World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:4, No:7, 2010

Some New Subclasses of Nonsingular H-matrices

Guangbin Wang+, Liangliang Li, and Fuping Tan

Abstract—In this paper, we obtain some new subclasses of nonsingular H-matrices by using α diagonally dominant matrix.

Keywords—H-matrix, diagonal dominance, α diagonally dominant matrix.

I. Introduction

LET $A \quad (a_{ij})_{n\times n} \in C^{n\times n}$, $M(A) \quad (m_{ij})$, where

$$m_{ij} \quad \begin{cases} |a_{ii}|, i \quad j, \\ -|a_{ij}|, i \;/\; j, \end{cases} \quad i \quad 1,2,...,n.$$

Then we call $M(A)$ is the comparison matrix of A. Suppose A is an n by n matrix over the field of real numbers. If A can be expressed in the form $A \quad \sigma I - B$ where B is a nonnegative matrix and $\sigma > \rho(B)$ the spectral radius of B, then A is called a nonsingular M-matrix. This class of matrices has been much studied [1].

If $M(A)$ is nonsingular M-matrix, then A is called a nonsingular H-matrix. If

$$|a_{ii}| > \sum_{j\neq i} |a_{ij}|, \; i \quad 1,2,\cdots,n,$$

then we say A is strictly diagonally dominant. If there exist positive number $x_1, x_2, \cdots x_n$ such that

$$x_i\,|a_{ii}| > \sum_{j\neq i} x_j\,|a_{ij}|, \; i \quad 1,2,\cdots,n \quad ,$$

then we say A is generalized strictly diagonally dominant[2].

A matrix A be a nonsingular H-matrix is equivalent to that A be a generalized strictly diagonally dominant matrix [3].

H-matrices have important applications, for instance, in iterative methods of numerical analysis, in the analysis of dynamical systems, in economics, and in mathematical programming. But how to determine whether an n by n complex matrix is a nonsingular H-matrix is not easy in practice. In this paper, we will give some new subclasses of nonsingular H-matrices.

II. Main Results

We will use the following notations:

$$R_i(\;) \quad \sum_{j\neq i} |a_{ij}|, \;\; S_i(\;) \quad \sum_{j\neq i} |a_{ji}|,$$
$$i \in \langle n\rangle \quad \{1,2,\cdots,n\},$$

$$I(A) \quad \left\{ \nu \in S(A) \,\middle|\, \begin{array}{l} \prod_{i\in\nu} |a_{ij}| \;/\; \prod_{i\in\nu} R_i(A) \\ or \; \prod_{i\in\nu} |a_{ij}| \;/\; \prod_{i\in\nu} C_i(A) \end{array} \right\},$$

$${}^{\alpha}_{2} \quad \{i \in <n> \mid |a_{ij}| > \alpha R_i(A) + (1-\alpha) S_i(A)\},$$
$${}^{\alpha}_{1} \quad <n> \setminus \; {}^{\alpha}_{2},$$

$$\beta_\alpha \quad \{i \in <n> |\alpha R_i(A) + (1-\alpha) S_i(A) > 0\}.$$

Definition [4] Let $A \quad (a_{ij}) \in C^{n\times n}$. If there exists $\alpha \in \; 0,1$, $|a_{ii}| \geq \alpha R_i + (1-\alpha) S_i (i \in N)$ holds, then we call A is α diagonally dominant and denote $A \in D_0(\alpha)$. If all the inequations are strict, we denote $A \in D(\alpha)$.

Lemma 1[4] Let $A \quad (a_{ij}) \in C^{n\times n}$. If $A \in D(\alpha)$, then A is a nonsingular H-matrix.

Lemma 2[4] Let $A \quad (a_{ij}) \in C^{n\times n}$. If for $\alpha \in \; 0,1$, $|a_{ii}| \geq \alpha R_i + (1-\alpha) S_i$ holds, and for i which satisfies $|a_{ii}| \quad \alpha R_i + (1-\alpha) S_i$ there exists a non-zero elements chain $a_{ii_1}, a_{i_1 i_2}, \ldots, a_{i_p j} \;/\; 0$ such that $j \in J \quad \{j \in N \mid |a_{ii}| > \alpha R_i + (1-\alpha) S_i\} \;/\; \Phi$, then A is a nonsingular H-matrix.

Theorem 1. Let $A \quad (a_{ij})_{n\times n} \in C^{n\times n}$, for $\alpha \in \; 0,1$, if

$$|a_{ii}| > \frac{\alpha}{x_i} \sum_{j\neq i} |a_{ij}|\, x_j + \frac{1-\alpha}{y_i} \sum_{j\neq i} |a_{ji}|\, y_j, \; i \in \; {}^{\alpha}_{1} \qquad (1)$$

$$|a_{ii}| \geq \tfrac{\alpha}{x_i} \left(\sum_{j\in N_1^\alpha} |a_{ij}|\, x_j + \sum_{j\in N_2^\alpha, j\neq i} |a_{ij}| \right)$$
$$+ \frac{1-\alpha}{y_i} \left(\sum_{j\in N_1^\alpha} |a_{ji}|\, y_j + \sum_{j\in N_2^\alpha, j\neq i} |a_{ji}| \right), \; i \in \; {}^{\alpha}_{2} \qquad (2)$$

where $0 < x_i < 1$, $0 < y_i < 1$, $i \in \langle n\rangle$. Then A is a nonsingular H-matrix.

Proof: Let

$$b_i \quad \frac{x_i y_i\,|a_{ii}| - y_i \alpha \sum_{j\neq i} |a_{ij}|\, x_j - (1-\alpha) \sum_{j\neq i} |a_{ji}|\, y_j x_i}{y_i \alpha \sum_{j\neq i} |a_{ij}|\, x_j + (1-\alpha) \sum_{j\neq i} |a_{ji}|\, y_j x_i}, \qquad (3)$$

$i \in \; {}^{\alpha}_{1}$.

From (1) we know that $0 < b_i < +\infty$. Let

$$c_i \quad b_i \frac{\sum_{j\neq i} |a_{ij}|\, x_j}{\sum_{j\in N_2^\alpha} |a_{ij}|} \quad , f_i \quad b_i \frac{\sum_{j\neq i} |a_{ji}|\, y_j}{\sum_{j\in N_2^\alpha} |a_{ji}|} \qquad (4)$$

Guangbin Wang and Liangliang Li are with the Department of Mathematics, Qingdao University of Science and Technology, Qingdao, 266061, China.

Fuping Tan is with the Department of Mathematics, Shanghai University, Shanghai, 200444, China.

+ Corresponding author. E-mail: wguangbin750828@sina.com. This work was supported by Natural Science Fund of Shandong Province of China (Y2008A13).

when $\sum_{j\in N_2^\alpha} |a_{ij}| \quad 0, \sum_{j\in N_2^\alpha} |a_{ji}| \quad 0$, we denote $c_i \quad \infty,\ f_i \quad \infty$, according to the hypothesis of this paper, we have $c_i > 0,\ f_i > 0$.

We denote

$$\ _x^\alpha \quad \{i \in \ _2^\alpha |x_i \quad 1\}, \quad \ _y^\alpha \quad \{i \in \ _2^\alpha |y_i \quad 1\}.$$

There must exits a small enough positive number ε, such that

$$0 < \varepsilon < \left\{ _{i\in N_1^\alpha}\{c_i\},\ _{i\in N_1^\alpha}\{f_i\},\ _{i\in N_2^\alpha\setminus N_x^\alpha}\{1-x_i\},\ _{i\in N_2^\alpha\setminus N_y^\alpha}\{1-y_i\} \right\}.$$

We choose positive diagonal matrix

$$diag\,(d_1, d_2, \cdots, d_n)$$

and

$$diag\,(e_1, e_2, \cdots, e_n),$$

where

$$d_i \quad \begin{cases} x_i & i \in \ _1^\alpha \\ x_i & i \in \ _x^\alpha \\ x_i+\varepsilon & i \in \ _2^\alpha \setminus \ _x^\alpha \end{cases} \qquad e_i \quad \begin{cases} y_i & i \in \ _1^\alpha \\ y_i & i \in \ _y^\alpha \\ y_i+\varepsilon & i \in \ _2^\alpha \setminus \ _y^\alpha \end{cases}$$

In the follows, we just need to prove that $\quad (b_{ij})$ is a strictly α diagonally dominant matrix.

For $\forall i \in \ _1^\alpha$, according to (3) we have

$$|a_{ij}|\, x_i y_i \quad (1+b_i)\left(\alpha \sum_{j\neq i} |a_{ij}|\, x_j y_i + (1-\alpha) \sum_{j\neq i} |a_{ji}|\, y_j x_i\right). \qquad (5)$$

There will be four cases:

Case one: $\sum_{j\in N_2^\alpha} |a_{ij}| \quad \sum_{j\in N_2^\alpha} |a_{ji}| \quad 0$, according to(1)we have:

$$b_{ii} \quad y_i\, |a_{ii}|\, x_i > y_i\alpha \sum_{j\neq i} |a_{ij}|\, x_j + (1-\alpha)\sum_{j\neq i} |a_{ji}|\, y_j \cdot x_i$$

$$\alpha \sum_{j\in N_1^\alpha} |b_{ij}| + (1-\alpha) \sum_{j\in N_1^\alpha} |b_{ji}|$$

$$\alpha R_i(\ ) + (1-\alpha)\, S_i(\ ).$$

Case two: $\sum_{j\in N_2^\alpha} |a_{ij}| \quad 0, \sum_{j\in N_2^\alpha} |a_{ji}| \ / \ 0$, in this case, $|a_{ij}| \quad 0$ for any $j \in \ _2^\alpha$. according to (4) we have:

$$\varepsilon < f_i \Leftrightarrow \varepsilon \sum_{j\in N_2^\alpha} |a_{ji}| < b_i \sum_{j\neq i} |a_{ji}| y_j$$

$$\Leftrightarrow (1+b_i)\sum_{j\neq i} |a_{ji}| y_j > \sum_{j\neq i} |a_{ji}| y_j + \varepsilon \sum_{j\in N_2^\alpha} |a_{ji}|. \qquad (6)$$

With (5), (6) and the hypothesis of the paper, we have:

$$b_{ii} \quad y_i\, |a_{ii}|\, x_i \quad y_i\alpha\,(1+b_i)\sum_{j\neq i} |a_{ij}|\, x_j + (1-\alpha)(1+b_i)\sum_{j\neq i} |a_{ji}|\, y_j x_i$$

$$> y_i\alpha \sum_{\substack{j\in \ _1^\alpha \\ j\ /\ i}} |a_{ij}|\, x_j + (1-\alpha)\left(\sum_{j\neq i} |a_{ji}|\, y_j + \varepsilon \sum_{j\in N_2^\alpha} |a_{ji}|\right) x_i$$

$$> y_i\alpha \sum_{\substack{j\in \ _1^\alpha \\ j\ /\ i}} |a_{ij}|\, x_j + x_i(1-\alpha) \times \left(\sum_{\substack{j\in \ _1^\alpha \\ j\ /\ i}} |a_{ji}|\, y_j + \sum_{j\in N_y^\alpha} |a_{ji}|\, y_j + \sum_{j\in N_2^\alpha\setminus N_y^\alpha} |a_{ji}|\,(y_j+\varepsilon)\right)$$

$$\alpha R_i(\ ) + (1-\alpha)\, S_i(\ ).$$

Case three: $\sum_{j\in N_2^\alpha} |a_{ij}| \ / \ 0, \sum_{j\in N_2^\alpha} |a_{ji}| \quad 0$. As the same proof of case two, we can obtain

$$|b_{ii}| > R_i(\ )^\alpha\, C_i(\ )^{1-\alpha}.$$

Case four: $\sum_{j\in N_2^\alpha} |a_{ij}| \ / \ 0, \sum_{j\in N_2^\alpha} |a_{ji}| \ / \ 0$ according to (4) we have:

$$\varepsilon < c_i \Leftrightarrow \varepsilon \sum_{j\in N_2^\alpha} |a_{ij}| < b_i \sum_{j\neq i} |a_{ij}| x_j$$

$$\Leftrightarrow (1+b_i)\sum_{j\neq i} |a_{ij}| x_j > \sum_{j\neq i} |a_{ij}| x_j + \varepsilon \sum_{j\in N_2^\alpha} |a_{ij}|.$$

From the above inequation and the inequation (6), we have

$$b_{ii} \quad y_i\, |a_{ii}|\, x_i \quad \alpha\,(1+b_i)\, y_i \sum_{j\neq i} |a_{ij}|\, x_j + (1-\alpha)(1+b_i)\, x_i \sum_{j\neq i} |a_{ji}|\, y_j$$

$$> \alpha y_i \left(\sum_{j\neq i} |a_{ij}|\, x_j + \varepsilon \sum_{j\in N_2^\alpha} |a_{ij}|\right) + (1-\alpha)\, x_i \left(\sum_{j\neq i} |a_{ji}|\, y_j + \varepsilon \sum_{j\in N_2^\alpha} |a_{ji}|\right)$$

$$\geq \alpha y_i \left(\sum_{j\in N_1^\alpha \bigcup N_y^\alpha} |a_{ij}|\, x_j + \sum_{j\in N_2^\alpha\setminus N_x^\alpha} (x_j+\varepsilon)\, |a_{ij}|\right)$$

$$+x_i\left(1-\alpha\right)\left(\sum_{j\in \mathrm{N}_1^\alpha\bigcup \mathrm{N}_y^\alpha}\left|a_{ji}\right|y_j+\sum_{j\in \mathrm{N}_2^\alpha\backslash \mathrm{N}_y^\alpha}\left(y_j+\varepsilon\right)\left|a_{ji}\right|\right)$$

$$\alpha R_i\left(\ \right)+\left(1-\alpha\right)S_i\left(\ \right).$$

For any $i\in\ \ {}_2^\alpha$, from the choice of ε and the positive diagonal matrices D and E, we know that $0<d_i,e_i\le 1$, for any $i\in\ \ {}_2^\alpha$.

Case one: $i\in\ \ {}_x^\alpha\cap\ \ {}_y^\alpha$

$$\left|b_{ii}\right|\quad \left|a_{ii}\right|\ge\alpha\left(\sum_{j\in \mathrm{N}_1^\alpha}\left|a_{ij}\right|x_j+\sum_{\substack{j\in\ \ {}_2^\alpha\\ j\ /\ i}}\left|a_{ij}\right|\right)$$

$$+\left(1-\alpha\right)\left(\sum_{j\in \mathrm{N}_1^\alpha}\left|a_{ji}\right|y_j+\sum_{\substack{j\in\ \ {}_2^\alpha\\ j\ /\ i}}\left|a_{ji}\right|\right)$$

$$>\alpha\left(\sum_{j\in\langle n\rangle\backslash \mathrm{N}_x^\alpha}\left|a_{ij}\right|x_j+\sum_{\substack{j\in\ \ {}_x^\alpha\\ j\ /\ i}}\left|a_{ij}\right|\right)$$

$$+\left(1-\alpha\right)\left(\sum_{j\in\langle n\rangle\backslash \mathrm{N}_y^\alpha}\left|a_{ji}\right|y_j+\sum_{\substack{j\in\ \ {}_y^\alpha\\ j\ /\ i}}\left|a_{ji}\right|\right)$$

$$\alpha R_i\left(\ \right)+\left(1-\alpha\right)S_i\left(\ \right).$$

Case two: $i\in\ \ {}_x^\alpha, i\notin\ \ {}_y^\alpha$, if $\alpha\ /\ 1$, from (2) we have

$$\begin{aligned}&\left(y_i+\varepsilon\right)\left|a_{ii}\right|\\ &\ge\left(y_i+\varepsilon\right)\alpha\left(\sum_{j\in \mathrm{N}_1^\alpha}\left|a_{ij}\right|x_j+\sum_{\substack{j\in\ \ {}_2^\alpha\\ j\ /\ i}}\left|a_{ij}\right|\right)\\ &+\left(1-\alpha\right)\left(\sum_{j\in \mathrm{N}_1^\alpha}\left|a_{ji}\right|y_j+\sum_{\substack{j\in\ \ {}_2^\alpha\\ j\ /\ i}}\left|a_{ji}\right|\right)\end{aligned},\qquad (7)$$

Hence

$$\left|b_{ii}\right|\quad\left(y_i+\varepsilon\right)\left|a_{ii}\right|$$

$$\ge\left(y_i+\varepsilon\right)\alpha\left(\sum_{j\in \mathrm{N}_1^\alpha}\left|a_{ij}\right|x_j+\sum_{\substack{j\in\ \ {}_2^\alpha\\ j\ /\ i}}\left|a_{ij}\right|\right)$$

$$+\left(1-\alpha\right)\left(\sum_{j\in \mathrm{N}_1^\alpha}\left|a_{ji}\right|y_j+\sum_{\substack{j\in\ \ {}_2^\alpha\\ j\ /\ i}}\left|a_{ji}\right|\right)$$

$$>\left(y_i+\varepsilon\right)\alpha\left(\sum_{j\in \mathrm{N}_1^\alpha}\left|a_{ij}\right|x_j+\sum_{j\in \mathrm{N}_2^\alpha\backslash \mathrm{N}_x^\alpha}\left|a_{ij}\right|\left(x_j+\varepsilon\right)+\sum_{j\in \mathrm{N}_x^\alpha}\left|a_{ij}\right|\right)$$

$$+\left(1-\alpha\right)\left(\sum_{j\in \mathrm{N}_1^\alpha}\left|a_{ji}\right|y_j+\sum_{\substack{j\in \mathrm{N}_2^\alpha\backslash \mathrm{N}_y^\alpha\\ j\neq i}}\left|a_{ji}\right|\left(y_j+\varepsilon\right)+\sum_{j\in \mathrm{N}_y^x}\left|a_{ji}\right|\right)$$

$$\alpha R_i\left(\ \right)+\left(1-\alpha\right)S_i\left(\ \right).$$

Case three: $i\notin\ \ {}_x^\alpha, i\in\ \ {}_y^\alpha$, as the same proof of case two, we can obtain

$$\left|b_{ii}\right|>\alpha R_i\left(\ \right)+\left(1-\alpha\right)S_i\left(\ \right).$$

Case four: $i\notin\ \ {}_x^\alpha, i\notin\ \ {}_y^\alpha$, from (2) we have

$$\left(y_i+\varepsilon\right)\left|a_{ii}\right|\left(x_i+\varepsilon\right)$$

$$\ge\left(y_i+\varepsilon\right)\alpha\left(\sum_{j\in \mathrm{N}_1^\alpha}\left|a_{ij}\right|x_j+\sum_{\substack{j\in\ \ {}_2^\alpha\\ j\ /\ i}}\left|a_{ij}\right|\right)$$

$$+\left(1-\alpha\right)\left(\sum_{j\in \mathrm{N}_1^\alpha}\left|a_{ji}\right|y_j+\sum_{\substack{j\in\ \ {}_2^\alpha\\ j\ /\ i}}\left|a_{ji}\right|\right)\left(x_i+\varepsilon\right)$$

Since

$$\left(y_i+\varepsilon\right)\alpha\left(\sum_{j\in \mathrm{N}_1^\alpha}\left|a_{ij}\right|x_j+\sum_{\substack{j\in\ \ {}_2^\alpha\\ j\ /\ i}}\left|a_{ij}\right|\right)>\alpha R_i\left(\ \right),$$

$$\left(1-\alpha\right)\left(\sum_{j\in \mathrm{N}_1^\alpha}\left|a_{ji}\right|y_j+\sum_{\substack{j\in\ \ {}_2^\alpha\\ j\ /\ i}}\left|a_{ji}\right|\right)\left(x_i+\varepsilon\right)$$

$$> (1-\alpha)\, S_i(\quad),$$

we have

$$|b_{ii}| \quad (y_i+\varepsilon)\,|a_{ii}|\,(x_i+\varepsilon)$$

$$\geq (y_i+\varepsilon)\,\alpha \left(\sum_{j\in \mathrm{N}_1^{\alpha}} |a_{ij}|\,x_j + \sum_{\substack{j \in \quad {}_2^{\alpha} \\ j \ / \ i}} |a_{ij}| \right)$$

$$+ (1-\alpha) \left(\sum_{j\in \mathrm{N}_1^{\alpha}} |a_{ji}|\,y_j + \sum_{\substack{j \in \quad {}_2^{\alpha} \\ j \ / \ i}} |a_{ji}| \right) (x_i+\varepsilon)$$

$$> \alpha R_i(\quad) + (1-\alpha)\, S_i(\quad).$$

We see that for any $i \in \langle n \rangle$, we have $|b_{ii}| > \alpha R_i(\quad) + (1-\alpha)\, S_i(\quad)$. According to Lemma 1, we know that matrix B is a nonsingular H-matrix, so matrix A is a nonsingular H-matrix.

Let $A \quad (a_{ij})_{n\times n} \in C^{n\times n}$, $0 < x_i,\ y_i < 1,\ i \in \langle n \rangle$ satisfy the equation (2), we denote

$$K_\alpha \quad \left\{ i \in < n > \,|\, |a_{ii}| > \frac{\alpha}{x_i} \sum_{j\neq i} |a_{ij}|\,x_j + \frac{1-\alpha}{y_i} \sum_{j\neq i} |a_{ji}|\,y_j \right\}.$$

Theorem 2 Let $A \quad (a_{ij})_{n\times n} \in C^{n\times n}$, for $\alpha \in \ 0, 1\ $, if $0 < x_i < 1,\ 0 < y_i < 1,\ i \in \langle n \rangle$ satisfy the inequations (2) and

$$|a_{ii}| \geq \frac{\alpha}{x_i} \sum_{j\neq i} |a_{ij}|\,x_j + \frac{1-\alpha}{y_i} \sum_{j\neq i} |a_{ji}|\,y_j,\ i \in \quad {}_1^{\alpha} \tag{8}$$

and $K_\alpha \ / \ \emptyset$, for any $i_0 \in (< n > \backslash K_\alpha)$, there exists a nonzero elements chain $a_{i_0 i_1} a_{i_1 i_2} \cdots a_{i_{k-1} i_k} \ / \ 0$ such that $i_k \in K_\alpha$, then A is a nonsingular H-matrix.

Proof: we structure two positive diagonal matrices: $D \quad diag(x_1, x_2, \cdots, x_n)$ and $\quad diag(y_1, y_2, \cdots, y_n)$, and notes $B \quad (b_{ij}) \quad EAD$. So for any $i \in < n >$, we have

$$|b_{ii}| \geq \alpha R_i(B) + (1-\alpha)\, S_i(B).$$

Obviously, K_α can be note

$$K_\alpha \quad \{ i \in < n > \,|\, |b_{ii}| > \alpha R_i(B) + (1-\alpha)\, S_i(B) \},$$

for any $i_0 \in K_\alpha$, we have $b_{i_0 i_1} b_{i_1 i_2} \cdots b_{i_{k-1} i_k} \ / \ 0$ such that $i_k \in K_\alpha$. So according to Lemma 2, we know that matrix B is a nonsingular H-matrix, so matrix A is a nonsingular H-matrix.

From Theorem 2, we can get the following corollary.

Corollary Let $A \quad (a_{ij})_{n\times n} \in C^{n\times n}$ be irreducible, for $\alpha \in (0,1)$, if $0 < x_i < 1,\ 0 < y_i < 1,\ i \in \langle n \rangle$ satisfy the inequations (2) and (8), $\widetilde{I} \ / \ \emptyset$, where

$$\widetilde{I} \quad \left\{ v \in S(A) \,|\, \sum_{i\in v} y_i\,|a_{ii}|\,x_i \ / \ \sum_{i\in v} \widetilde{R}_i(A) \right.$$
$$\left. or \ \sum_{i\in v} y_i\,|a_{ii}|\,x_i \ / \ \sum_{i\in v} \widetilde{C}_i(A) \right\},$$

$$\widetilde{R}_i(A) \quad y_i \sum_{j\neq i} |a_{ij}| x_j, \quad \widetilde{C}_i(A) \quad x_i \sum_{j\neq i} |a_{ji}| y_j,$$

then A is a nonsingular H-matrix.

References

[1] Horn R A, Johnson C A, *Matrix Analysis*, Cambridge: Cambridge University Press, 1985.
[2] Gao Y M, Wang X H, "Criteria for generalized diagonally dominant matrices and M-matrices", Lin. Alg. Appl., 1992, Vol.169, pp. 257-268.
[3] Berman A, Plemmons R J, Nonnegative Matrices in the Mathematical Sciences, New York: Academic Press, 1979.
[4] Sun Y X, "Sufficient Conditions of Generalized Diagonal Dominance Matrix", Numerical Mathematics (a Journal of Chinese Universities), 1997, Vol. 15(3), pp. 573-578.