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# On Graded Semiprime Submodules

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*Abstract*—Let $G$ be an arbitrary group with identity $e$ and let $R$ be a $G$-graded ring. In this paper we define graded semiprime submodules of a graded $R$-module $M$ and we give a number of results concerning such submodules. Also, we extend some results of graded semiprime submoduls to graded weakly semiprime submodules.

*Keywords*—graded semiprime, graded weakly semiprime, graded secondary.

## I. Introduction

Weakly prime ideals in a commutative ring with nonzero identity have been introduced and studied by D. D. Anderson and S. Smith (see [1]). Weakly primary ideals in a commutative ring with nonzero identity have been introduced and studied in [4]. Also, weakly prime submodules have been studied in [5]. Graded prime ideals in a commutative $G$-graded ring with nonzero identity have been introduced and studied by M. Refaei and K. Alzobi in [11]. Also, graded weakly prime ideals in a commutative graded ring with nonzero identity have been studied by S. Ebrahimi Atani (see [2]). Graded prime submodules and graded weakly prime submoduled have been studied in [6] and [3] respectively. Here we study graded semiprime and graded weakly semiprime submodules of a graded $R$-module. For example, we show that graded semiprime submodules of graded secondary modules are graded secondary. Throughout this work $R$ will denote a commutative $G$-graded ring with nonzero identity and $M$ a graded $R$-module.

Before we state some results let us introduce some notation and terminology. A ring $(R, G)$ is called a $G$-graded ring if there exists a family $\{R_g : g \in G\}$ of additive subgroups of $R$ such that $R = \bigoplus_{g\in G} R_g$ such that $R_gR_h \subseteq R_{gh}$ for each $g$ and $h$ in $G$. For simplicity, we will denote the graded ring $(R, G)$ by $R$. If $a \in R$, then $a$ can written uniquely as $\sum_{g\in G} a_g$ where $a_g$ is the component of $a$ in $R_g$. Also, we write $h(R) = \cup_{g\in G} R_g$. Moreover, if $R = \bigoplus_{g\in G} R_g$, is a graded ring, then $R_e$ is a subring of $R$, $1_R \in R_e$ and $R_g$ is an $R_e$-module for all $g \in G$. A ideal $I$ of $R$, where $R$ is $G$-graded, is called $G$-graded if $I = \bigoplus_{g\in G}(I \cap R_g)$ or if, equivalently, $I$ is generated by homogeneous elements. Moreover, $R/I$ becomes a $G$-graded ring with $g$-component $(R/I)_g = (R_g + I)/I$ for $g \in G$. Let $I$ be a graded ideal of $R$, graded radical $I$ of $R$, $Grad(R) = \{r \in R : x_g^{n_g} \in I$ for some $n_g \in N\}$. A graded ideal $I$ of $R$ is said to be graded prime if $I \neq R$; and whenever $ab \in I$, we have $a \in I$ or $b \in I$, where $a, b \in h(R)$. A proper graded ideal $P$ of $R$ is said to be graded weakly prime if $0 \neq ab \in P$ where $a, b \in h(R)$, implies $a \in P$ or $b \in P$. A graded ideal $I$ of $R$ is said to be graded maximal if $I \neq R$ and if $J$ is a graded ideal of $R$ such that $I \subseteq J \subseteq R$, then $I = J$ or $J = R$. A graded ring $R$ is called a graded integral domain if $ab = 0$ for $a, b \in h(R)$, then $a = 0$ or $b = 0$. A graded ring $R$ is called a graded local ring if it has a unique graded maximal ideal $P$, and denoted by $(R, P)$. Let $R_1$ and $R_2$ be graded rings. Let $R = R_1 \times R_2$, clearly $R$ is a graded ring. We write $h(R) = h(R_1) \times h(R_2)$. If $R$ is $G$-graded, then an $R$-module $M$ is said to be $G$-graded if it has a direct sum decomposition $M = \bigoplus_{g\in G} M_g$ such that for all $g, h \in G$; $R_gM_h \subseteq M_{gh}$. An element of some $R_g$ or $M_g$ is said to be homogeneous element. A submodule $N \subseteq M$, where $M$ is $G$-graded, is called $G$-graded if $N = \bigoplus_{g\in G}(N \cap M_g)$ or if, equivalently, $N$ is generated by homogeneous elements. Moreover, $M/N$ becomes a $G$-graded module with $g$-component $(M/N)_g = (M_g + N)/N$ for $g \in G$. A proper graded submodule $N$ of a graded module $M$ over a commutative graded ring $R$ is said to be graded prime if whenever $r^k m \in N$, for some $r \in h(R)$, $m \in h(M)$, then $rM \subseteq N$ or $m \in N$. A proper graded submodule $N$ of a graded $R$-module $M$ is said to be graded weakly prime if $0 \neq rm \in N$ where $r \in h(R)$, $m \in h(M)$, then $m \in N$ or $rM \subseteq N$. Let $R$ be a $G$-graded ring and $S \subseteq h(R)$ be a multiplicatively closed subset of $R$. Then the ring of fraction $S^{-1}R$ is a graded ring which is called the graded ring of fractions. Indeed, $S^{-1}R = \bigoplus_{g\in G}(S^{-1}R)_g$ where $(S^{-1}R)_g = \{r/s : r \in R, s \in S$ and $g = (degs)^{-1}(degr)\}$. We write $h(S^{-1}R) = \bigcup_{g\in G}(S^{-1}R)_g$. Let $M$ be a graded $R$-module. The module of fraction $S^{-1}M$ over a graded ring $S^{-1}R$ is a graded module which is called the module of fractions, if $S^{-1}M = \bigoplus_{g\in G}(S^{-1}M)_g$ where $(S^{-1}M)_g = \{m/s : m \in M, s \in S$ and $g = (degs)^{-1}(degm)\}$. Let $P$ be any graded prime ideal of a graded ring $R$ and consider the multiplicatively closed subset of $S = h(R) - P$. We denote the graded ring of fraction $S^{-1}R$ of $R$ by $R_P^g$ and we call it the graded localization of $R$. This ring is graded local with the unique graded maximal ideal $S^{-1}P$ which will be denoted by $PR_P^g$. Moreover, $R_P^g$-module $S^{-1}M$ is denoted by $M_P^g$ (see [9]).

## II. Graded Semiprime Submodules

In this section, we define the graded semiprime submodules of a graded $R$-module $M$ and give some of their basic properties.

*Definition 2.1:* Let $R$ be a graded ring and $M$ a graded $R$-module. A proper graded submodule $N$ of $M$ is said to be graded semiprime, if $r^k m \in N$ for some $r \in h(R)$, $m \in h(M)$ and $k \in Z^+$, then $rm \in N$.

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It is clear that every graded prime submodule is a graded semiprime submodule, but the converse is not true in general. For example, let $R = Z_{30}[i] = \{a + bi : a, b \in Z_{30}\}$ that $Z_{30}$ is the ring of integers modulo 30 and let $G = Z_2$. Then $R$ is a $G$-graded ring with $R_0 = Z_{30}$, $R_1 = iZ_{30}$. Let $I =< 6 > \bigoplus < 0 >$. The graded ideal $I$ is graded semiprime, but it is not graded prime. Because $(2, 0).(3, 0) \in I$, but $(2, 0) \notin I$ and $(3, 0) \notin I$.

*Definition 2.2:* Let $N$ be a graded submodule of graded $R$-module $M$ and $g \in G$. We say that $N_g$ is a semiprime submodule of $R_e$-module $M_g$, if $r_e^k m_g \in N_g$ where $r_e \in R_e$, $m_g \in M_g$, then $r_e m_g \in N_g$.

*Proposition 2.3:* Let $M$ be a $G$-graded $R$-module and $N = \bigoplus_{g\in G} N_g$ a graded submodule of $M$. If $N$ is a graded semiprime submodule of $M$, then $N_g$ is a semiprime submodule of $R_e$-module $M_g$ for any $g \in G$.

*Proof:* Let $r_e^k m_g \in N_g$ where $r_e \in R_e$, $m_g \in M_g$ and $k \in Z^+$. So $r_e^k m_g \in N_g \subseteq N$, hence $r_e m_g \in N$ since $N$ is a graded semiprime submodule. Since $R_e M_g \subseteq M_{eg} = M_g$, so $r_e m_g \in N_g$, as required. ■

The following Lemma is known, but we write it here for the sake of references.

*Lemma 2.4:* Let $M$ be a graded module over a graded ring $R$. Then the following hold:
(i) If $I$ and $J$ are graded ideals of $R$, then $I + J$ and $I \bigcap J$ are graded ideals.
(ii) If $N$ is a graded submodule, $r \in h(R)$ and $x \in h(M)$, then $Rx$, $IN$ and $rN$ are graded submodules of $M$.
(iii) If $N$ and $K$ are graded submodules of $M$, then $N + K$ and $N \bigcap K$ are also graded submodules of $M$ and $(N :_R M)$ is a graded ideal of $R$.
(iv) Let $N_\lambda$ be a collection of graded submodules of $M$. Then $\sum_\lambda N_\lambda$ and $\bigcap_\lambda N_\lambda$ are graded submodues of $M$.

*Proposition 2.5:* Let $M$ be a graded $R$-module, $N$ a graded semiprime submodule of $M$ and $m \in h(M)$. Then
(i) If $m \in N$, then $(N : m) = R$.
(ii) If $m \notin N$, then $(N : m)$ is a graded semiprime submodule of $M$.

*Proof:* (i) It is clear.
(ii) Let $x^k y \in (N : m)$ where $x, y \in h(R)$ and $k \in Z^+$. Hence $x^k ym \in N$, so $xym \in N$ since $N$ is graded semiprime. Therefore $xy \in (N : m)$, as needed. ■

*Proposition 2.6:* Let $M$ be a graded $R$-module and $I$ a graded ideal of $R$. If $N$ is a graded semiprime submodule of $M$ such that $I^k M \subseteq N$ for some $k \in Z^+$, then $IM \subseteq N$.

*Proof:* Let $am \in IM$ where $a \in I$ and $m \in M$. So $a = \sum_{g\in G} a_g$ that $a_g \in I \cap h(R)$ and $m = \sum_{g\in G} m_h$ that $m_h \in h(M)$. Hence for any $g, h \in G$, $a_g^k m_h \in I^k M \subseteq N$, so $a_g m_h \in N$ since $N$ is a graded semiprime submodule. Therefore $am \in N$, as needed. ■

A graded $R$-module $M$ is called graded multiplication if for any graded submodule $N$ of $M$, $N = IM$ for some graded ideal $I$ of $R$ (see [9]).

*Proposition 2.7:* Let $M$ be a graded multiplication $R$-module and $K$ a graded submodule of $M$. If $N$ is a graded semiprime submodule of $M$ such that $K^n \subseteq N$ for some $n \in Z^+$, then $K \subseteq N$. Moreover, if $K^n = N$ for some $n \in Z^+$, then $K = N$.

*Proof:* Since $M$ is a graded multiplication module, so $K = IM$ for some graded ideal $I$ of $R$. Hence $K^n = (IM)^n = I^n M \subseteq N$, then $K \subseteq N$ by Proposition 2.6. Clearly, if $K^n = N$ for some $n \in Z^+$, then $K = N$. ■

*Proposition 2.8:* Let $R = R_1 \times R_2$ where $R_i$, $i = 1, 2$, is a graded commutative ring with identity for $i = 1, 2$. Let $M_i$ be a graded $R_i$-module and let $M = M_1 \times M_2$ be the graded $R$-module with action $(r_1, r_2)(m_1, m_2) = (r_1 m_1, r_2 m_2)$ where $r_i \in R_i$ and $m_i \in M_i$. Then the following hold:
(i) $N_1$ is a graded semiprime submodule of $M_1$ if and only if $N_1 \times M_2$ is a graded semiprime submodule of $M$.
(ii) $N_2$ is a graded semiprime submodule of $M_2$ if and only if $M_1 \times N_2$ is a graded semiprime submodule of $M$.

*Proof:* (i) Let $N_1$ be a graded semiprime submodule of $M_1$. Suppose $(a, b)^k(m, n) \in N_1 \times M_2$ where $(a, b) \in h(R) = h(R_1) \times h(R_2)$, $(m, n) \in h(M) = h(M_1) \times h(M_2)$ and $k \in Z^+$. So $a^k m \in N_1$, and $am \in N_1$ since $N_1$ is a graded semiprime submodule. Hence $(a, b)(m, n) \in N_1 \times M_2$, as required. Let $N_1 \times M_2$ is a graded semiprime submodule of $M$. Let $a^k m \in N_1$ where $a \in h(R_1)$, $m \in h(M_1)$ and $k \in Z^+$. So $(a, 1)^k(m, 0) \in N_1 \times M_2$ where $(a, 1) \in h(R)$ and $(m, 0) \in h(M)$, thus $(a, 1)(m, 0) \in N_1 \times M_2$ since $N_1 \times M_2$ is a graded semiprime submodule. Hence $am \in N_1$, as needed.
(ii) The proof is similar to that in case (i) and we omit it. ■

A graded $R$-module $M$ is called a graded secondary module provided that for every homogeneous element $r \in h(R)$, $rM = M$ or $r^n M = 0$ for some positive integer $n$ (see [7]).

*Theorem 2.9:* Let $M$ be a graded secondary $R$-module and $N$ a nonzero graded semiprime $R$-submodule of $M$. Then $N$ is graded secondary $R$-module.

*Proof:* Let $r \in h(R)$. If $r^n M = 0$ for some positive integer $n$, then $r^n N \subseteq r^n M = 0$, so $r$ is nilpotent on $N$. Suppose that $rM = M$; we show that $r$ divides $N$. Let $n \in N$. We may assume that $n = \sum_{g\in G} n_g$ where $n_g \neq 0$. So for every $g \in G$, $n_g = rm$ for some $m \in h(M)$. We have $rm' = m$ for some $m' \in h(M)$, hence $rm = r^2 m' \in N$, so $m = rm' \in N$ since $N$ is graded semiprime. Hence $n = rm \in rN$. Thus $rN = N$, as needed. ■

*Corollary 2.10:* Let $M$ be a graded $R$-module, $N$ a graded secondary $R$-submodule of $M$ and $K$ a graded semiprime submodule of $M$. Then $N \cap K$ is graded secondary.

*Proof:* The proof is straightforward by Theorem 2.7. ■

*Proposition 2.11:* Let $R$ be a graded ring and $S \subseteq h(R)$ be a multiplication closed subset of $R$. If $N$ is a graded semiprime submodule of $M$, then $S^{-1}N$ is a graded semiprime submodule of $S^{-1}M$.

*Proof:* Let $(r/s)^k.m/t \in S^{-1}N$ where $r/s \in h(S^{-1}R), m/t \in h(S^{-1}M)$ and $k \in Z^+$. So $r^k m/s^k t = n/t'$ for some $n \in N \cap h(M)$ and $t' \in S$, hence there exists $s' \in S$ such that $s't'r^k m = s's^k tn \in N$, so $N$ graded semiprime gives $rms't' \in N$. Hence $rm/st = rms't'/sts't' \in S^{-1}N$, as needed. ■

*Proposition 2.12:* Let $(R, P)$ be a graded local ring with graded maximal ideal $P$ and $S = h(R) - P$. Then $N$ is a graded semiprime submodule of graded $R$-module $M$ if and only if $N^g_P$ is a graded semiprime submodule of graded $R^g_P$-module $M^g_P$.

*Proof:* Let $N$ be a graded semiprime submodule of $M$, then $N^g_P$ is a graded semiprime submodule of $M^g_P$ by Proposition 2.11. Let $r^k m \in N$ where $r \in h(R)$, $m \in h(M)$ and $k \in Z^+$. So $r^k m/1 = (r/1)^k m/1 \in N^g_P$. Hence $rm/1 \in N^g_P$, and $rm/1 = c/s$ for some $c \in N \cap h(M)$ and $s \in S$. So there exists $t \in S$ such that $strm = tc \in N$. So $rm \in N$, because if $rm \notin N$, then $(N : rm) \neq R$, and $st \in (N : rm) \cap S \subseteq P \cap S = \emptyset$, which is a contradiction. Therefore $N$ is a graded semiprime submodule of $M$. ■

*Proposition 2.13:* Let $K \subseteq N$ be proper graded submodules of a graded $R$-module $M$. Then $N$ is a graded semiprime submodule of $M$ if and only if $N/K$ is a graded semiprime submodule of $M/N$.

*Proof:* $(\Rightarrow)$ Let $r^k(m+K) \in N/K$ where $r \in h(R), m \in h(M)$ and $Z^+$. So $r^k m \in N$, $N$ graded semiprime gives $rm \in N$. Hence $r(m + K) \in N/K$.
$(\Leftarrow)$ Let $r^k m \in N$ where $r \in h(R)$, $m \in h(M)$ and $k \in Z^+$. So $r^k m + K = r^k(m + K) \in N/K$. Then $r(m + K) \in N/K$ since $N/K$ is graded semiprime. Hence $rm \in N$, as required. ■

## III. Graded weakly semiprime submodules

In this section, we define the graded weakly semiprime submodules of a graded $R$-module and we extend some results of graded semiprime submodules to graded weakly semiprime submodules.

*Definition 3.1:* Let $R$ be a graded ring and $M$ a graded $R$-module. A proper graded submodule $N$ of $M$ is said to be graded weakly semiprime, if $0 \neq r^k m \in N$ for some $r \in h(R)$, $m \in h(M)$ and $k \in Z^+$, then $rm \in N$.

It is clear that every graded semiprime submodule is a graded weakly semiprime submodule. However, since $0$ is always graded weakly semiprime, a graded weakly semiprime submodule need not be graded semiprime, but if $R$ be a graded integral domain and $M$ a faithful graded prime module, then every graded weakly semiprime is graded semiprime.

*Definition 3.2:* Let $N$ be a graded submodule of a graded $R$-module $M$ and $g \in G$. We say that $N_g$ is a weakly semiprime submodule of $R_e$-module $M_g$, if $r_e^k m_g \in N_g$ where $r_e \in R_e$, $m_g \in M_g$ and $k \in Z^+$, then $r_e m_g \in N_g$.

*Proposition 3.3:* Let $M$ be a graded $R$-module and $N = \bigoplus_{g \in G} N_g$ a graded submodule of $M$. If $N$ is a graded weakly semiprime submodule of $M$, then $N_g$ is a weakly semiprime submodule of $R_e$-module $M_g$ for any $g \in G$.

*Proof:* Let $0 \neq r_e^k m_g \in N_g$ where $r_e \in R_e$, $m_g \in M_g$ and $k \in Z^+$. So $r_e^k m_g \in N_g \subseteq N$, hence $r_e m_g \in N$ since $N$ is a graded weakly semiprime submodule. Since $R_e M_g \subseteq M_{eg} = M_g$, so $r_e m_g \in N_g$, as required. ■

*Theorem 3.4:* Let $R$ be a graded ring, $M$ a graded $R$-module, $N$ a graded submodule of $M$ and $g \in G$. Consider the following assertion.
(i) $N_g$ is a weakly semiprime submodule of $M_g$.
(ii) For $a \in M_g$, $Rad(N_g :_{R_e} a) = (N_g :_{R_e} a) \cup Rad(0 :_{R_e} a)$.
(iii) For $a \in M_g$, $Rad(N_g :_{R_e} a) = (N_g :_{R_e} a)$ or $Rad(N_g :_{R_e} a) = Rad(0 :_{R_e} a)$.
Then $(i) \Rightarrow (ii) \Rightarrow (iii)$.

*Proof:* $(i) \Rightarrow (ii)$ It is clear that $(N_g :_{R_e} a) \cup Rad(0 :_{R_e} a) \subseteq Rad(N_g :_{R_e} a)$. Let $r \in Rad(N_g :_{R_e} a)$. So $r^n a \in N_g$ for some positive integer $n$. If $r^n a = 0$, then $r \in Rad(0 :_{R_e} a)$. If $0 \neq r^n a \in N_g$, then $ra \in N_g$ since $N_g$ is a weakly semiprime submodule of $M_g$. Hence $Rad(N_g :_{R_e} a) \subseteq (N_g :_{R_e} a) \cup Rad(0 :_{R_e} a)$. Therefore the proof is complete.
$(ii) \Rightarrow (i)$ It is well known that if an ideal (a subgroup) is the union of two ideals (two subgroups), then it is equal to one of them. ■

An $R_e$-module $M_g$ is called prime module if the zero submodule is prime.

*Remark 3.5:* An $R_e$-module $M_g$ is prime if and only if $(0 :_{R_e} M_g) = (0 :_{R_e} m_g)$ for any $0 \neq m_g \in M_g$).

*Theorem 3.6:* Let $R$ be a graded ring, $M$ a graded $R$-module, $N$ a graded submodule of $M$, and $g \in G$. Then the following assertion are equivalent.
(i) $N_g$ is a weakly semiprime submodule of $M_g$.
(ii) For $a \in M_g$, $Rad(N_g :_{R_e} a) = (N_g :_{R_e} a) \cup Rad(0 :_{R_e} a)$.
(iii) For $a \in M_g$, $Rad(N_g :_{R_e} a) = (N_g :_{R_e} a)$ or $Rad(N_g :_{R_e} a) = Rad(0 :_{R_e} a)$.

*Proof:* It is enough to show that $(iii) \Rightarrow (i)$. Let $0 \neq r^k m \in N_g$ where $r \in R_e$, $m \in M_g$ and $k \in Z^+$. So $r \in Rad(N_g :_{R_e} m)$. If $r \in Rad(0 : m)$, then $r^n m = 0$ for some $n \in Z^+$. Let $t$ be the smallest integer such that $r^t m = 0$. If $t > k$, then $0 < t - k < t$; $r^t m = r^k(r^{t-k} m) = 0$; $r^k \in (0 :_{R_e} r^{t-k} m) = (0 :_{R_e} M_g)$ since $M_g$ is a graded prime module. Hence $r^k M_g = 0$, so $r^k m = 0$, a contradiction. Let $k \geq t$. Thus $r^k m = r^{k-t}(r^t m) = 0$ which is a contradiction. Therefore $r \notin Rad(0 :_{R_e} m)$. So $r \in (N_g : m)$, hence $rm \in N_g$, as needed. ■

*Proposition 3.7:* Let $R = R_1 \times R_2$ where $R_i$ for $i = 1, 2$, is a commutative graded ring with identity. Let $M_i$ be a graded $R_i$-module and let $M = M_1 \times M_2$ be the graded $R$-module.

Then the following hold:
(i) If $N_1 \times M_2$ is a graded weakly semiprime submodule of $M$, then $N_1$ is a graded weakly semiprime submodule of $M_1$.
(ii) If $M_1 \times N_2$ is a graded weakly semiprime submodule of $M$, then $N_2$ is a graded weakly semiprime submodule of $M_2$.

*Proof:* (i) Let $N_1 \times M_2$ is a graded weakly semiprime submodule of $M$. Suppose $0 \neq a^k m \in N_1$ where $a \in h(R_1)$, $m \in h(M_1)$ and $k \in Z^+$. So $0 \neq (a,1)^k(m,0) \in N_1 \times M_2$, then $(a,1)(m,0) \in N_1 \times M_2$ since $N_1 \times M_2$ is a graded weakly semiprime. Hence $am \in N_1$, so $N_1$ is a graded weakly semiprime submodule of $M_1$.
(ii) The proof is similar to that in case (i). ■

*Theorem 3.8:* Let $M$ be a graded secondary $R$-module and $N$ a nonzero graded weakly semiprime $R$-submodule of $M$. Then $N$ is graded secondary.

*Proof:* Let $r \in h(R)$. If $r^n M = 0$ for some positive integer $n$, then $r^n N \subseteq r^n M = 0$, so $r$ is nilpotent on $N$. Suppose that $rM = M$; we show that $r$ divides $N$. Let $0 \neq n \in N$. We may assume that $n = \sum_{g \in G} n_g$ where $n_g \neq 0$. So for any $g \in G$, $n_g = rm$ for some $m \in h(M)$. We have $rm' = m$ for some $m' \in h(M)$, hence $0 \neq rm = r^2 m' \in N$, so $m = rm' \in N$ since $N$ is a graded weakly semiprime submodule. Thus $n_g \in rN$, so $n \in rN$. Therefore $rN = N$, as needed. ■

*Corollary 3.9:* Let $M$ be a graded $R$-module, $N$ a graded secondary $R$-submodule of $M$ and $K$ a graded weakly semiprime submodule of $M$. Then $N \cap K$ is graded secondary.

*Proof:* The proof is straightforward by Theorem 3.8. ■

*Proposition 3.10:* Let $R$ be a graded ring and $S \subseteq h(R)$ be a multiplication closed subset of $R$. If $N$ is a graded weakly semiprime submodule of $M$, then $S^{-1}N$ is a graded weakly semiprime submodule of $S^{-1}M$.

*Proof:* Let $0/1 \neq (r/s)^k.m/t \in S^{-1}N$ where $r/s \in h(S^{-1}R), m/t \in h(S^{-1}M)$ and $k \in Z^+$. So $0/1 \neq r^k m/s^k t = n/t'$ for some $n \in N \cap h(M)$ and $t' \in S$, hence there exists $s' \in S$ such that $0 \neq s't'r^k m = s's^k tn \in N$ (because if $s't'r^k m = 0$, $r^k m/s^k t = s't'r^k m/s't's^k t = 0/1$, a contradiction), so $N$ graded weakly semiprime gives $rms't' \in N$. Hence $rm/st = rms't'/sts't' \in S^{-1}N$, as needed. ■

*Proposition 3.11:* Let $(R, P)$ be a graded local ring with graded maximal ideal $P$ and $S = h(R) - P$. Then $N$ is a graded weakly semiprime submodule of graded $R$-module $M$ if and only if $N_P^g$ is a graded weakly semiprime submodule of graded $R_P^g$-module $M_P^g$.

*Proof:* Let $N$ be a graded weakly semiprime submodule of $M$, then $N_P^g$ is a graded weakly semiprime submodule of $M_P^g$ by Proposition 3.10. Let $0 \neq r^k m \in N$ where $r \in h(R)$, $m \in h(M)$ and $k \in Z^+$. So $0/1 \neq r^k m/1 = (r/1)^k m/1 \in N_P^g$ because if $0/1 = r^k m/1$, then $s(r^k m) = 0$ for some $s \in S$, so $s \in (0 : r^k m) \cap S \subseteq P \cap S = \emptyset$, a contradiction. Hence $rm/1 \in N_P^g$, and $rm/1 = c/s$ for some $c \in N \cap h(M)$ and $s \in S$. So there exists $t \in S$ such that $strm = tc \in N$. So $rm \in N$, because if $rm \notin N$, then $(N : rm) \neq R$, and $st \in (N : rm) \cap S \subseteq P \cap S = \emptyset$, which is a contradiction. Therefore $N$ is a graded weakly semiprime submodule of $M$. ■

*Proposition 3.12:* Let $K \subseteq N$ be proper graded submodules of a graded $R$-module $M$. Then the following hold:
$(i)$ If $N$ is a graded weakly semiprime submodule of $M$, then $N/K$ is a graded weakly semiprime $R$-submodule of $M/N$.
$(ii)$ If $K$ and $N/K$ are graded weakly semiprime submodules of $M$ and $M/K$ respectively, then $N$ is a graded weakly semiprime submodule of $M$.

*Proof:* $(i)$ Let $0 \neq r^k(m+K) \in N/K$ where $r \in h(R)$, $m+K \in h(M/K)$ and $k \in Z^+$. So $0 \neq r^k m \in N$, $N$ weakly semiprime gives $rm \in N$. Hence $r(m+K) \in N/K$.
$(ii)$ Let $0 \neq r^k m \in N$ where $r \in h(R), m \in h(M)$ and $k \in Z^+$. So $r^k m + K = r^k(m+K) \in N/K$. If $0 \neq r^k m \in K$, then $rm \in K \subseteq N$ since $K$ is graded weakly semiprime, as needed. Let $0 \neq r^k(m+K) \in N/K$, then $r(m+K) \in N/K$ since $N/K$ is graded weakly semiprime. Hence $rm \in N$, as required. ■

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## References

[1] D. D. Anderson and E. Smith, *Weakly prime ideals*, Hoston J. of Math. 29 (2003), 831-840.
[2] S. Ebrahimi Atani, *On graded weakly prime ideals*, Turk. J. of Math. 30 (2006), 351-358.
[3] S. Ebrahimi Atani, *On graded weakly prime submodules*, Int. Math. Forum. 1 (2006), 61-66.
[4] S. Ebrahimi Atani and F.Farzalipour, *On weakly primary ideals*, Georgian Math. Journal. 3 (2003), 705-709.
[5] S. Ebrahimi Atani and F.Farzalipour, *On weakly prime submodules*, Tamkang J. of Math. 38 (2007), 247-252.
[6] S. Ebrahimi Atani and F. Farzalipour , *Notes On the graded prime submodules*, Int. Math. Forum. 1 (2006), 1871-1880.
[7] Ebrahimi Atani and F. Farzalipour, *On graded secondary modules*, Turk. J. Math. 31 (2007), 371-378.
[8] F. Farzalipour and P. Ghiasvand, *Quasi Multiplication Modules*, Thai J. of Math. 1 (2009), 361-366.
[9] P. Ghiasvand and F. Farzalipour, *Some Properties of Graded Multiplication Modules*, Far East J. Math. Sci. 34 (2009), 341-359.
[10] C. Nastasescu and F. Van Oystaeyen, *Graded Ring Theory*, Mathematical Library 28, North Holand, Amsterdam, 1982.
[11] M. Refaei and K. Alzobi, *On graded primary ideals*, Turk. J. Math. 28 (2004), 217-229.