Construction Methods for Sign Patterns Allowing Nilpotence of Index $k$

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Abstract—In this paper, the smallest such integer $k$ is called by the index (of nilpotence) of $B$ such that $B^k = 0$. In this paper, we study sign patterns allowing nilpotence of index $k$ and obtain four methods to construct sign patterns allowing nilpotence of index at most $k$, which generalizes some recent results.

Keywords—Sign pattern, Nilpotence, Jordan block.

I. INTRODUCTION

THE sign of a real number $a$, denoted by $\text{sgn}(a)$, is defined to be $1, -1$ or $0$, according to $a > 0, a < 0, a = 0$, respectively. A sign pattern matrix (or a sign pattern, for short) is a matrix whose entries are from the set $\{1, -1, 0\}$. The sign pattern of a real matrix $B$, denoted by $\text{sgn}(B)$, is the sign pattern matrix obtained from $B$ by replacing each entry by its sign.

Let $m$ be a positive integer. The integers $a$ and $b$ are congruent modulo $m$ if and only if there is an integer $t$ such that $a = b + tm$ (for short, written as $a \equiv b \pmod{m}$).

Let $Q_n$ be the set of all sign patterns of order $n$. For $A \in Q_n$, the set of all real matrices with the same sign pattern as $A$ is called the qualitative class of $A$, and is denoted by $Q(A)$ (\cite{1}).

Suppose that a real matrix has the property $p$. Then a sign pattern $A$ is said to require $p$ if every real matrix in $Q(A)$ has property $p$, or to allow $p$ if some real matrix in $Q(A)$ has property $p$ (\cite{1}).

In this paper, we investigate the property $N$ of being nilpotent. Recall that a real matrix $B$ is said to be nilpotent if $B^k = 0$ for some positive integer $k$. The smallest such integer $k$ is called the index of nilpotence ($N$).

Let $k$ be a positive integer. We now consider sign patterns that allow nilpotence of index at most $k$. These sign patterns that allow nilpotence, are also referred to as the potentially nilpotent sign patterns (see \cite{1,4,5,6}). For convenience, we denote the class of all sign patterns that allow nilpotence of index at most $k$ by $N_k$. In \cite{7}, it is reported that it is an open problem to determine necessary and/or sufficient conditions for a sign pattern to allow nilpotence of index $k \geq 4$. Eschenbach and Li \cite{4} studied $N_2$ and Gao, Li and Shao \cite{1} studied $N_3$.

In this paper, we mainly extend these results to any $N_k$.

II. PRELIMINARY

Lemma 1\cite{4}. The set $N_k$ is closed under the following operations:

1) negation;

2) transposition;

3) permutational similarity, and

4) signature similarity.

As defined in \cite{1}, two sign patterns are equivalent if one can be obtained from the other by performing a sequence of operations listed in Lemma 1. This is indeed an equivalence relation.

Lemma 2\cite{1}. A real matrix $B$ is nilpotent if and only if its eigenvalues are equal to zero.

Recall that a reducible (real or sign pattern) matrix is permutationally similar to a matrix in Frobenius normal form (see page 57 in \cite{8}). Consequently, a reducible sign pattern $A$ allows nilpotence if and only if each irreducible component (see \cite{8}) of $A$ allows nilpotence.

Lemma 3. Let $B$ be a nilpotent real matrix of index at most $k$, and $J$ the Jordan form of $B$. Then each Jordan block in $J$ is one of the following:

$$J_i = \begin{cases}
0 & \text{for } i = 2, 3, \ldots, k. \\
0 & \text{for } i = 2, 3, \ldots, k.
\end{cases}$$

Let $A$ be a sign pattern matrix. The minimal rank of $A$, denoted by $\text{mr}(A)$, is defined as $\text{mr}(A) = \min \{\text{rank} B : B \in Q(A)\}$ (\cite{3}).

Theorem 1. Let $A \in Q_n$. If $A \in N_k$, then

$$\text{mr}(A) \leq \frac{k-1}{k}n.$$ 

Proof. Let $A \in Q_n$ and $A \in N_k$. Then there exists a real matrix $B \in Q(A)$ such that $B^k = 0$. By Lemma 3 we can assume that the Jordan form $J$ of $B$ is a direct sum of $k_i$ copies of $J_i$ ($i = 1, 2, \ldots, k$), where $\sum_{i=1}^k i k_i = n$. Then

$$\text{rank}(B) = \text{rank}(J) = \sum_{i=1}^k (i-1)k_i \leq \frac{k-1}{k}k_1 + \frac{k-1}{k}2k_2 + \cdots + \frac{k-1}{k}kk_k = \frac{k-1}{k}n.$$ 

Hence $\text{mr}(A) \leq \text{rank}(B) \leq \frac{k-1}{k}n$. □
Remark 1. Note that the sign pattern
\[
A = \begin{bmatrix}
-1 & 1 & 1 & 1 \\
-1 & 1 & 1 & 1 \\
-1 & -1 & -1 & -1 \\
1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}
\]
satisfies \( mr(A) = 3 \leq \frac{2}{3} \times 5 \). However, \( A^1 \neq 0 \), \( A \notin N_4 \). So the condition in Theorem 1 is not a sufficient one.

Theorem 2. Let \( B \) be a real matrix of order \( n \) with \( \text{rank}(B) = r \). Then \( B^4 = 0 \) if and only if there exist nonnegative integers \( l, m \) and nonzero real column vectors \( \alpha_1, \alpha_2, \cdots, \alpha_r \) and \( \beta_1, \beta_2, \cdots, \beta_r \) of order \( n \) with \( l \leq \frac{r}{2} \), \( m \leq \frac{r}{2} \), \( 2r - 2l - m \leq n \) and
\[
\beta_j^T \alpha_i = \begin{cases}
1 & j \equiv 1 \pmod{3}, 1 \leq j \leq 3l - 1, \text{and } i = j + 1, \\
1 & j \equiv 2 \pmod{3}, 1 \leq j \leq 3l - 1, \text{and } i = j + 1, \\
onothe rwise,
\end{cases}
\]
such that
\[
B = \sum_{1 \leq i \leq r} \alpha_i \beta_i^T.
\]

Proof. Sufficiency. Let \( B = \alpha_1 \beta_1^T + \alpha_2 \beta_2^T + \cdots + \alpha_r \beta_r^T. \)
By (1), we have
\[
B^2 = (\alpha_1 \beta_1^T + \alpha_2 \beta_2^T + \cdots + \alpha_r \beta_r^T)
\]
\[
= \alpha_1 \beta_1^T + \alpha_2 \beta_2^T + \cdots + \alpha_r \beta_r^T
\]
\[
B^3 = (\alpha_1 \beta_1^T + \alpha_2 \beta_2^T + \cdots + \alpha_r \beta_r^T)
\]
\[
= \alpha_1 \beta_1^T + \alpha_2 \beta_2^T + \cdots + \alpha_r \beta_r^T
\]
and
\[
B^4 = (\alpha_1 \beta_1^T + \alpha_2 \beta_2^T + \cdots + \alpha_r \beta_r^T)
\]
\[
= \alpha_1 \beta_1^T + \alpha_2 \beta_2^T + \cdots + \alpha_r \beta_r^T
\]
\[
= 0.
\]

Necessity. Let \( B^4 = 0 \) with \( \text{rank}(B) = r \). By Lemma 3, the Jordan form \( J \) of \( B \) is a direct sum of \( l \) copies of \( J_4 \), \( m \) copies of \( J_3 \), \( r - 3l - 3m \) copies of \( J_2 \) and \( n - 4l - 3m - 2(r - 3l - 2m) = n - 2r + 2l + m \) copies of \( J_1 \), where \( 0 \leq l \leq \frac{r}{2} \), \( 0 \leq m \leq \frac{r}{2} \), and \( 2r - 2l - m \leq n \). It implies that there exists a nonsingular real matrix \( D \) of order \( n \) such that
\[
D^{-1}BD = J
\]
\[
= \begin{bmatrix}
J_{11} & J_{12} \\
J_{21} & J_{22} \\
\vdots & \vdots \\
J_{n-r+2m+4l, n-r+2m+4l} & \end{bmatrix}
\]
where
\[
J_{11} = \cdots = J_{l} = J_{4}, \quad J_{l+1,l+1} = \cdots = J_{l+m+l+m} = J_{3},
\]
\[
J_{l+m+1,l+m+1} = \cdots = J_{r-m-2l+r-m-2l+1} = J_{2},
\]
and
\[
J_{r-m-2l+1,r-m-2l+1} = J_{r-m-2l+2,r-m-2l+2} = \cdots = J_{n-r+2m+4l, n-r+2m+4l} = J_{1}.
\]

Write
\[
D = (u_1, u_2, \cdots, u_n) \text{ and } D^{-1} = (v_1, v_2, \cdots, v_n)^T,
\]
where \( u_1, u_2, \cdots, u_n \) are column vectors of \( D \) and \( v_1, v_2, \cdots, v_n \) are column vectors of \( D^{-1} \). Clearly, \( v_i^T u_i = 1 \), for \( i = 1, 2, \cdots, n \), and \( v_j^T u_i = 0 \), for \( i \neq j \). Let
\[
\alpha_{3l-2} = u_{4l-3}, \quad \alpha_{3l-1} = u_{4l-2}, \quad \alpha_{3l} = u_{4l-1} \text{ for } i = 1, 2, \cdots, l,
\]
\[
\alpha_{3l+2j-1} = u_{4l+j-3-2}, \quad \alpha_{3l+2j} = u_{4l+j-3-1}, \text{ for } j = 1, 2, \cdots, m.
\]
\[
\alpha_{3l+2m+s} = u_{4l+3m+2s-1} \text{ for } s = 1, 2, \cdots, r - 3l - 2m,
\]
\[
\beta_{3l-2} = v_{4l-3}, \quad \beta_{3l-1} = v_{4l-2}, \quad \beta_{3l} = v_{4l-1} \text{ for } i = 1, 2, \cdots, l,
\]
\[
\beta_{3l+2j-1} = v_{4l+j-3-2}, \quad \beta_{3l+2j} = v_{4l+j-3-1}, \text{ for } j = 1, 2, \cdots, m.
\]
\[
\beta_{3l+2m+s} = v_{4l+3m+2s-1} \text{ for } s = 1, 2, \cdots, r - 3l - 2m.
\]
It is easy to see that \( \alpha_i \) and \( \beta_i \) satisfy the condition (1). By (3), we have
\[
B = DJD^{-1} = \alpha_1 \beta_1^T + \alpha_2 \beta_2^T + \cdots + \alpha_r \beta_r^T.
\]
The conclusion follows. \( \square \)

Next, we generalize the above result to any \( B^k = 0 \), that is, \( N_k \).

Theorem 3. Let \( B \) be a real matrix of order \( n \) with \( \text{rank}(B) = r \). Then \( B^k = 0 \) if and only if there exist nonnegative integers \( l_1, l_2, \cdots, l_k \) and nonzero real column vectors \( \alpha_1, \alpha_2, \cdots, \alpha_r \) and \( \beta_1, \beta_2, \cdots, \beta_r \) of order \( n \) with \( \sum_{i=1}^{k} l_i = n \),
\[
k \sum_{i=1}^{k} (i-1)l_i = r \text{ and }\]
\[
\beta_j^T \alpha_i = \begin{cases}
1, & j \equiv s \pmod{(k-1)}, s = 1, 2, \cdots, k - 2, \\
1 \leq j \leq (k-1)l_i - 1, & i = j + 1, \\
onothe rwise,
\end{cases}
\]
such that
\[
B = \sum_{1 \leq i \leq r} \alpha_i \beta_i^T.
\]

Proof. Sufficiency. Let \( B = \alpha_1 \beta_1^T + \alpha_2 \beta_2^T + \cdots + \alpha_r \beta_r^T. \)
By (4), we have
\[
B^2 = (\alpha_1 \beta_1^T + \alpha_2 \beta_2^T + \cdots + \alpha_r \beta_r^T)
\]
\[
= (\alpha_1 \beta_1^T + \alpha_2 \beta_2^T + \cdots + \alpha_r \beta_r^T)
\]
\[
= \alpha_1 \beta_1^T + \alpha_2 \beta_2^T + \cdots + \alpha_r \beta_r^T
\]
\[
= 0.
\]

and
\[
B^k = BB^{k-1} = (\alpha_1\beta_1^T + \alpha_2\beta_2^T + \cdots + \alpha_k\beta_k^T)(\alpha_1\beta_1^T + \alpha_2\beta_2^T + \cdots + \alpha_k\beta_k^T) + \alpha_{k+1}\beta_{k+1} + \cdots + \alpha_{(k-2)l+1}\beta_{(k-1)l}^T
\]
\[
= 0.
\]

Necessity. Let \(B^k = 0\) with \(\text{rank}(B) = r\). By Lemma \(3\), the Jordan form \(J\) of \(B\) is a direct sum of \(l_i\) copies of \(J_i\), where \(\sum_{i=1}^k l_i = n\), \(\sum_{i=1}^k (i-1)l_i = r\) and it implies that there exists a nonsingular real matrix \(D\) of order \(n\) such that
\[
D^{-1}BD = J = \begin{bmatrix}
J_{11} & J_{12} & \cdots & J_{1n-r,1,n-r} \\
J_{21} & J_{22} & \cdots & J_{2n-r,1,n-r} \\
& \ddots & \ddots & \ddots \\
J_{n-r,1,n-r-1} & J_{n-r,1,n-r-1} & \cdots & J_{n-r,n-r}
\end{bmatrix},
\]
where
\[
J_{11} = \cdots = J_{l_1l_1} = J_{k},
\]
\[
J_{l_1+1,l_1+1} = \cdots = J_{l_1+l_1-1,l_1+l_1-1} = J_{k-1},
\]
\[
\cdots,
\]
\[
J \begin{bmatrix}
1 & \sum_{i=2}^k l_i, 1 + \sum_{i=2}^k l_i
\end{bmatrix} = \cdots = J_{n-r,n-r} = J_1.
\]

Write
\[
D = (u_1, u_2, \ldots, u_n) \quad \text{and} \quad D^{-1} = (v_1, v_2, \ldots, v_n)^T,
\]
where \(u_1, u_2, \ldots, u_n\) are column vectors of \(D\) and \(v_1, v_2, \ldots, v_n\) are column vectors of \(D^{-1}\), \(v_i^T u_i = 1\), for \(i = 1, 2, \ldots, n\), and \(v_i^T u_i = 0\), for \(i \neq j\). Let
\[
\alpha_{(s-1)i+m+\sum_{j=i+1}^s l_j} = u_{i+m+\sum_{j=i+1}^s l_j},
\]
\[
\alpha_{(s-1)i+m+\sum_{j=i+1}^k l_j} = u_{i+m+\sum_{j=i+1}^k l_j},
\]
for \(s = 1, 2, \ldots, k\), \(i = 1, 2, \ldots, l_s - 1\), and \(m = 1, 2, \ldots, s - 1\). It is easy to see that \(\alpha_i\) and \(\beta_i\) satisfy the condition \((4)\). By \((6)\), we have that
\[
B = DJD^{-1} = \alpha_1\beta_1^T + \alpha_2\beta_2^T + \cdots + \alpha_r\beta_r^T.
\]
The proof is completed. \(\square\)

### III. MAIN RESULTS

Based on the above analysis, one can obtain the following construction methods to find a sign pattern in \(N_k\).

#### A. Construction Method 1—Jordan Method

By Lemma \(3\), we may obtain the Jordan form method to construct a sign patterns in \(N_k\). For example, let
\[
J = \begin{bmatrix}
J_4 & J_2 \\
J_2 & J_4
\end{bmatrix},
\]
\[
D = \begin{bmatrix}
1 & 2 & 2 & 0 & 1 & 3 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 2 \\
1 & 0 & 2 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1
\end{bmatrix},
\]
\[
D^{-1} = \begin{bmatrix}
-1 & -1 & -2 & 2 & 2 & 3 \\
0 & 1 & 0 & 0 & -1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1
\end{bmatrix},
\]
\[
D^{-1}BD = J = \begin{bmatrix}
1 & 2 & 2 & 0 & 1 & 3 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 2 \\
1 & 0 & 2 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1
\end{bmatrix},
\]
\[
D^{-1} = \begin{bmatrix}
-1 & -1 & -2 & 2 & 2 & 3 \\
0 & 1 & 0 & 0 & -1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1
\end{bmatrix},
\]
\[
B = DJD^{-1} = \begin{bmatrix}
0 & 2 & 2 & 0 & -2 & -1 \\
1 & 0 & -1 & -1 & -2 \\
0 & 1 & 1 & 0 & -1 & -1 \\
-2 & 1 & 0 & 2 & 1 & 3 \\
1 & 0 & 1 & -1 & -1 & -2 \\
1 & 1 & 1 & -1 & -1 & -2
\end{bmatrix},
\]
\[
B^4 = 0,
\]
\[
A = \text{sgn}(B) = \begin{bmatrix}
0 & 1 & 1 & 0 & -1 & -1 \\
1 & 0 & 1 & -1 & -1 & -1 \\
0 & 1 & 1 & 0 & -1 & -1 \\
-1 & 1 & 0 & 1 & 1 & 1 \\
1 & 0 & 1 & -1 & -1 & -1 \\
1 & 1 & 1 & -1 & -1 & -1
\end{bmatrix}.
\]

#### B. Construction Method 2—vectors spanning method

Let \(l_1, l_2, \ldots, l_k\) be nonnegative integers with \(\sum_{i=1}^k l_i = n, \sum_{i=1}^k (i-1)l_i = r\). Let real column vectors \(\alpha_1, \alpha_2, \ldots, \alpha_r\) and \(\beta_1, \beta_2, \ldots, \beta_r\) of order \(n\) satisfy the condition
\[
\beta_i^T \alpha_i = \begin{cases}
1 & j \equiv s \pmod{k-1}, s = 1, 2, \ldots, k, \\
1 \leq j \leq (k-1)l_k - 1, i = j + 1, \\
0 & \text{otherwise}.
\end{cases}
\]

By Theorem \(3\), the real matrix
\[
B = \sum_{i=1}^r \alpha_i \beta_i^T
\]
is nilpotent of index at most \(k\), and its sign pattern is in \(N_k\). For example, let \(n = 8, r = 6, l = m = 1,\)
\[
\alpha_1 = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \quad \alpha_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix},
\]
\[
\alpha_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \alpha_5 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \alpha_6 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
\]
\[
\begin{align*}
\alpha_4 &= \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \\
\alpha_5 &= \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \\
\alpha_6 &= \begin{bmatrix} 2 \\ 2 \\ 0 \\ 2 \\ 2 \\ 2 \end{bmatrix},
\end{align*}
\]
\[
\begin{align*}
\beta_1 &= (1, 1, 0, -1, 0, 0, -1, -1), \\
\beta_2 &= (-1, 1, 2, -1, 1, -1, -1, -1), \\
\beta_3 &= (-1, 0, 0, 0, 0, 0, 1, 0), \\
\beta_4 &= (2, 0, 0, -1, 1, 0, -1, -1), \\
\beta_5 &= (1, -1, -1, -1, -1, 1, 0, 0, -1, 0, 0, 0, 0, -1, 1), \\
\beta_6 &= (-1, 0, 0, 0, 0, 0, 0, 1),
\end{align*}
\]
\[
B = \sum_{1 \leq i \leq 6} \alpha_i \beta_i = \begin{bmatrix}
1 & 2 & 1 & -3 & 1 & 0 & -4 & 1 \\
2 & 5 & 5 & -7 & 4 & -2 & -8 & 1 \\
1 & 2 & 2 & -3 & 2 & -1 & -3 & -1 \\
1 & 3 & 1 & -3 & 1 & 0 & -4 & 0 \\
-1 & 2 & 3 & -3 & 2 & -1 & -4 & 3 \\
1 & 2 & 1 & -3 & 1 & 0 & -4 & 1 \\
1 & 2 & 1 & -3 & 1 & 0 & -4 & 1
\end{bmatrix}
\]
\[
D^4 = 0.
\]
Then
\[
A = \text{sgn} (B) \in N_4.
\]

C. Construction Method 3—block method

**Theorem 4.** Suppose \( A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \in N_k \), where \( A_1 \) and \( A_4 \) are square, then for any positive integer \( m \), we have
\[
\tilde{A} = \begin{bmatrix} A_1 & A_2 & \cdots & A_2 \\ A_3 & A_4 & \cdots & A_4 \\ \vdots & \vdots & \ddots & \vdots \\ A_3 & A_4 & \cdots & A_4 \end{bmatrix} \in N_{k^2}
\]
where \( \tilde{A} \) has \((m + 1)^2\) blocks.

**Proof.** Note that, if \( A \in N_k \), then there is a real matrix
\[
B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \in Q(A),
\]
where \( B_i \in Q(A_i) \), such that \( B^k = 0 \).

Let
\[
B^j = \begin{bmatrix} f_{ij}(B_1, B_2, B_3, B_4) & f_{sj}(B_1, B_2, B_3, B_4) & f_{3j}(B_1, B_2, B_3, B_4) & f_{s3j}(B_1, B_2, B_3, B_4) \end{bmatrix}
\]
for \( j = 1, 2, \ldots, k - 1 \).

For short, we denote
\[
B^j = \begin{bmatrix} f_{ij} & f_{sj} & f_{3j} & f_{s3j} \end{bmatrix}
\]
for \( j = 1, 2, \ldots, k - 1 \). And
\[
\tilde{B} = \begin{bmatrix} B_1 & \frac{1}{m}B_2 & \cdots & \frac{1}{m}B_2 \\ B_3 & \frac{1}{m}B_4 & \cdots & \frac{1}{m}B_4 \\ \vdots & \vdots & \ddots & \vdots \\ B_3 & \frac{1}{m}B_4 & \cdots & \frac{1}{m}B_4 \end{bmatrix}
\]
When \( k = 2 \), it follows that
\[
B^2 = \begin{bmatrix} B_1^2 + B_2B_3 & B_1B_2 + B_2B_4 \\ B_3B_1 + B_4B_3 & B_3B_2 + B_4B_4 \end{bmatrix}
\]
\[
\tilde{B}^2 = \begin{bmatrix} B_1^2 + B_2B_3 & \frac{1}{m}(B_1B_2 + B_2B_3) & \cdots & \frac{1}{m}(B_1B_2 + B_2B_4) \\ B_3B_1 + B_4B_3 & \frac{1}{m}(B_3B_2 + B_3B_4) & \cdots & \frac{1}{m}(B_3B_2 + B_3B_4) \\ \vdots & \vdots & \ddots & \vdots \\ f_{32} & \frac{1}{m}f_{32} & \cdots & \frac{1}{m}f_{32} \end{bmatrix}
\]
So \( \tilde{B}^2 = 0 \). Then \( \tilde{A} \in N_{k^2} \).

**Suppose that we have**
\[
\tilde{B}^s = \begin{bmatrix} f_{is} & \frac{1}{m}f_{is} & \cdots & \frac{1}{m}f_{is} \\ f_{s3} & \frac{1}{m}f_{s3} & \cdots & \frac{1}{m}f_{s3} \\ \vdots & \vdots & \ddots & \vdots \\ f_{s3} & \frac{1}{m}f_{s3} & \cdots & \frac{1}{m}f_{s3} \end{bmatrix}
\]
for \( 2 \leq s < k \),

then
\[
B^{s+1} = B^sB = \begin{bmatrix} f_{is} & f_{is} & \cdots & f_{is} \\ f_{s3} & f_{s3} & \cdots & f_{s3} \\ \vdots & \vdots & \ddots & \vdots \\ f_{s3} & f_{s3} & \cdots & f_{s3} \end{bmatrix} \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}
\]
\[
= \begin{bmatrix} f_{is}B_1 + f_{is}B_3 & f_{is}B_2 + f_{is}B_4 \\ f_{s3}B_1 + f_{s3}B_3 & f_{s3}B_2 + f_{s3}B_4 \end{bmatrix}
\]
and
\[
\tilde{B}^{s+1} = \begin{bmatrix} f_{is} & \frac{1}{m}f_{is} & \cdots & \frac{1}{m}f_{is} \\ f_{s3} & \frac{1}{m}f_{s3} & \cdots & \frac{1}{m}f_{s3} \\ \vdots & \vdots & \ddots & \vdots \\ f_{s3} & \frac{1}{m}f_{s3} & \cdots & \frac{1}{m}f_{s3} \end{bmatrix} \begin{bmatrix} B_1 & \frac{1}{m}B_2 & \cdots & \frac{1}{m}B_2 \\ B_3 & \frac{1}{m}B_4 & \cdots & \frac{1}{m}B_4 \end{bmatrix}
\]
\[
= \begin{bmatrix} f_{is}B_1 + f_{is}B_3 & \frac{1}{m}(f_{is}B_2 + f_{is}B_4) & \cdots & \frac{1}{m}(f_{is}B_2 + f_{is}B_4) \\ f_{s3}B_1 + f_{s3}B_3 & \frac{1}{m}(f_{s3}B_2 + f_{s3}B_4) & \cdots & \frac{1}{m}(f_{s3}B_2 + f_{s3}B_4) \\ \vdots & \vdots & \ddots & \vdots \\ f_{s3}B_1 + f_{s3}B_3 & \frac{1}{m}(f_{s3}B_2 + f_{s3}B_4) & \cdots & \frac{1}{m}(f_{s3}B_2 + f_{s3}B_4) \end{bmatrix}
\]
So
\[
B^k = \begin{bmatrix} f_{1(k-1)}B_1 + f_{2(k-1)}B_3 & f_{1(k-1)}B_2 + f_{2(k-1)}B_4 \\ f_{3(k-1)}B_1 + f_{4(k-1)}B_3 & f_{3(k-1)}B_2 + f_{4(k-1)}B_4 \end{bmatrix} \in N_k.
\]
\[
\tilde{B}^k = 0.
\]

By the principle of mathematical induction, we have \( \tilde{A} \in N_{k^2} \).

D. Construction Method 4—null space method

**Theorem 5.** Let \( B \) and \( C \) be nilpotent real matrices of indices of \( k \) with order \( n_1 \) and \( n_2 \), respectively. Let \( p \) be a positive integer. The kernel of a matrix \( B \), denoted by \( \text{Ker}(B) \), also called the null space, is the kernel of the linear map defined by the matrix \( B \). Suppose that the following conditions hold:
\[
u_1, \ldots, v_p \in \text{Ker}(B^p), \quad v_1, v_2, \ldots, v_p \in \text{Ker}((C^T)^T).
\]
where $1 \leq i < k$, $1 \leq j < k$, and $i + j \leq k$. Then the following partitioned block real matrix of order $n_1 + n_2$

$$D = \begin{bmatrix} B & X \\ 0 & C \end{bmatrix}$$

is nilpotent of index at most $k$ and $A = \text{sgn}(D) \in N_k$, where $X = u_1 v_1^T + u_2 v_2^T + \cdots + u_p v_p^T$.

**Proof.** In fact, let $D = \begin{bmatrix} B & X \\ 0 & C \end{bmatrix}$, where $B$ and $C$ are square. Then

$$D^k = \begin{bmatrix} B^k & B^{k-1}X + B^{k-2}XC + \cdots + XC^{k-1} \\ 0 & C^k \end{bmatrix}.$$  

Thus $D^k = 0$ if and only if $B^k = 0, C^k = 0$ and $B^{k-1}X + B^{k-2}XC + \cdots + XC^{k-1} = 0$.

It is obvious that $B^2 = 0$ and $C^2 = 0$. In addition, we observe that

$$B^{k-1}X + B^{k-2}XC + \cdots + XC^{k-1} = B^{k-2}(Bu_1v_1^T + Bu_2v_2^T + \cdots + Bu_pv_p^T) + B^{k-3}(u_1, \ldots, u_p) \begin{pmatrix} v_1^T \\ \vdots \\ v_p^T \end{pmatrix} C + \cdots + (u_1, \ldots, u_p) \begin{pmatrix} v_1^T \\ \vdots \\ v_p^T \end{pmatrix} C^{k-1}.$$

Therefore, we get the desired result with the above condition (9). □

**IV. CONCLUSION**

In this paper, sign patterns allowing nilpotence of index at most $k$ are researched and four methods to construct sign patterns under the condition that allows nilpotence of index at most $k$ are obtained, which generalizes some recent results in [1], [4] and has a certain theoretical and practical value.

**ACKNOWLEDGMENT**

The work is part supported by National Nature Science Foundation of China (11026085, 11101071, 1117105, 51175443) and the Fundamental Research Funds for the Central universities (ZYGX2009J103).

**REFERENCES**


