On Graded Semiprime Submodules

Farkhonde Farzalipour and Peyman Ghiyasvand

Abstract—Let $G$ be an arbitrary group with identity $e$ and let $R$ be a $G$-graded ring. In this paper we define graded semiprime submodules of a graded $R$-module $M$ and we give a number of results concerning such submodules. Also, we extend some results of graded semiprime submodules to graded weakly semiprime submodules.

Keywords—graded semiprime, graded weakly semiprime, graded secondary.

I. INTRODUCTION

W EAKLY prime ideals in a commutative ring with nonzero identity have been introduced and studied by D. D. Anderson and S. Smith (see [1]). Weakly primary ideals in a commutative ring with nonzero identity have been introduced and studied in [4]. Also, weakly prime submodules have been studied in [5]. Graded prime ideals in a commutative $G$-graded ring with nonzero identity have been introduced and studied by M. Refaei and K. Alzobi in [11]. Also, graded weakly prime ideals in a commutative graded ring with nonzero identity have been studied by S. Ebrahimi Atani (see [2]). Graded prime submodules and graded weakly prime submodule have been studied in [6] and [3] respectively. Here we study graded semiprime and graded weakly semiprime submodules of a graded $R$-module. For example, we show that graded semiprime submodules of graded secondary modules are graded secondary. Throughout this work $R$ will denote a commutative $G$-graded ring with nonzero identity and $M$ a graded $R$-module.

Before we state some results let us introduce some notation and terminology. A ring $(R, G)$ is called a $G$-graded ring if there exists a family $\{R_g : g \in G\}$ of additive subgroups of $R$ such that $R = \bigoplus_{g \in G} R_g$ such that $R_g R_h \subseteq R_{gh}$ for each $g$ and $h$ in $G$. For simplicity, we will denote the graded ring $(R, G)$ by $R$. If $a \in R$, then $a$ can written uniquely as $\sum_{g \in G} a_g$ where $a_g$ is the component of $a$ in $R_g$. Also, we write $h(R) = \cup_{g \in G} R_g$. Moreover, if $R = \bigoplus_{g \in G} R_g$, is a graded ring, then $R_g$ is a subring of $R$, $1_R \in R_g$, and $R_g$ is an $R$-module for all $g \in G$. An ideal $I$ of $R$, where $R$ is $G$-graded, is called $G$-graded if $I = \bigoplus_{g \in G} (I \cap R_g)$ or if, equivalently, $I$ is generated by homogeneous elements.

Moreover, $R/I$ becomes a $G$-graded ring with $g$-component $(R/I)_g = (R_g + I)/I$ for $g \in G$. Let $I$ be a graded ideal of $R$, graded radical ideal of $R$, $\text{rad}(R) = \{r \in R : x^n_r \in I \text{ for some } n \in N\}$. A graded ideal $I$ of $R$ is said to be graded prime if $I \neq R$; and whenever $ab \in I$, we have $a \in I$ or $b \in I$, where $a, b \in h(R)$. A proper graded ideal $P$ of $R$ is said to be graded weakly prime if $0 \neq ab \in P$ where $a, b \in h(R)$.

F. Farzalipour and P. Ghiyasvand are with the Department of Mathematics, Payame Noor University, Tehran 19395-3697, Iran, e-mail: (f_farzalipour@pnu.ac.ir and p_ghiasvand@pnu.ac.ir).

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implies $a \in P$ or $b \in P$. A graded ideal $I$ of $R$ is said to be graded maximal if $I \neq R$ and if $J$ is a graded ideal of $R$ such that $I \subseteq J \subseteq R$, then $I = J$ or $J = R$. A graded ring $R$ is called a graded integral domain if $ab = 0$ for $a, b \in h(R)$, then $a = 0$ or $b = 0$. A graded ring $R$ is called a graded local ring if it has a unique graded maximal ideal $P$, and denoted by $(R, P)$. Let $R_1$ and $R_2$ be graded rings. Let $R = R_1 \times R_2$, clearly $R$ is a graded ring. We write $h(R) = h(R_1) \times h(R_2)$.

If $R$ is $G$-graded, then an $R$-module $M$ is said to be $G$-graded if it has a direct sum decomposition $M = \bigoplus_{g \in G} M_g$ such that for all $g, h \in G; R_g M_h \subseteq M_{gh}$. An element of some $R_g$ or $M_g$ is said to be homogeneous element. A submodule $N \subseteq M$, where $M$ is $G$-graded, is called $G$-graded if $N = \bigoplus_{g \in G} (N \cap M_g)$ or if, equivalently, $N$ is generated by homogeneous elements. Moreover, $M/N$ becomes a $G$-graded module with $g$-component $(M/N)_g = (M_g + N)/N$ for $g \in G$. A proper graded submodule $N$ of a graded module $M$ over a commutative graded ring $R$ is said to be graded prime if whenever $\left(\frac{r^m}{n}\right) \in N$, for some $r \in h(R), m \in h(M)$, then $r \in h(M)$ or $m = n \in N$. A proper graded submodule $N$ of a graded $R$-module $M$ is said to be graded weakly prime if $0 \neq rm \in N$ where $r \in h(R), m \in h(M)$, then $r \in h(M)$ or $m \in N$. Let $R$ be a $G$-graded ring and $S \subseteq h(R)$ be a multiplicatively closed subset of $R$. Then the ring of fraction $S^{-1}R$ is a graded ring which is called the graded ring of fractions. Indeed, $S^{-1}R = \bigoplus_{g \in G} (S^{-1}R)_g$ where $(S^{-1}R)_g = \{r/s : r \in R, s \in S \text{ and } g = (\text{degs})^{-1}(\text{degr})\}$. We write $h(S^{-1}R) = \bigcup_{g \in G} (S^{-1}R)_g$. Let $M$ be a graded $R$-module. The module of fraction $S^{-1}M$ over a graded ring $S^{-1}R$ is a graded module which is called the module of fractions, if $S^{-1}M = \bigoplus_{g \in G} (S^{-1}M)_g$ where $(S^{-1}M)_g = \{m/s : m \in M, s \in S \text{ and } g = (\text{degs})^{-1}(\text{degr})\}$. Let $P$ be any graded prime ideal of a graded ring $R$ and consider the multiplicatively closed subset of $S = h(R) - P$. We denote the graded ring of fraction $S^{-1}R$ of $R$ by $R_P$ and we call it the graded localization of $R$. This ring is graded local with the unique graded maximal ideal $S^{-1}P$ which will be denoted by $PR^*_P$. Moreover, $R^*_P$-module $S^{-1}M$ is denoted by $M_P^*$ (see [9]).

II. GRADED SEMIPRIME SUBMODULES

In this section, we define the graded semiprime submodules of a graded $R$-module $M$ and give some of their basic properties.

Definition 2.1: Let $R$ be a graded ring and $M$ a graded $R$-module. A proper graded submodule $N$ of $M$ is said to be graded semiprime, if $\left(\frac{r}{n}\right) \in N$ for some $r \in h(R), m \in h(M)$ and $k \in Z^+$, then $rm \in N$.}

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It is clear that every graded prime submodule is a graded semiprime submodule, but the converse is not true in general. For example, let \( R = \mathbb{Z}_{10}[t] = \{a + bt : a, b \in \mathbb{Z}_{10}\} \) that \( \mathbb{Z}_{10} \) is the ring of integers modulo 30 and let \( G = \mathbb{Z}_2 \). Then \( R \) is a \( G \)-graded ring with \( R_0 = \mathbb{Z}_{10}, R_1 = i\mathbb{Z}_{10} \). Let \( I = 6 > \oplus < 0 > \). The graded ideal \( I \) is graded semiprime, but it is not graded prime. Because \((2, 0), (3, 0) \in I\), but \((2, 0) \notin I\) and \((3, 0) \notin I\).

**Definition 2.2**: Let \( N \) be a graded submodule of graded \( R \)-module \( M \) and \( g \in G \). We say that \( N_g \) is a semiprime submodule of \( R_g \)-module \( M_g \), if \( r^2m_g \in N_g \) where \( r \in R_g \), \( m_g \in M_g \), then \( rm_g \in N_g \).

**Proposition 2.3**: Let \( M \) be a \( G \)-graded \( R \)-module and \( N = \bigoplus_{g \in G} N_g \) a graded submodule of \( M \). If \( N \) is a graded semiprime submodule of \( M \), then \( N_g \) is a graded submodule of \( R_g \)-module for any \( g \in G \).

**Proof**: Let \( r^2m_g \in N_g \) where \( r \in R_g \), \( m_g \in M_g \) and \( k \in \mathbb{Z}^+ \). Then \( r^2m_g \in N_g \subseteq N \), hence \( rm_g \in N \) since \( N \) is a graded semiprime submodule. Since \( R_g \)-module \( M_g \subseteq M_g \), so \( rm_g \in N_g \), as required.

The following Lemma is known, but we write it here for the sake of references.

**Lemma 2.4**: Let \( M \) be a graded module over a graded ring \( R \). Then the following hold:

1. If \( I \) and \( J \) are graded ideals of \( R \), then \( I + J \) and \( I \cap J \) are graded ideals.
2. If \( N \) is a graded submodule, \( r \in h(R) \) and \( x \in h(M) \), then \( Rrx \) and \( rN \) are graded submodules of \( M \).
3. If \( N \) and \( K \) are graded submodules of \( M \), then \( N + K \) and \( N \cap K \) are also graded submodules of \( M \) and \( N :_R M \) is a graded ideal of \( R \).

**Proposition 2.5**: Let \( M \) be a graded \( R \)-module, \( N \) a graded semiprime submodule of \( M \) and \( m \in h(M) \). Then

1. If \( m \in N \), then \((N : m) = R \).
2. If \( m \notin N \), then \((N : m) \) is a graded semiprime submodule of \( M \).

**Proof**: (i) It is clear.

(ii) Let \( x^k y \in (N : m) \) where \( x,y \in h(R) \) and \( k \in \mathbb{Z}^+ \). Hence \( x^km \in N \), so \( xym \in N \) since \( N \) is graded semiprime. Therefore \( xy \in (N : m) \), as needed.

**Proposition 2.6**: Let \( M \) be a graded \( R \)-module and \( I \) a graded ideal of \( R \). If \( N \) is a graded semiprime submodule of \( M \) such that \( I^k M \subseteq N \) for some \( k \in \mathbb{Z}^+ \), then \( IM \subseteq N \).

**Proof**: Let \( am \in IM \) where \( a \in I \) and \( m \in M \). So \( a = \sum_{g \in G} a_g \) where \( a_g \in I \cap h(R) \) and \( m = \sum_{g \in G} m_g \) where \( m_g \in h(M) \). Hence for any \( g,h \in G \), \( a_g m_h \in I^k M \subseteq N \), so \( a_mh \in N \) since \( N \) is a graded semiprime submodule. Therefore \( am \in N \), as needed.

A graded \( R \)-module \( M \) is called graded multiplication if for any graded submodule \( N \) of \( M \), \( N = IM \) for some graded ideal \( I \) of \( R \) (see [9]).

**Proposition 2.7**: Let \( M \) be a graded multiplication \( R \)-module and \( K \) a graded submodule of \( M \). If \( N \) is a graded semiprime submodule of \( M \) such that \( K^n \subseteq N \) for some \( n \in \mathbb{Z}^+ \), then \( K \subseteq N \). Moreover, if \( K^n = N \) for some \( n \in \mathbb{Z}^+ \), then \( K = N \).

**Proof**: Since \( M \) is a graded multiplication module, so \( K = IM \) for some graded ideal \( I \) of \( R \). Hence \( K^n = (IM)^n = I^m M \subseteq N \), then \( K \subseteq N \) by Proposition 2.6. Clearly, if \( K^n = N \) for some \( n \in \mathbb{Z}^+ \), then \( K = N \).

**Proposition 2.8**: Let \( R = R_1 \times R_2 \) where \( R_i, i = 1, 2 \) is a graded commutative ring with identity for \( i = 1,2 \). Let \( M \) be a graded \( R \)-module and let \( M = M_1 \times M_2 \) be the graded \( R \)-module with action \( (r_1, r_2)(m_1, m_2) = (r_1 m_1, r_2 m_2) \) where \( r_1 \in R_1 \) and \( m_i \in M_i \). Then the following hold:

1. \( N_1 \) is a graded semiprime submodule of \( M_1 \) if and only if \( N_1 \times M_2 \) is a graded semiprime submodule of \( M \).
2. \( N_2 \) is a graded semiprime submodule of \( M_2 \) if and only if \( M_1 \times N_2 \) is a graded semiprime submodule of \( M \).

**Proof**: (i) \( N_1 \) is a graded submodule of \( M_1 \). Suppose \((a, b)(m, n) \in N_1 \times M_2 \) where \((a, b) \in h(R) = h(R_1) \times h(R_2), (m, n) \in h(M) = h(M_1) \times h(M_2) \) and \( k \in \mathbb{Z}^+ \). So \( a^m \in N_1 \), and \( am \in N_1 \) since \( N_1 \) is a graded semiprime submodule. Hence \((a, b)(m, n) \in N_1 \times M_2 \), as required. Let \( N_1 \times M_2 \) be a graded submodule of \( M \). Let \( a^m \in N_1 \) where \( a \in h(R_1), m \in h(M_2) \) and \( k \in \mathbb{Z}^+ \). So \((a, 1)(m, 0) \in N_1 \times M_2 \), hence \((a, 1)(m, 0) \in N_1 \times M_2 \) since \( N_1 \times M_2 \) is a graded submodule of \( M \). Hence \( am \in N_1 \), as needed.

(ii) The proof is similar to that in case (i) and we omit it.

A graded \( R \)-module \( M \) is called a graded secondary module provided that for every homogeneous element \( r \in h(R) \), \( rM = M \) or \( r^m M = 0 \) for some positive integer \( n \) (see [7]).

**Theorem 2.9**: Let \( M \) be a graded secondary \( R \)-module and \( N \) a nonzero graded semiprime \( R \)-submodule of \( M \). Then \( N \) is graded secondary \( R \)-module.

**Proof**: Let \( r \in h(R) \). If \( r^m M = 0 \) for some positive integer \( n \), then \( r^m N \subseteq r^m M = 0 \), so \( rN \) is nilpotent on \( N \). Suppose that \( rM = M \); we show that \( r \) divides \( N \). Let \( n \in N \). We may assume that \( n = \sum_{g \in G} n_g \) where \( n_g \neq 0 \). So for every \( g \in G \), \( n_g = r^m \) for some \( m \in h(M) \). We have \( rm = m \) for some \( m \in h(M) \), hence \( rm = r^m m \), so \( m = rm \in N \) since \( N \) is graded semiprime. Hence \( n = rm \in rN \). Thus \( rN = N \), as needed.

**Corollary 2.10**: Let \( M \) be a graded \( R \)-module, \( N \) a graded secondary \( R \)-submodule of \( M \) and \( K \) a graded semiprime submodule of \( M \). Then \( N \cap K \) is graded secondary.

**Proof**: The proof is straightforward by Theorem 2.7.

**Proposition 2.11**: Let \( R \) be a graded ring and \( S \subseteq h(R) \) be a multiplication closed subset of \( R \). If \( N \) is a graded semiprime
submodule of $M$, then $S^{-1}N$ is a graded semiprime submodule of $S^{-1}M$.

**Proof:** Let $(r/s)^k m/t \in S^{-1}N$ where $r/s \in h(S^{-1}R)$, $m/t \in h(S^{-1}M)$ and $k \in Z^+$. So $r^km/s^k t = n/t'$ for some $n \in N \cap h(M)$ and $t' \in S$, hence there exists $s' \in S$ such that $s't'r^km = s's^k t'n \in N$, so $N$ graded semiprime gives $rm/s't' \in N$. Hence $rm/st = rm/s't'/st's' \in S^{-1}N$, as needed.

**Proposition 2.12:** Let $(R, P)$ be a graded local ring with graded maximal ideal $P$ and $S = h(R) - P$. Then $N$ is a graded semiprime submodule of graded $R$-module $M$ if and only if $N^g_p$ is a graded semiprime submodule of graded $R^g_p$-module $M^g_p$.

**Proof:** Let $N$ be a graded semiprime submodule of $M$, then $N^g_p$ is a graded semiprime submodule of $M^g_p$ by Proposition 2.11. Let $r^km \in N$ where $r \in h(R)$, $m \in h(M)$ and $k \in Z^+$. So $r^km/1 = (r/1)^k m/1 \in N^g_p$. Hence $rm/1 \in N^g_p$ and $rm/1 = c/s$ for some $c \in N \cap h(N)$ and $s \in S$. So there exists $t \in S$ such that $stm = te \in N$. So $rm \in N$, because if $rm \notin N$, then $(N : rm) \cap R \neq R$, and $st \in (N : rm) \cap S \subseteq P \cap S = \emptyset$, which is a contradiction. Therefore $N$ is a graded semiprime submodule of $M$.

**Proposition 2.13:** Let $K \subseteq N$ be proper graded submodules of a graded $R$-module $M$. Then $N$ is a graded semiprime submodule of $M$ if and only if $N/K$ is a graded semiprime submodule of $M/N$.

**Proof:** ($\Rightarrow$) Let $r^k(m + K) \in N/K$ where $r \in h(R), m \in h(M)$ and $K \subseteq Z^+$. So $r^km \in N$, $N$ graded semiprime gives $rm \in N$. Hence $r(m + K) \in N/K$.

($\Leftarrow$) Let $r^km \in N$ where $r \in h(R), m \in h(M)$ and $k \in Z^+$. So $r^km + K = r^k(m + K) \in N/K$. Then $r(m + K) \in N/K$ since $N/K$ is graded semiprime. Hence $rm \in N$, as required.

### III. Graded Weakly Semiprime Submodules

In this section, we define the graded weakly semiprime submodules of a graded $R$-module and we extend some results of graded semiprime submodules to graded weakly semiprime submodules.

**Definition 3.1:** Let $R$ be a graded ring and $M$ a graded $R$-module. A proper graded submodule $N$ of $M$ is said to be graded weakly semiprime, if $0 \neq r^km \in N$ for some $r \in h(R)$, $m \in h(M)$ and $k \in Z^+$, then $rm \in N$.

It is clear that every graded semiprime submodule is a graded weakly semiprime submodule. However, since 0 is always graded weakly semiprime, a graded weakly semiprime submodule need not be graded semiprime, but if $R$ is a graded integral domain and $M$ a faithful graded prime module, then every graded weakly semiprime is graded semiprime.

**Definition 3.2:** Let $N$ be a graded submodule of a graded $R$-module $M$ and $g \in G$. We say that $N_g$ is a weakly semiprime submodule of $R_e$-module $M_g$, if $r^km \in N_g$ where $r_e \in R_e$, $m \in M_g$ and $k \in Z^+$, then $rm \in N_g$.

**Proposition 3.3:** Let $M$ be a graded $R$-module and $N = \bigoplus_{g \in G} N_g$ a graded submodule of $M$. If $N$ is a graded weakly semiprime submodule of $M$, then $N_g$ is a weakly semiprime submodule of $R_g$-module $M_g$ for any $g \in G$.

**Proof:** Let $0 \neq r^km \in N_g$ where $r_e \in R_e$, $m \in M_g$ and $k \in Z^+$. So $r^km \in N_g \subseteq N$, hence $rm \in N$ since $N$ is a graded weakly semiprime submodule. Since $R_gM_g \subseteq M_{eg} = M_g$, so $rm \in N_g$, as required.

**Theorem 3.4:** Let $R$ be a graded ring, $M$ a graded $R$-module, $N$ a graded submodule of $M$ and $g \in G$. Consider the following assertion.

(i) $N_g$ is a weakly semiprime submodule of $M_g$, (ii) For $a \in M_g$, $Rad(N_g : R_g a) = (N_g : R_g a) \cup Rad(0 : R_g a)$, (iii) For $a \in M_g$, $Rad(N_g : R_g a) = (N_g : R_g a) \cup Rad(0 : R_g a)$.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

**Proof:** (i) $\Rightarrow$ (ii) It is clear that $(N_g : R_g a) \cup Rad(0 : R_g a) \subseteq Rad(N_g : R_g a)$. Let $r \in Rad(N_g : R_g a)$. So $r^na \in N_g$ for some positive integer $n$. If $r^na = 0$, then $r \in Rad(0 : R_g a)$. If $0 \neq r^na \in N_g$, then $r^na \in N_g$ since $N_g$ is a weakly semiprime submodule of $M_g$. Hence $Rad(N_g : R_g a) \subseteq (N_g : R_g a) \cup Rad(0 : R_g a)$. Therefore the proof is complete.

(ii) $\Rightarrow$ (i) It is well known that if an ideal (a subgroup) is the union of two ideals (two subgroups), then it is equal to one of them.

An $R_e$-module $M_g$ is called prime module if the zero submodule is prime.

**Remark 3.5:** An $R_e$-module $M_g$ is prime if and only if $(0 : R_e M_g) = (0 : R_e m_g)$ for any $0 \neq m_g \in M_g$.

**Theorem 3.6:** Let $R$ be a graded ring, $M$ a graded $R$-module, $N$ a graded submodule of $M$, and $g \in G$. Then the following assertion are equivalent.

(i) $N_g$ is a weakly semiprime submodule of $M_g$, (ii) For $a \in M_g$, $Rad(N_g : R_g a) = (N_g : R_g a) \cup Rad(0 : R_g a)$. (iii) For $a \in M_g$, $Rad(N_g : R_g a) = (N_g : R_g a)$ or $Rad(N_g : R_g a) = Rad(0 : R_g a)$.

**Proof:** It is enough to show that (iii) $\Rightarrow$ (i) Let $0 \neq r^km \in N_g$ where $r_e \in R_e$, $m \in M_g$ and $k \in Z^+$. So $r \in Rad(N_g : R_g m)$. If $r \in Rad(0 : m)$, then $r^km = 0$ for some $n \in Z^+$. Let $t$ be the smallest integer such that $r^tm = 0$. If $t > k$, then $0 < t - k < t$; $r^km = r^{k}(r^t-m) = 0$; $r^km \in (0 : R_t r^k-m) \subseteq (0 : R_e M_g)$ since $M_g$ is a graded prime module. Hence $r^km = 0$, so $r^km = 0$, a contradiction. Therefore $r \in Rad(0 : R_g m)$. So $r \in (N_g : m)$, hence $rm \in N_g$, as needed.

**Proposition 3.7:** Let $R = R_1 \times R_2$ where $R_i$ for $i = 1, 2$, is a commutative graded ring with identity. Let $M_i$ be a graded $R_i$-module and let $M = M_1 \times M_2$ be the graded $R$-module.
Then the following hold:
(i) If $N_1 \times M_2$ is a graded weakly semiprime submodule of $M$, then $N_1$ is a graded weakly semiprime submodule of $M_1$.
(ii) If $M_1 \times N_2$ is a graded weakly semiprime submodule of $M$, then $N_2$ is a graded weakly semiprime submodule of $M_2$.

Proof: Let $N_1 \times M_2$ be a graded weakly semiprime submodule of $M$. Suppose $0 \neq a^k m \in N_1$ where $a \in h(R_1)$, $m \in h(M_2)$ and $k \in \mathbb{Z}^+$. So $0 \neq (a, 1)^k (m, 0) \in N_1 \times M_2$, then $(a, 1)^k (m, 0) \in N_1 \times M_2$ since $N_1 \times M_2$ is a graded weakly semiprime. Hence $am \in N_1$, so $N_1$ is a graded weakly semiprime submodule of $M_1$.

(ii) The proof is similar to that in case (i).

Theorem 3.8: Let $M$ be a graded secondary $R$-module and $N$ a nonzero graded weakly semiprime submodule of $M$. Then $N \cap K$ is graded secondary.

Proof: Let $r \in h(R)$. If $r^n M = 0$ for some positive integer $n$, then $r^n N \subseteq r^n M = 0$, so $r$ is nilpotent on $N$. Suppose that $r M = M$; we show that $r$ divides $N$. Let $0 \neq n \in N$. We may assume that $n = \sum_{g \in G} n_g$ where $n_g \neq 0$. So for any $g \in G$, $n_g = rm$ for some $m \in h(M)$. We have $rm' = m$ for some $m' \in h(M)$, so $m = m' N$ since $N$ is a graded weakly semiprime submodule. Hence $n = rN$, so $n \in rN$. Therefore $rN = N$, as needed.

Corollary 3.9: Let $M$ be a graded $R$-module, $N$ a graded secondary $R$-module of $M$ and $K$ a graded weakly semiprime submodule of $M$. Then $N \cap K$ is graded secondary.

Proof: The proof is straightforward by Theorem 3.8.

Proposition 3.10: Let $R$ be a graded ring and $S \subseteq h(R)$ be a multiplicity closed subset of $R$. If $N$ is a graded weakly semiprime submodule of $M$, then $N^* - N$ is a graded weakly semiprime submodule of $S^* - M$.

Proposition 3.10: Let $R$ be a graded ring and $S \subseteq h(R)$ be a multiplicity closed subset of $R$. If $N$ is a graded weakly semiprime submodule of $M$, then $N^* - N$ is a graded weakly semiprime submodule of $S^* - M$.

Proof: Let $0 / 1 \neq (r/s)^{k} t^m / t^j \in S^* - N$ where $r / s \in h(S^* - R)$, $t^j / t^m \in h(S^* - M)$ and $k \in \mathbb{Z}^+$. So $0 / 1 \neq r^{k} s^{-k} t^m = n / t^j$ for some $n \in N \cap h(M)$ and $t^j / t^m \in S^* - M$ (because $s^t r^m = 0$, $r^{k} s^{-k} t = s^t r^m / s^t s^k t = 0 / 1$, a contradiction), so $N$ graded weakly semiprime gives $r \in N$. Hence $r / s \in S$, which is a contradiction.

Therefore $N$ is a graded weakly semiprime submodule of $M$.

Proposition 3.12: Let $K \subseteq N$ be proper graded submodules of a graded $R$-module $M$. Then the following hold:
(i) If $N$ is a graded weakly semiprime submodule of $M$, then $N / K$ is a graded weakly semiprime submodule of $M / N$.
(ii) If $K$ and $N / K$ are graded weakly semiprime submodules of $M$ and $M / K$ respectively, then $N$ is a graded weakly semiprime submodule of $M$.

Proof: (i) Let $0 \neq r^k (m + K) \in N / K$ where $r \in h(R)$, $m + K \in h(M / K)$ and $k \in \mathbb{Z}^+$. So $0 \neq r^k m \in N$, $N$ weakly semiprime gives $r m N \in K$. Hence $r (m + K) \in K / N$.

(ii) Let $0 \neq r^k m \in N$ where $r \in h(R), m \in h(M)$ and $k \in \mathbb{Z}^+$. So $r^k m + K = r^k (m + K) \in K / N$. If $0 \neq r^k m \in K$, then $r m \in K \subseteq N$ since $K$ is graded weakly semiprime, as needed.

Therefore $0 \neq r^k (m + K) \in K / N$, then $r (m + K) \in K / N$ since $N$ is graded weakly semiprime. Hence $r m \in N$, as required.

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