 1, 2, \dots, n,$$

then we say A is generalized strictly diagonally dominant [2].

A matrix A be a nonsingular H-matrix is equivalent to that A be a generalized strictly diagonally dominant matrix [3].

H-matrices have important applications, for instance, in iterative methods of numerical analysis, in the analysis of dynamical systems, in economics, and in mathematical programming. But how to determine whether an n by n complex matrix is a nonsingular H-matrix is not easy in practice. In this paper, we will give some new subclasses of nonsingular H-matrices.

II. MAIN RESULTS

We will use the following notations:

$$R_i(\cdot) = \sum_{j \neq i} |a_{ij}|, \quad S_i(\cdot) = \sum_{j \neq i} |a_{ji}|, \\ i \in \langle n \rangle = \{1, 2, \dots, n\},$$

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$$I(A) = \left\{ \nu \in S(A) \left| \begin{array}{l} \prod_{i \in \nu} |a_{ij}| / \prod_{i \in \nu} R_i(A) \\ \text{or} \\ \prod_{i \in \nu} |a_{ij}| / \prod_{i \in \nu} C_i(A) \end{array} \right. \right\},$$

$$\frac{\alpha}{1} \{i \in \langle n \rangle \mid |a_{ij}| > \alpha R_i(A) + (1 - \alpha) S_i(A)\},$$

$$\beta_\alpha \{i \in \langle n \rangle \mid |\alpha R_i(A) + (1 - \alpha) S_i(A)| > 0\}.$$

Definition [4] Let $A = (a_{ij}) \in C^{n \times n}$. If there exists $\alpha \in (0, 1)$, $|a_{ii}| \geq \alpha R_i + (1 - \alpha) S_i (i \in N)$ holds, then we call A is α diagonally dominant and denote $A \in D_0(\alpha)$. If all the inequations are strict, we denote $A \in D(\alpha)$.

Lemma 1 [4] Let $A = (a_{ij}) \in C^{n \times n}$. If $A \in D(\alpha)$, then A is a nonsingular H-matrix.

Lemma 2 [4] Let $A = (a_{ij}) \in C^{n \times n}$. If for $\alpha \in (0, 1)$, $|a_{ii}| \geq \alpha R_i + (1 - \alpha) S_i$ holds, and for i which satisfies $|a_{ii}| < \alpha R_i + (1 - \alpha) S_i$ there exists a non-zero elements chain $a_{i i_1}, a_{i_1 i_2}, \dots, a_{i_{p-1} i_p} \neq 0$ such that $j \in J = \{j \in N \mid |a_{ii}| < \alpha R_i + (1 - \alpha) S_i\} \cap \Phi$, then A is a nonsingular H-matrix.

Theorem 1. Let $A = (a_{ij})_{n \times n} \in C^{n \times n}$, for $\alpha \in (0, 1)$, if

$$|a_{ii}| > \frac{\alpha}{x_i} \sum_{j \neq i} |a_{ij}| x_j + \frac{1 - \alpha}{y_i} \sum_{j \neq i} |a_{ji}| y_j, \quad i \in \langle n \rangle \quad (1)$$

$$|a_{ii}| \geq \frac{\alpha}{x_i} \left(\sum_{j \in N_1^\alpha} |a_{ij}| x_j + \sum_{j \in N_2^\alpha, j \neq i} |a_{ij}| \right) \\ + \frac{1 - \alpha}{y_i} \left(\sum_{j \in N_1^\alpha} |a_{ji}| y_j + \sum_{j \in N_2^\alpha, j \neq i} |a_{ji}| \right), \quad i \in \langle n \rangle \quad (2)$$

where $0 < x_i < 1$, $0 < y_i < 1$, $i \in \langle n \rangle$. Then A is a nonsingular H-matrix.

Proof: Let

$$b_i = \frac{x_i y_i |a_{ii}| - y_i \alpha \sum_{j \neq i} |a_{ij}| x_j - (1 - \alpha) \sum_{j \neq i} |a_{ji}| y_j x_i}{y_i \alpha \sum_{j \neq i} |a_{ij}| x_j + (1 - \alpha) \sum_{j \neq i} |a_{ji}| y_j x_i}, \quad (3)$$

$$i \in \langle n \rangle.$$

From (1) we know that $0 < b_i < +\infty$. Let

$$c_i = \frac{\sum_{j \neq i} |a_{ij}| x_j}{\sum_{j \in N_2^\alpha} |a_{ij}|}, \quad f_i = \frac{\sum_{j \neq i} |a_{ji}| y_j}{\sum_{j \in N_2^\alpha} |a_{ji}|} \quad (4)$$

when $\sum_{j \in N_2^\alpha} |a_{ij}| = 0$, $\sum_{j \in N_2^\alpha} |a_{ji}| = 0$, we denote $c_i = \infty$, $f_i = \infty$, according to the hypothesis of this paper, we have $c_i > 0$, $f_i > 0$.

We denote

$$\alpha \left\{ i \in \frac{\alpha}{2} |x_i| = 1 \right\}, \quad \alpha \left\{ i \in \frac{\alpha}{2} |y_i| = 1 \right\}.$$

There must exist a small enough positive number ε , such that

$$0 < \varepsilon < \left\{ \begin{array}{l} \left\{ c_i \right\}, \quad \left\{ f_i \right\}, \\ \left\{ 1 - x_i \right\}, \quad \left\{ 1 - y_i \right\} \end{array} \right\}.$$

We choose positive diagonal matrix

$$\text{diag}(d_1, d_2, \dots, d_n)$$

and

$$\text{diag}(e_1, e_2, \dots, e_n),$$

where

$$d_i = \begin{cases} x_i & i \in \frac{\alpha}{2} \\ x_i & i \in \frac{\alpha}{2} \\ x_i + \varepsilon & i \in \frac{\alpha}{2} \setminus \frac{\alpha}{2} \end{cases} \quad e_i = \begin{cases} y_i & i \in \frac{\alpha}{2} \\ y_i & i \in \frac{\alpha}{2} \\ y_i + \varepsilon & i \in \frac{\alpha}{2} \setminus \frac{\alpha}{2} \end{cases} \quad (b_{ij})$$

In the follows, we just need to prove that is a strictly α diagonally dominant matrix.

For $\forall i \in \frac{\alpha}{2}$, according to (3) we have

$$|a_{ij}| x_i y_i = (1 + b_i) \left(\alpha \sum_{j \neq i} |a_{ij}| x_j y_j + (1 - \alpha) \sum_{j \neq i} |a_{ji}| y_j x_i \right). \quad (5)$$

There will be four cases:

Case one: $\sum_{j \in N_2^\alpha} |a_{ij}| = \sum_{j \in N_2^\alpha} |a_{ji}| = 0$, according to (1) we have:

$$b_{ii} = y_i |a_{ii}| x_i > y_i \alpha \sum_{j \neq i} |a_{ij}| x_j + (1 - \alpha) \sum_{j \neq i} |a_{ji}| y_j \cdot x_i$$

$$\alpha \sum_{j \in N_1^\alpha} |b_{ij}| + (1 - \alpha) \sum_{j \in N_1^\alpha} |b_{ji}|$$

$$\alpha R_i(\cdot) + (1 - \alpha) S_i(\cdot).$$

Case two: $\sum_{j \in N_2^\alpha} |a_{ij}| = 0$, $\sum_{j \in N_2^\alpha} |a_{ji}| \neq 0$, in this case, $|a_{ij}| = 0$ for any $j \in \frac{\alpha}{2}$. according to (4) we have:

$$\varepsilon < f_i \Leftrightarrow \varepsilon \sum_{j \in N_2^\alpha} |a_{ji}| < b_i \sum_{j \neq i} |a_{ji}| y_j$$

$$\Leftrightarrow (1 + b_i) \sum_{j \neq i} |a_{ji}| y_j > \sum_{j \neq i} |a_{ji}| y_j + \varepsilon \sum_{j \in N_2^\alpha} |a_{ji}|. \quad (6)$$

With (5), (6) and the hypothesis of the paper, we have:

$$b_{ii} = y_i |a_{ii}| x_i = y_i \alpha \sum_{j \neq i} |a_{ij}| x_j + (1 - \alpha) (1 + b_i) \sum_{j \neq i} |a_{ji}| y_j x_i$$

$$> y_i \alpha \sum_{\substack{j \in \frac{\alpha}{2} \\ j \neq i}} |a_{ij}| x_j$$

$$+ (1 - \alpha) \left(\sum_{j \neq i} |a_{ji}| y_j + \varepsilon \sum_{j \in N_2^\alpha} |a_{ji}| \right) x_i$$

$$> y_i \alpha \sum_{\substack{j \in \frac{\alpha}{2} \\ j \neq i}} |a_{ij}| x_j + x_i (1 - \alpha) \times$$

$$\left(\sum_{\substack{j \in \frac{\alpha}{2} \\ j \neq i}} |a_{ji}| y_j + \sum_{j \in N_y^\alpha} |a_{ji}| y_j + \sum_{j \in N_2^\alpha \setminus N_y^\alpha} |a_{ji}| (y_j + \varepsilon) \right)$$

$$\alpha R_i(\cdot) + (1 - \alpha) S_i(\cdot).$$

Case three: $\sum_{j \in N_2^\alpha} |a_{ij}| \neq 0$, $\sum_{j \in N_2^\alpha} |a_{ji}| = 0$. As the same proof of case two, we can obtain

$$|b_{ii}| > R_i(\cdot)^\alpha C_i(\cdot)^{1-\alpha}.$$

Case four: $\sum_{j \in N_2^\alpha} |a_{ij}| \neq 0$, $\sum_{j \in N_2^\alpha} |a_{ji}| \neq 0$ according to (4) we have:

$$\varepsilon < c_i \Leftrightarrow \varepsilon \sum_{j \in N_2^\alpha} |a_{ij}| < b_i \sum_{j \neq i} |a_{ij}| x_j$$

$$\Leftrightarrow (1 + b_i) \sum_{j \neq i} |a_{ij}| x_j > \sum_{j \neq i} |a_{ij}| x_j + \varepsilon \sum_{j \in N_2^\alpha} |a_{ij}|.$$

From the above inequation and the inequation (6), we have

$$b_{ii} = y_i |a_{ii}| x_i = \alpha (1 + b_i) y_i \sum_{j \neq i} |a_{ij}| x_j + (1 - \alpha) (1 + b_i) x_i \sum_{j \neq i} |a_{ji}| y_j$$

$$> \alpha y_i \left(\sum_{j \neq i} |a_{ij}| x_j + \varepsilon \sum_{j \in N_2^\alpha} |a_{ij}| \right) + (1 - \alpha) x_i \left(\sum_{j \neq i} |a_{ji}| y_j + \varepsilon \sum_{j \in N_2^\alpha} |a_{ji}| \right)$$

$$\geq \alpha y_i \left(\sum_{j \in N_1^\alpha \cup N_y^\alpha} |a_{ij}| x_j + \sum_{j \in N_2^\alpha \setminus N_y^\alpha} (x_j + \varepsilon) |a_{ij}| \right)$$

$$+x_i(1-\alpha)\left(\sum_{j \in N_1^\alpha \cup N_y^\alpha} |a_{ji}|y_j + \sum_{j \in N_2^\alpha \setminus N_y^\alpha} (y_j + \varepsilon)|a_{ji}|\right)$$

$$\alpha R_i(\cdot) + (1-\alpha)S_i(\cdot).$$

For any $i \in \frac{\alpha}{2}$, from the choice of ε and the positive diagonal matrices D and E, we know that $0 < d_i, e_i \leq 1$, for any $i \in \frac{\alpha}{2}$.

Case one: $i \in \frac{\alpha}{x} \cap \frac{\alpha}{y}$

$$|b_{ii}| - |a_{ii}| \geq \alpha \left(\sum_{j \in N_1^\alpha} |a_{ij}|x_j + \sum_{\substack{j \in \frac{\alpha}{2} \\ j \neq i}} |a_{ij}| \right)$$

$$+ (1-\alpha) \left(\sum_{j \in N_1^\alpha} |a_{ji}|y_j + \sum_{\substack{j \in \frac{\alpha}{2} \\ j \neq i}} |a_{ji}| \right)$$

$$> \alpha \left(\sum_{j \in \langle n \rangle \setminus N_x^\alpha} |a_{ij}|x_j + \sum_{\substack{j \in \frac{\alpha}{x} \\ j \neq i}} |a_{ij}| \right)$$

$$+ (1-\alpha) \left(\sum_{j \in \langle n \rangle \setminus N_y^\alpha} |a_{ji}|y_j + \sum_{\substack{j \in \frac{\alpha}{y} \\ j \neq i}} |a_{ji}| \right)$$

$$\alpha R_i(\cdot) + (1-\alpha)S_i(\cdot).$$

Case two: $i \in \frac{\alpha}{x}, i \notin \frac{\alpha}{y}$, if $\alpha < 1$, from (2) we have

$$\begin{aligned} & (y_i + \varepsilon)|a_{ii}| \\ & \geq (y_i + \varepsilon)\alpha \left(\sum_{j \in N_1^\alpha} |a_{ij}|x_j + \sum_{\substack{j \in \frac{\alpha}{2} \\ j \neq i}} |a_{ij}| \right) \\ & + (1-\alpha) \left(\sum_{j \in N_1^\alpha} |a_{ji}|y_j + \sum_{\substack{j \in \frac{\alpha}{2} \\ j \neq i}} |a_{ji}| \right) \end{aligned} \quad (7)$$

Hence

$$|b_{ii}| - (y_i + \varepsilon)|a_{ii}|$$

$$\geq (y_i + \varepsilon)\alpha \left(\sum_{j \in N_1^\alpha} |a_{ij}|x_j + \sum_{\substack{j \in \frac{\alpha}{2} \\ j \neq i}} |a_{ij}| \right)$$

$$+ (1-\alpha) \left(\sum_{j \in N_1^\alpha} |a_{ji}|y_j + \sum_{\substack{j \in \frac{\alpha}{2} \\ j \neq i}} |a_{ji}| \right)$$

$$> (y_i + \varepsilon)\alpha \left(\sum_{j \in N_1^\alpha} |a_{ij}|x_j + \sum_{j \in N_2^\alpha \setminus N_x^\alpha} |a_{ij}|(x_j + \varepsilon) + \sum_{j \in N_x^\alpha} |a_{ij}| \right)$$

$$+ (1-\alpha) \left(\sum_{j \in N_1^\alpha} |a_{ji}|y_j + \sum_{\substack{j \in N_2^\alpha \setminus N_y^\alpha \\ j \neq i}} |a_{ji}|(y_j + \varepsilon) + \sum_{j \in N_y^\alpha} |a_{ji}| \right)$$

$$\alpha R_i(\cdot) + (1-\alpha)S_i(\cdot).$$

Case three: $i \notin \frac{\alpha}{x}, i \in \frac{\alpha}{y}$, as the same proof of case two, we can obtain

$$|b_{ii}| > \alpha R_i(\cdot) + (1-\alpha)S_i(\cdot).$$

Case four: $i \notin \frac{\alpha}{x}, i \notin \frac{\alpha}{y}$, from (2) we have

$$\begin{aligned} & (y_i + \varepsilon)|a_{ii}|(x_i + \varepsilon) \\ & \geq (y_i + \varepsilon)\alpha \left(\sum_{j \in N_1^\alpha} |a_{ij}|x_j + \sum_{\substack{j \in \frac{\alpha}{2} \\ j \neq i}} |a_{ij}| \right) \end{aligned}$$

Since

$$(y_i + \varepsilon)\alpha \left(\sum_{j \in N_1^\alpha} |a_{ij}|x_j + \sum_{\substack{j \in \frac{\alpha}{2} \\ j \neq i}} |a_{ij}| \right) > \alpha R_i(\cdot),$$

$$(1-\alpha) \left(\sum_{j \in N_1^\alpha} |a_{ji}|y_j + \sum_{\substack{j \in \frac{\alpha}{2} \\ j \neq i}} |a_{ji}| \right) (x_i + \varepsilon)$$

$$> (1 - \alpha) S_i (),$$

we have

$$\begin{aligned} & |b_{ii}| (y_i + \varepsilon) |a_{ii}| (x_i + \varepsilon) \\ & \geq (y_i + \varepsilon) \alpha \left(\sum_{j \in N_1^\alpha} |a_{ij}| x_j + \sum_{\substack{j \in \frac{\alpha}{2} \\ j \neq i}} |a_{ij}| \right) \\ & + (1 - \alpha) \left(\sum_{j \in N_1^\alpha} |a_{ji}| y_j + \sum_{\substack{j \in \frac{\alpha}{2} \\ j \neq i}} |a_{ji}| \right) (x_i + \varepsilon) \\ & > \alpha R_i () + (1 - \alpha) S_i (). \end{aligned}$$

We see that for any $i \in \langle n \rangle$, we have $|b_{ii}| > \alpha R_i () + (1 - \alpha) S_i ()$. According to Lemma 1, we know that matrix B is a nonsingular H-matrix, so matrix A is a nonsingular H-matrix.

Let $A = (a_{ij})_{n \times n} \in C^{n \times n}$, $0 < x_i, y_i < 1, i \in \langle n \rangle$ satisfy the equation (2), we denote

$$K_\alpha = \left\{ i \in \langle n \rangle \mid |a_{ii}| > \frac{\alpha}{x_i} \sum_{j \neq i} |a_{ij}| x_j + \frac{1 - \alpha}{y_i} \sum_{j \neq i} |a_{ji}| y_j \right\}.$$

Theorem 2 Let $A = (a_{ij})_{n \times n} \in C^{n \times n}$, for $\alpha \in (0, 1)$, if $0 < x_i < 1, 0 < y_i < 1, i \in \langle n \rangle$ satisfy the inequations (2) and

$$|a_{ii}| \geq \frac{\alpha}{x_i} \sum_{j \neq i} |a_{ij}| x_j + \frac{1 - \alpha}{y_i} \sum_{j \neq i} |a_{ji}| y_j, \quad i \in \frac{\alpha}{1} \quad (8)$$

and $K_\alpha \neq \emptyset$, for any $i_0 \in (\langle n \rangle \setminus K_\alpha)$, there exists a nonzero elements chain $a_{i_0 i_1} a_{i_1 i_2} \cdots a_{i_{k-1} i_k} \neq 0$ such that $i_k \in K_\alpha$, then A is a nonsingular H-matrix.

Proof: we structure two positive diagonal matrices: $D = \text{diag}(x_1, x_2, \dots, x_n)$ and $E = \text{diag}(y_1, y_2, \dots, y_n)$, and notes $B = (b_{ij}) = EAD$. So for any $i \in \langle n \rangle$, we have

$$|b_{ii}| \geq \alpha R_i (B) + (1 - \alpha) S_i (B).$$

Obviously, K_α can be note

$$K_\alpha = \{ i \in \langle n \rangle \mid |b_{ii}| > \alpha R_i (B) + (1 - \alpha) S_i (B) \},$$

for any $i_0 \in K_\alpha$, we have $b_{i_0 i_1} b_{i_1 i_2} \cdots b_{i_{k-1} i_k} \neq 0$ such that $i_k \in K_\alpha$. So according to Lemma 2, we know that matrix B is a nonsingular H-matrix, so matrix A is a nonsingular H-matrix.

From Theorem 2, we can get the following corollary.

Corollary Let $A = (a_{ij})_{n \times n} \in C^{n \times n}$ be irreducible, for $\alpha \in (0, 1)$, if $0 < x_i < 1, 0 < y_i < 1, i \in \langle n \rangle$ satisfy the inequations (2) and (8), $\tilde{I} \neq \emptyset$, where

$$\tilde{I} = \left\{ v \in S(A) \mid \begin{aligned} & \frac{y_i}{i \in v} |a_{ii}| x_i / \frac{\tilde{R}_i(A)}{i \in v} \\ & \text{or } \frac{y_i}{i \in v} |a_{ii}| x_i / \frac{\tilde{C}_i(A)}{i \in v} \end{aligned} \right\},$$

$$\tilde{R}_i(A) = y_i \sum_{j \neq i} |a_{ij}| x_j, \quad \tilde{C}_i(A) = x_i \sum_{j \neq i} |a_{ji}| y_j,$$

then A is a nonsingular H-matrix.

REFERENCES

- [1] Horn R A, Johnson C A, *Matrix Analysis*, Cambridge: Cambridge University Press, 1985.
- [2] Gao Y M, Wang X H, "Criteria for generalized diagonally dominant matrices and M-matrices", *Lin. Alg. Appl.*, 1992, Vol.169, pp. 257-268.
- [3] Berman A, Plemmons R J, *Nonnegative Matrices in the Mathematical Sciences*, New York: Academic Press, 1979.
- [4] Sun Y X, "Sufficient Conditions of Generalized Diagonal Dominance Matrix", *Numerical Mathematics (a Journal of Chinese Universities)*, 1997, Vol. 15(3), pp. 573-578.