Abstract—This paper deals with the optimal design of two-channel recursive parallelogram quadrature mirror filter (PQMF) banks. The analysis and synthesis filters of the PQMF bank are composed of two-dimensional (2-D) recursive digital all-pass filters (DAFs) with nonsymmetric half-plane (NSHP) support region. The design problem can be facilitated by using the 2-D doubly complementary half-band (DC-HB) property possessed by the analysis and synthesis filters. For finding the coefficients of the 2-D recursive NSHP DAFs, we appropriately formulate the design problem to result in an optimization problem that can be solved by using a weighted least-squares (WLS) algorithm in the minimax ($L_{\infty}$) optimal sense. The designed 2-D recursive PQMF bank achieves perfect magnitude response and possesses satisfactory phase response without requiring extra phase equalizer. Simulation results are also provided for illustration and comparison.

Keywords—Parallelogram Quadrature Mirror Filter Bank, Doubly Complementary Filter, Nonsymmetric Half-Plane Filter, Weighted Least Squares Algorithm, Digital All-Pass Filter.

I. INTRODUCTION

TWO-DIMENSIONAL (2-D) quadrature mirror filter (QMF) banks have been widely considered for high quality coding of image and video data at low bit rates and spatial image analysis like feature extraction and matching [1], [2]. The design problem of 2-D QMF banks with the perfect reconstruction (PR) characteristics has been considered in the literature [3], [4].

A parallel structure using the 2-D recursive NSHP DAF to construct the analysis and synthesis filters of a recursive two-channel parallelogram QMF (PQMF) bank has been presented in [5]. For image processing, the 2-D DC-based PQMF bank can avoid transmission nulls [6] without using additional delays and satisfy the frequency constraints [7], [8] that avoid the aliasing artifacts. However, an extra NSHP DAF used as a phase equalizer must be added in the synthesis system to eliminate the phase distortion induced by the 2-D recursive NSHP DAFs in the analysis and synthesis systems [5].

In this paper, we present a novel method for the minimax design of two-channel recursive PQMF banks using 2-D recursive NSHP DAFs. The proposed 2-D recursive PQMF bank possesses the advantages as those presented by [5], namely, (i) the resulting analysis/synthesis filters possess the 2-D DC properties, i.e., 2-D all-pass complementary and power complementary properties, (ii) the proposed 2-D DC-based analysis/synthesis filters possess an attractive DC symmetry with respect to the half-band frequency ($\omega_1, \omega_2$) = $(0, \pi/2)$ in the upper half plane of the frequency plane, (iii) the proposed 2-D DC-based recursive PQMF bank have no magnitude distortion. The frequency characteristics totally depend on the phase responses of the 2-D recursive NSHP DAFs. However, the proposed 2-D recursive PQMF bank avoids the need of extra NSHP DAF used as a phase equalizer like [5] to achieve satisfactory linear-phase response. Using the stability constraints presented by [9] to guarantee the stability of the 2-D recursive NSHP DAFs, we then derive a novel objective function for minmax phase approximation. The problem of minimizing the objective function can be solved by using the weighted least-squares algorithm developed in [10] (termed as the LLCY algorithm). Simulation results show that the proposed design method provides more satisfactory results than the method of [5].

II. PROBLEM FORMULATION

A. PQMF Bank with Linear-Phase Response

The conventional 2-D two-channel PQMF system is shown in Fig. 1, where $H_d(z_1,z_2) = \hat{H}_d(z_1,-z_2)$ and $H_s(z_1,z_2)$ are the low-pass and high-pass analysis filters, respectively, $F_d(z_1,z_2)$ and $F_s(z_1,z_2)$ are the low-pass and high-pass synthesis filters, respectively. M denotes the decimation/interpolation matrix. The ideal frequency specifications for the 2-D two-channel analysis and synthesis systems are given in Fig. 2. Setting the synthesis filters $F_d(z_1,z_2) = H_d(z_1,\ -z_2)$, $F_s(z_1,z_2) = -H_d(z_1,\ -z_2)$, and using the mirror-image symmetry about the frequency $(\omega_1, \omega_2) = (0, \pi/2)$, we have $H_s(z_1,z_2) = H_d(z_1,\ -z_2)$ and the input/output relationship of the 2-D PQMF bank in the z-transform is as follows:

$$\hat{X}(z_1,z_2) = \frac{1}{2} [H_d^*(z_1, z_2) - H_d^*(z_1, -z_2)]X(z_1, z_2)$$

Let $\Phi(e^{j\omega_1}, e^{j\omega_2}) = H_d^*(e^{j\omega_1}, e^{j\omega_2}) - H_d^*(e^{j\omega_1}, e^{-j\omega_2})$ be the frequency response of the PQMF bank. Equation (1) reveals that producing an output $\hat{X}(m,n)$ that is a delayed replica of $x(m,n)$ requires...
for $\forall (\omega_1, \omega_2)$, where $g_1$ and $g_2$ are the system delays of the PQMF bank. Designing of PQMF banks with conventional FIR or IIR structures for $H_d(z_1, z_2)$ usually induces both magnitude and phase distortions.

B. Digital All-Pass Based PQMF Bank

First, we used the widely considered decimation/interpolation matrix $M = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$. Then, a new two-channel recursive PQMF bank is constructed by setting the analysis filters $H_d(z_1, z_2)$ and $H(z_1, z_2)$ of Fig. 1 as follows:

$$H_0(z_1, z_2) = \frac{A_1(z_1, z_2) + z_2^{-1}A_2(z_1, z_2)}{2}$$

$$H_1(z_1, z_2) = \frac{A_1(z_1, z_2) - z_2^{-1}A_2(z_1, z_2)}{2} \tag{3}$$

where $A(z_1, z_2)$, for $i = 1, 2$, are two 2-D DAFs with transfer function given by

$$A(z_1, z_2) = z_1^{-M}z_2^{-N} \frac{D(z_1^{-1}, z_2^{-1})}{D_0(z_1, z_2)}, \tag{4}$$

where the denominator polynomial $D(z_1, z_2)$ of the $(M \times N)$th-order DAF $A(z_1, z_2)$ with nonsymmetric half-plane (NSHP) support regions for its coefficients is given by

$$D_1(z_1, z_2) = \sum_{m=0}^{M} d_1(m,0)z_1^{-m} + \sum_{m=-M,n=1}^{M-N,1} d_1(m,n)z_1^{-m}z_2^{-n}$$

$$= \sum_{(m,n) \subset \mathbb{R}} d_1(m,n)z_1^{-m}z_2^{-n} \tag{5}$$

where $\mathbb{R}$ represents the 2-D NSHP region on the 2-D $(m,n)$ plane. We note from (3) and (4) that

$$\left|H_0(e^{j\omega_1}, e^{j\omega_2}) + H_1(e^{j\omega_1}, e^{j\omega_2})\right| = 1, \text{ for } \forall (\omega_1, \omega_2),$$

$$\left|H_0(e^{j\omega_1}, e^{j\omega_2})\right|^2 + \left|H_1(e^{j\omega_1}, e^{j\omega_2})\right|^2 = 1, \text{ for } \forall (\omega_1, \omega_2, \omega_2). \tag{6}$$

Hence, $H_d(z_1, z_2)$ and $H(z_1, z_2)$ simultaneously satisfy the all-pass complementary and power complementary properties, i.e., $H_d(z_1, z_2)$ and $H(z_1, z_2)$ form a 2-D doubly-complementary (DC) filter pair. The $H_d(z_1, z_2)$ and $H(z_1, z_2)$ of Fig. 1 are replaced by (3) to form a new 2-D DC-based recursive PQMF bank of Fig. 3. From (1) and (3), we can show that the proposed 2-D two-channel PQMF bank has the transfer function given by

$$T(z_1, z_2) = \frac{1}{2} \left[H^*_0(z_1, z_2) - H^*_1(z_1, z_2)\right] + \frac{1}{2} z_1^{-1}A_1(z_1, z_2)A_2(z_1, z_2) \tag{7}$$

Equation (7) reveals that the 2-D two-channel recursive PQMF bank possesses a magnitude response without distortion.

C. Frequency Characteristics of 2-D DC Analysis Filters

The frequency response of (4) with phase response $\theta_1(\omega_1, \omega_2)$ can be expressed by

$$A_1(e^{j\omega_1}, e^{j\omega_2}) = e^{j\theta_1(\omega_1, \omega_2)} = e^{j(-M\omega_1 - N\omega_2 - 2\phi(\omega_1, \omega_2))} \tag{8}$$

where

$$\phi(\omega_1, \omega_2) = \text{arg}\{D(e^{j\omega_1}, e^{j\omega_2})\} = \frac{1}{2}\left[1 + \frac{M_1 + M_2 - N_1 - N_2 + 1}{N_2 - N_1 - 1} \theta_1(\omega_1, \omega_2) + \phi(\omega_1, \omega_2)\right] \tag{9}$$

is the phase of $D(z_1, z_2)$. Substituting (8) into (3), we have

$$H_0(e^{j\omega_1}, e^{j\omega_2}) = \frac{1}{2} \left[\frac{A_1(e^{j\omega_1}, e^{j\omega_2}) + e^{j\theta_1(\omega_1, \omega_2)}}{A_1(e^{j\omega_1}, e^{j\omega_2}) + e^{j\theta_1(\omega_1, \omega_2)}}\right] = e^{j\theta_1(\omega_1, \omega_2)} \tag{10}$$

Moreover, we have from (11) that

$$\phi(\omega_1, \omega_2) = \frac{M_1 - M_2 - N_1 + N_2 - 1}{2N_2 - N_1} \omega_1 + \frac{\phi(\omega_1, \omega_2)}{2} \tag{11}$$

Hence, $H_d(z_1, z_2)$ and $H(z_1, z_2)$ simultaneously satisfy the all-pass complementary and power complementary properties, i.e., $H_d(z_1, z_2)$ and $H(z_1, z_2)$ form a 2-D doubly-complementary (DC) filter pair. The $H_d(z_1, z_2)$ and $H(z_1, z_2)$ of Fig. 1 are replaced by (3) to form a new 2-D DC-based recursive PQMF bank of Fig. 3. From (1) and (3), we can show that the proposed 2-D two-channel PQMF bank has the transfer function given by

$$T(z_1, z_2) = \frac{1}{2} \left[H^*_0(z_1, z_2) - H^*_1(z_1, z_2)\right] + \frac{1}{2} z_1^{-1}A_1(z_1, z_2)A_2(z_1, z_2) \tag{7}$$

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where

$$\phi(\omega_1, \omega_2) = \text{arg}\{D(e^{j\omega_1}, e^{j\omega_2})\} = \frac{1}{2}\left[1 + \frac{M_1 + M_2 - N_1 - N_2 + 1}{N_2 - N_1 - 1} \theta_1(\omega_1, \omega_2) + \phi(\omega_1, \omega_2)\right] \tag{9}$$

is the phase of $D(z_1, z_2)$. Substituting (8) into (3), we have

$$H_0(e^{j\omega_1}, e^{j\omega_2}) = \frac{1}{2} \left[\frac{A_1(e^{j\omega_1}, e^{j\omega_2}) + e^{j\theta_1(\omega_1, \omega_2)}}{A_1(e^{j\omega_1}, e^{j\omega_2}) + e^{j\theta_1(\omega_1, \omega_2)}}\right] = e^{j\theta_1(\omega_1, \omega_2)} \tag{10}$$

Moreover, we have from (11) that

$$\phi(\omega_1, \omega_2) = \frac{M_1 - M_2 - N_1 + N_2 - 1}{2N_2 - N_1} \omega_1 + \frac{\phi(\omega_1, \omega_2)}{2} \tag{11}$$

for $i = 1, 2$. (12) shows that $\phi(\omega_1, \omega_2)$ can be determined by $\theta_1(\omega_1, \omega_2)$ and $\phi(\omega_1, \omega_2)$. Hence, the desired response $\phi(\omega_1, \omega_2)$ can be obtained from (12) by specifying the desired $\theta_1(\omega_1, \omega_2)$ and $\phi(\omega_1, \omega_2)$. Since $H_1(z_1, z_2) = H_0(z_1, z_2)$, we have $H_0(e^{j\omega_1}, e^{j\omega_2}) = H_1(e^{j\omega_1}, e^{j\omega_2})$. Hence, (6) becomes

$$\left|H_0(e^{j\omega_1}, e^{j\omega_2}) + H_1(e^{j\omega_1}, e^{j\omega_2})\right|^2 = 1, \text{ for } \forall (\omega_1, \omega_2),$$

$$\left|H_0(e^{j\omega_1}, e^{j\omega_2})\right|^2 + \left|H_1(e^{j\omega_1}, e^{j\omega_2})\right|^2 = 1, \text{ for } \forall (\omega_1, \omega_2) \tag{13}$$
The attractive properties indicate that \( H_0(e^{j\omega}, e^{j\omega_m}) \) possesses the DC symmetry with respect to the half-band frequency \((\omega_0,\omega_0) = (0,\pi/2)\) in the upper half of the 2-D frequency plane. Moreover, (13) reveals that if \(|H_0(e^{j\omega}, e^{j\omega_m})|\) equals zero in the stop-band, then \(|H_0(e^{j\omega}, e^{j\omega_m})|\) equals one in the pass-band, and vice versa. Therefore, we only need to approximate the pass-band or stop-band response when designing \(H_0(\omega_1,\omega_2)\). These advantages reduce the amount of numerical computation required for design and implementation.

D. Minimax Design Formulation

From (7) and (8), we note that the frequency response of the proposed 2-D PQMF bank is given by

\[
T(e^{j\omega}, e^{j\omega_m}) = \frac{1}{2} \exp\{j(-M_1 + M_2)\omega_1 - (N_1 + N_2 + 1)\omega_2)\}.
\]

From (13) and (18), we can see that the design problem for the proposed 2-D recursive PQMF bank is finding the real coefficients \(d(m,n)\) of \(A(\omega_1,\omega_2)\) such that the following two constraints can be approximately met in the minimax sense:

(i) \(\phi_1(\omega_1, \omega_2) + \phi_2(\omega_1, \omega_2) = 0\), for \(\forall (\omega_1, \omega_2)\),

(ii) \(H_0(e^{j\omega}, e^{j\omega_m}) = 0\), for \((\omega_1, \omega_2)\in\Omega\).

For a practical design, the ideal frequency band splitting for 2-D PQMF banks with sampling pattern given by \(M\) is shown by Fig. 4, where \(\omega_0\) and \(\omega_1\) are two band edge frequencies. As to the stability of \(A(\omega_1,\omega_2)\), the stability constraints guaranteeing the stability of \(A(\omega_1,\omega_2)\) are summarized as follows [9]:

(i) \(\theta(\omega_1, \omega_2)\) is monotonically decreasing along \(\omega_1\) axis and \(\theta(\pi,\omega_2) = \theta(0,\omega_2)\) - \(M\pi\) for \(-\pi\leq \omega_1\leq \pi\); (ii) \(\theta(\omega_1, \pi)\) is monotonically decreasing along \(\omega_2\) axis and \(\theta(0,\pi) = \theta(0,0) - N\pi\) for \(-\pi\leq \omega_2\leq \pi\).

For simplicity, by specifying the desired phase response \(\phi_{\pi}(\omega_1, \omega_2)\) to satisfy the stability constraints, we can neglect the stability problem and focus on the approximation problem given by (20) only. As a result, we can formulate the minimax design problem as follows:

\[
\text{Minimize} |\phi(\omega_1, \omega_2) + \phi(\omega_1, \omega_2)|_\gamma + c\|W(\omega_1, \omega_2)H_0(\omega_1, \omega_2)\|_\gamma, (21)
\]

where \(|\gamma|\) denotes the Chebyshev norm of \(\gamma\), \(\theta\) is the relative weight between the two peak error terms, and \(W(\omega_1, \omega_2)\) is a preset weighting function with value one in \(\Omega_0\) and zero in \(\Omega_2\).

III. PROPOSED DESIGN METHOD

A. Frequency Sampling and Approximation Scheme

Using the property of \(\tan^{-1}(x) + \tan^{-1}(y) = \tan^{-1}\{xy/(1-xy)\}\), we rewrite the first constraint of (20) for \(\forall (\omega_1, \omega_2)\), as follows:

\[
|\phi_1(\omega_1, \omega_2) + \phi_2(\omega_1, \omega_2)|_\gamma = |\tan^{-1}\{\text{Aprix}(d(\omega_1, \omega_2))\}|_\gamma = |\tan^{-1}\{\text{Aprix}(d(\omega_1, \omega_2))\}|_\gamma, (22)
\]

\[
\text{Aprix}(d_{(0,0)}) = 1 \text{ for simplicity, } d = [d_1(1,0), d_1(2,0),..., d_1(1,0), d_2(2,0),...]^T \text{ is the } (M_1 + 2M_2 + 1)N_1/2 + M_2 + 2M_1 + 1 N_2/2 \text{ vector, where the superscript T denotes the transospse operation. And}
\]

\[
S_i(\omega_1, \omega_2) = \sum_{(m,n):\in\Omega_i} d_{(m,n)}\sin(m_0\omega_1 + n_0\omega_2), \text{ } i = 1, 2.
\]

\[
C_i(\omega_1, \omega_2) = \sum_{(m,n):\in\Omega_i} d_{(m,n)}\cos(m_0\omega_1 + n_0\omega_2), \text{ } i = 1, 2.
\]
Minimizing (22) is equivalent to minimizing the following term
\[
\left| S_1(\omega_1, \omega_2)C_1(\omega_1, \omega_2) + \frac{C_1(\omega_1, \omega_2)S_2(\omega_1, \omega_2)}{C_1(\omega_1, \omega_2)S_2(\omega_1, \omega_2)} - S_1(\omega_1, \omega_2)C_2(\omega_1, \omega_2) \right|
\]
(24)
since the inverse tangent function is a monotonic function. The second condition of (20) can be rewritten as
\[
H_\alpha(e^{j\omega_1}, e^{j\omega_2}) = |\text{Aprx2}(d_{\omega_1, \omega_2})|^2
\]
(25)
where
\[
\text{Aprx2}(d_{\omega_1, \omega_2}) = [\psi_2(\omega_1, \omega_2, 1) \psi_2(\omega_1, \omega_2, 2) \ldots]^T
\]
(28)
Utilizing the LLCY algorithm of [10], we reformulate the minimization problem of (21) as follows:
\[
\text{Minimize} \left\{ W_{ls}(\omega_1, \omega_2) | \text{Aprx1}(d_{\omega_1, \omega_2}) \right\}^2
\]
(26)
where
\[
W_{ls}(\omega_1, \omega_2) = W_{ls}(\omega_1, \omega_2) + aW_{ls}(\omega_1, \omega_2)
\]
\[
\text{Aprx2}(d_{\omega_1, \omega_2})
\]
and
\[
W_{ls}(\omega_1, \omega_2)
\]
are two required least-squares weighting functions. Let the frequency pair \((\omega_1, \omega_2)\) represent the \(r\)th uniformly sampled frequency grid point in the interested frequency band. And \(1 \leq r \leq R\), where \(R\) represents the total number of uniformly sampled frequency grid points. The design process of the proposed method is then performed on the \(R\) frequency grid points. If the number of \(R\) of grid points is sufficiently large, the obtained best approximation solution of the objective function based on the grid points will be close to the best solution found based on (26). This conclusion can be justified by the theorem due to Cheney [11, Chapter 3]. We utilize a linearization scheme to approximate the related errors of (22) and (25) due to a perturbation in the filter coefficient vector \(d\) in the linear subspace spanned by the gradient matrix associated with Aprx1(d_{\omega_1, \omega_2}) and Aprx2(d_{\omega_1, \omega_2}). As a result, the approximation for minimizing (21) can be formulated as finding the increments \(\delta d = [\delta d_1(1,0), \delta d_1(2,0), \ldots, \delta d_2(1,0), \delta d_2(2,0) \ldots]^T\) of the filter coefficient vectors \(d\) at the \(k\)th iteration such that
\[
[\text{Aprx1}(d_{\omega_1, \omega_2}) + \delta d]^T \cdot \text{Aprx1}(d_{\omega_1, \omega_2}) + aW_{ls}(\omega_1, \omega_2)
\]
(27)
is minimized for all \((\omega_1, \omega_2)\), where the subscript \(k\) denotes the results obtained at the \(k\)th iteration. \(\nabla \text{Aprx1}(d_{\omega_1, \omega_2})\) and \(\nabla \text{Aprx2}(d_{\omega_1, \omega_2})\) are the gradient vectors of Aprx1(d_{\omega_1, \omega_2}) and Aprx2(d_{\omega_1, \omega_2}) with respect to the filter coefficient vector \(d_i\), respectively, and are given by
\[
\nabla \text{Aprx1}(d_{\omega_1, \omega_2}) = [\psi_1(\omega_1, \omega_2, 1) \psi_1(\omega_1, \omega_2, 2) \ldots]^T,
\]
\[
\nabla \text{Aprx2}(d_{\omega_1, \omega_2}) = [\psi_2(\omega_1, \omega_2, 1) \psi_2(\omega_1, \omega_2, 2) \ldots]^T,
\]
\[
\text{Aprx1}(d_{\omega_1, \omega_2}) = [\psi_1(\omega_1, \omega_2, 1) \psi_1(\omega_1, \omega_2, 2) \ldots]^T,
\]
\[
\text{Aprx2}(d_{\omega_1, \omega_2}) = [\psi_2(\omega_1, \omega_2, 1) \psi_2(\omega_1, \omega_2, 2) \ldots]^T.
\]

\textbf{B. Iterative Design Procedure}

\textbf{Step 1: Initiation:}

\textit{<1.1>} Determine the design parameters: the orders \(M_i\) and \(N_i\), \(i = 1,2\), the relative weight \(a\), band edge frequencies \(\omega_0\) and \(\omega_e\).

\textit{<1.2>} Set the zero vector as an initial guess for the vector \(d\).

\textit{<1.3>} Set the iteration number \(k = 0\).

\textbf{Step 2: Perform a test for stopping the iteration process:}

\textit{<2.1> Compute} \(V_k = [\nabla \text{Aprx1}(d_{\omega_1, \omega_2})]_x + a[\nabla \text{Aprx2}(d_{\omega_1, \omega_2})]_x\).

\textit{<2.2> Terminate the design process and take the coefficient vector \(d_i\) as the desired filter coefficient vector if} \(V_k < 1/\|V_k\|\), otherwise go to Step 3.

\textbf{Step 3: Calculate the increment} \(\delta d_i\) \textit{of} \(d_i\) \textit{at the} \(k\) \textit{iteration from (30) according to the LLCY iterative algorithm as follows:}

\textit{<3.1> Set the initial weighting matrix} \(W_{ls}\) \textit{to the} \(R \times R\) \textit{identity matrix} \(I\) \textit{and an inner iteration index} \(p = 0\).

\textit{<3.2> Compute the WLS solution} \(\delta d_i\) \textit{from (30). Set} \(\delta d_i^{(p)} = \delta d_i\), \textit{where the superscript} \(p\) \textit{denotes the} \(\delta d_i\) \textit{obtained at the} \(p\) \textit{th iteration during the inner iteration process.}
<3.3> Compute the error functions $e_{1}^{(\beta)}(\omega_{1}, \omega_{2}) = (U_{1}d_{1}(\omega_{1}, \omega_{2}) - s_{1})$ and $e_{2}^{(\beta)}(\omega_{1}, \omega_{2}) = (U_{2}d_{2}(\omega_{1}, \omega_{2}) - s_{2})$.

<3.4> Let $G_{1}^{(\beta)}(\omega_{1}, \omega_{2}) = |e_{1}^{(\beta)}(\omega_{1}, \omega_{2})|$ and $G_{2}^{(\beta)}(\omega_{1}, \omega_{2}) = |e_{2}^{(\beta)}(\omega_{1}, \omega_{2})|$. The threshold values $G_{1,2}^{(\beta)}(\omega_{1}, \omega_{2})$ are obtained by substituting $\theta_{d}(\alpha_{1}, \alpha_{2}) = 0$ and $\theta_{p}(\alpha_{1}, \alpha_{2}) = -\frac{(M_{1}+M_{2})}{(N_{1}+N_{2}+1)}\alpha_{2}/2$ into (12). The resulting $\phi_{d}(\alpha_{1}, \alpha_{2})$ are obtained by

$$
\phi_{d}(\alpha_{1}, \alpha_{2}) = \begin{cases} 
\frac{\alpha_{2}}{4}, & \text{for } (\alpha_{1}, \alpha_{2}) \in \Omega_{p} \\
\frac{\alpha_{2}}{4} - \frac{\pi}{4}, & \text{for } (\alpha_{1}, \alpha_{2}) \in \Omega_{p}
\end{cases}$$

(31)

Since $\phi_{d}(\alpha_{1}, \alpha_{2})$ of (31) are two functions of $\alpha_{2}$ only and satisfy the stability conditions, we can omit the stability issue and focus the design problem shown by (21). Moreover, the designed analysis/synthesis filters can achieve approximately linear-phase responses in the pass-band because of the ideal phase specification given by $\theta_{d}(\alpha_{1}, \alpha_{2}) = -\frac{(M_{1}+M_{2})}{(N_{1}+N_{2}+1)}\alpha_{2}/2$. The parameters used for this design are chosen by experiment and given as follows: $\eta_{1} = \eta_{2} = 1.0 \times 10^{-4}$, $\alpha = 5$, and $\varepsilon = 1 \times 10^{-1}$. The spacing between two adjacent frequency grid points is the same as that used by [5] and set to $2\pi/64$. Significant design results are shown in Table 1 for comparison.

The magnitude responses of the analysis filters are depicted in Fig. 5 and the resulting phase error of $A(z_{1}, z_{2})$ is depicted in Fig. 6. The resulting phase error of the designed 2-D recursive PQMF bank is shown in Fig. 7. From Table I, we observe that the proposed method can design the filter coefficients for the 2-D NSHP DAFs $A(z_{1}, z_{2})$, for $i = 1, 2$ simultaneously and provide a 2-D recursive PQMF bank with frequency responses better than those of [5].

IV. SIMULATION RESULTS

We present simulation results for illustration and comparison. The performance of the designed 2-D recursive PQMF bank is evaluated in terms of the minimum stop-band attenuation (MSA) of $H_{d}(z_{1}, z_{2})$, the pass-band magnitude mean-squared error (PMSE) of $H_{d}(z_{1}, z_{2})$, the stop-band magnitude mean-squared error (SMSE) of $H_{d}(z_{1}, z_{2})$, the maximum variation in phase response (MVPR$_{r}$) of $D(e^{j\omega_{1}}, e^{j\omega_{2}})$ in the pass-band, and the peak phase error (PPE) of the designed PQMF bank $T(e^{j\omega_{1}}, e^{j\omega_{2}})$. They are defined as follows:

$$
\text{MSA} = -\max_{(\omega_{1}, \omega_{2})} \left[ 10 \log_{10} \left( \left\| H_{d}(e^{j\omega_{1}}, e^{j\omega_{2}}) \right\|^2 \right) \right] \text{ (dB)}
$$

$$
\text{PMSE} = \sum_{(\omega_{1}, \omega_{2})} \frac{\left\| H_{d}(e^{j\omega_{1}}, e^{j\omega_{2}}) \right\|^2}{\text{number of grid points in the pass-band}}
$$

$$
\text{SMSE} = \sum_{(\omega_{1}, \omega_{2})} \frac{\left\| H_{d}(e^{j\omega_{1}}, e^{j\omega_{2}}) \right\|^2}{\text{number of grid points in the stop-band}}
$$

$$
\text{MVPR}_{r} = \max_{(\omega_{1}, \omega_{2})} \left[ \arg\{D(e^{j\omega_{1}}, e^{j\omega_{2}})\} - \phi_{d}(\alpha_{1}, \alpha_{2}) \right] \text{ (radian)}
$$

$$
\text{PPE} = \max_{(\omega_{1}, \omega_{2})} \left[ \arg\{T(e^{j\omega_{1}}, e^{j\omega_{2}})\} + g_{1}\alpha_{1} + g_{2}\alpha_{2} \right] \text{ (radian)}
$$

Example: Using the same magnitude characteristic as that of [5], we set the band edge frequencies $\omega_{0} = \omega_{1} = 0.8\pi$. Accordingly, the desired magnitude response of $H_{d}(z_{1}, z_{2})$ possesses the symmetry with respect to the frequency $(0, 0.5\pi)$. Therefore, about half of the coefficients can be set to zero and only uniformly sampled pass-band frequency grid points are considered during the design procedure. The 2-D DC filters $H_{d}(z_{1}, z_{2})$ and $H_{d}(z_{1}, z_{2})$ are obtained by setting $M_{1} = M_{2} = 6$ and $N_{1} = N_{2} = 8$. Then, the ideal group delays are $g_{1} = M_{1} + M_{2} = 12$ and $g_{2} = N_{1} + N_{2} + 1 = 17$. We design the 2-D NSHP DAF $A(z_{1}, z_{2})$ whose denominator phase $\arg\{D(e^{j\omega_{1}}, e^{j\omega_{2}})\}$ with the desired passband phase response $\phi_{d}(\alpha_{1}, \alpha_{2})$ obtained by

$$
\phi_{d}(\alpha_{1}, \alpha_{2}) = \begin{cases} 
\frac{\alpha_{2}}{4}, & \text{for } (\alpha_{1}, \alpha_{2}) \in \Omega_{p} \\
\frac{\alpha_{2}}{4} - \frac{\pi}{4}, & \text{for } (\alpha_{1}, \alpha_{2}) \in \Omega_{p}
\end{cases}
$$

(31)

TABLE I

<table>
<thead>
<tr>
<th></th>
<th>Proposed Method</th>
<th>Method of [5]</th>
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</thead>
<tbody>
<tr>
<td>MSA (dB)</td>
<td>28.1350</td>
<td>23.2292</td>
</tr>
<tr>
<td>PMSE</td>
<td>3.9039×10^{-4}</td>
<td>7.3647×10^{-7}</td>
</tr>
<tr>
<td>SMSE</td>
<td>4.5983×10^{-4}</td>
<td>1.2800×10^{-3}</td>
</tr>
<tr>
<td>MVPR$_{r}$</td>
<td>3.9142×10^{-2}</td>
<td>5.9370×10^{-2}</td>
</tr>
<tr>
<td>MVPR$_{2}$</td>
<td>2.3284×10^{-2}</td>
<td>1.8531×10^{-2}</td>
</tr>
<tr>
<td>PPE</td>
<td>0.0826</td>
<td>0.1386</td>
</tr>
<tr>
<td>Number of Independent Coefficients</td>
<td>58,58</td>
<td>45,45,22</td>
</tr>
</tbody>
</table>

This paper has presented a novel technique for the minimax design of two-channel recursive parallelogram quadrature mirror filter (PQMF) banks. The proposed analysis/synthesis filters possess a 2-D doubly complementary (DC) property. The PQMF bank can achieve the nearly linear-phase response without magnitude distortion, the aliasing artifacts, and an extra phase equalizer. Using the weighted least-squares (WLS) algorithm, the proposed PQMF bank can be efficiently designed. The design results show that the proposed method provides more satisfactory design than the existing techniques.
Fig. 1 The conventional two-channel PQMF bank

(a) Analysis Bank

(b) Synthesis Bank

Fig. 2 The Ideal Frequency Band-Splitting for 2-D PQMF bank
Pass-band region: Shadow area. Stop-band region: White area

(a) Analysis Bank

(b) Synthesis Bank

Fig. 3 The proposed Analysis/Synthesis banks

Fig. 4 The ideal frequency band splitting. $\Omega^+_i$: gray shadow area. $\Omega^-_i$: dark shadow area. The white area represents the transition region

(a) $i = 1$

Fig. 5 Magnitude responses of analysis filters

(a) Perspective plot of $|H_i(e^{j\omega_x}, e^{j\omega_y})|$

(b) Perspective plot of $|H_i(e^{j\omega_x}, e^{j\omega_y})|$

(c) Contour plot of $|H_i(e^{j\omega_x}, e^{j\omega_y})|$ (the solid lines: the band edges).
Fig. 6 Phase errors $\frac{\arg\left(D(e^{j\omega}), e^{i\omega}\right)}{\phi_i(\omega, \omega_0)}$ for $i = 1, 2$

Fig. 7 Phase error $\left|\frac{\arg\left(G(e^{j\omega}), e^{i\omega}\right)}{g_1, g_2}\right|

REFERENCES


