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Construction Methods for Sign Patterns Allowing Nilpotence of Index$k$

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Abstract—In this paper, the smallest such integer k is called by the index (of nilpotence) of B such that $B^k = 0$. In this paper, we study sign patterns allowing nilpotence of index k and obtain four methods to construct sign patterns allowing nilpotence of index at most k, which generalizes some recent results.

Keywords—Sign pattern, Nilpotence, Jordan block.

I. Introduction

THE sign of a real number a, denoted by $\text{sgn}(a)$, is defined to be $1, -1$ or 0, according to $a > 0$, $a < 0$, $a = 0$, respectively. A *sign pattern matrix* (or a *sign pattern*, for short) is a matrix whose entries are from the set $\{1, -1, 0\}$. The sign pattern of a real matrix B, denoted by $\text{sgn}(B)$, is the sign pattern matrix obtained from B by replacing each entry by its sign.

Let m be a positive integer. The integers a and b are congruent modulo m if and only if there is an integer t such that $a = b + tm$ (for short, written as $a \equiv b(\text{mod} m)$).

Let Q_n be the set of all sign patterns of order n. For $A \in Q_n$, the set of all real matrices with the same sign pattern as A is called the *qualitative class* of A, and is denoted by $Q(A)$ ([2]).

Suppose that a real matrix has the property p. Then a sign pattern A is said to *require* p if every real matrix in $Q(A)$ has property p, or to allow p if some real matrix in $Q(A)$ has property p ([1]).

In this paper, we investigate the property N of being nilpotent. Recall that a real matrix B is said to be *nilpotent* if $B^k = 0$ for some positive integer k. The smallest such integer k is called the index (of nilpotence) of B.

Let k be a positive integer. We now consider sign patterns that allow nilpotence of index at most k. These sign patterns that allow nilpotence, are also referred to as the potentially nilpotent sign patterns (see [1], [4], [5], [6]). For convenience, we denote the class of all sign patterns that allow nilpotence of index at most k by N_k. In [7], it is reported that it is an open problem to determine necessary and/or sufficient conditions for a sign pattern to allow nilpotence of index $k \geq 4$. Eschenbach and Li [4] studied N_2 and Gao, Li and Shao [1] studied N_3. In this paper, we mainly extend these results to any N_k.

II. Preliminary

Lemma 1([4]). *The set N_k is closed under the following operations:*

1) negation;

2) transposition;

3) permutational similarity, and

4) signature similarity.

As defined in [1], two sign patterns are equivalent if one can be obtained from the other by performing a sequence of operations listed in Lemma 1. This is indeed an equivalence relation.

Lemma 2 ([1]). *A real matrix B is nilpotent if and only if its eigenvalues are equal to zero.*

Recall that a reducible (real or sign pattern) matrix is permutationally similar to a matrix in Frobenius normal form (see page 57 in [8]). Consequently, a reducible sign pattern A allows nilpotence if and only if each irreducible component (see [8]) of A allows nilpotence.

Lemma 3. *Let B be a nilpotent real matrix of index at most k, and J the Jordan form of B. Then each Jordan block in J is one of the following:*

$$J_1 = [0], \quad J_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \quad for\ i = 2, 3, \cdots, k.$$

Let A be a sign pattern matrix. The minimal rank of A, denoted by $\text{mr}(A)$, is defined as $\text{mr}(A) = \min\{\text{rank}B : B \in Q(A)\}$ ([3]).

Theorem 1. *Let $A \in Q_n$. If $A \in N_k$, then*

$$\text{mr}(A) \leq \frac{k-1}{k} n.$$

Proof. Let $A \in Q_n$ and $A \in N_k$. Then there exists a real matrix $B \in Q(A)$ such that $B^k = 0$. By Lemma 3 we can assume that the Jordan form J of B is a direct sum of k_i copies of J_i $(i = 1, 2, \cdots, k)$, where $\sum\limits_{i=1}^{k} ik_i = n$. Then

$$\begin{aligned} \text{rank}(B) &= \text{rank}(J) = \sum_{i=1}^{k} (i-1)k_i \\ &\leq \tfrac{k-1}{k}k_1 + \tfrac{k-1}{k}2k_2 + \cdots + \tfrac{k-1}{k}kk_k = \tfrac{k-1}{k}n. \end{aligned}$$

Hence $\text{mr}(A) \leq \text{rank}(B) \leq \frac{k-1}{k}n$. $\square$

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Remark 1. Note that the sign pattern

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

satisfies $\mathrm{mr}(A) = 3 \le \frac{3}{4} \times 5$. However, $A^4 \ne 0, A \notin N_4$. So the condition in Theorem 1 is not a sufficient one.

Theorem 2. *Let B be a real matrix of order n with rank$(B) = r$. Then $B^4 = 0$ if and only if there exist nonnegative integers l, m and nonzero real column vectors $\alpha_1, \alpha_2, \cdots, \alpha_r$ and $\beta_1, \beta_2, \cdots, \beta_r$ of order n with $l \le \frac{r}{3}$, $m \le \frac{r}{2}$, $2r - 2l - m \le n$ and*

$$\beta_j^T \alpha_i = \begin{cases} 1 & j \equiv 1 (\mathrm{mod} 3), 1 \le j \le 3l-1, \textit{and } i = j+1, \\ 1 & j \equiv 2 (\mathrm{mod} 3), 1 \le j \le 3l-1, \textit{and } i = j+1, \\ 0 & \textit{otherwise,} \end{cases} \quad (1)$$

such that

$$B = \sum_{1 \le i \le r} \alpha_i \beta_i^T. \quad (2)$$

Proof. *Sufficiency*. Let $B = \alpha_1\beta_1^T + \alpha_2\beta_2^T + \cdots + \alpha_r\beta_r^T$. By (1), we have

$$\begin{aligned} B^2 &= (\alpha_1\beta_1^T + \alpha_2\beta_2^T + \cdots + \alpha_r\beta_r^T) \\ &\quad (\alpha_1\beta_1^T + \alpha_2\beta_2^T + \cdots + \alpha_r\beta_r^T) \\ &= \alpha_1\beta_2^T + \alpha_2\beta_3^T + \alpha_4\beta_5^T \cdots + \alpha_{3l-1}\beta_{3l}^T, \\ B^3 &= (\alpha_1\beta_1^T + \alpha_2\beta_2^T + \cdots + \alpha_r\beta_r^T) \\ &\quad (\alpha_1\beta_2^T + \alpha_2\beta_3^T + \alpha_4\beta_5^T + \cdots + \alpha_{3l-1}\beta_{3l}^T) \\ &= \alpha_1\beta_3^T + \alpha_4\beta_6^T + \cdots + \alpha_{3l-2}\beta_{3l}^T \end{aligned}$$

and

$$\begin{aligned} B^4 &= (\alpha_1\beta_1^T + \alpha_2\beta_2^T + \cdots + \alpha_r\beta_r^T) \\ &\quad (\alpha_1\beta_3^T + \alpha_4\beta_6^T + \cdots + \alpha_{3l-2}\beta_{3l}^T) \\ &= 0. \end{aligned}$$

Necessity. Let $B^4 = 0$ with rank$(B) = r$. By Lemma 3, the Jordan form J of B is a direct sum of l copies of J_4, m copies of J_3, $r-3l-2m$ copies of J_2 and $n-4l-3m-2(r-3l-2m) = n-2r+2l+m$ copies of J_1, where $0 \le l \le \frac{r}{3}$, $0 \le m \le \frac{r}{2}$ and $2r-2l-m \le n$. It implies that there exists a nonsingular real matrix D of order n such that

$$D^{-1}BD = J = \begin{bmatrix} J_{11} & & & \\ & J_{22} & & \\ & & \ddots & \\ & & & J_{n-r+2m+4l,n-r+2m+4l} \end{bmatrix}. \quad (3)$$

where

$$J_{11} = \cdots = J_{ll} = J_4,\ J_{l+1,l+1} = \cdots = J_{l+m,l+m} = J_3,$$

$$J_{l+m+1,l+m+1} = \cdots = J_{r-m-2l,r-m-2l} = J_2,$$

and

$$\begin{aligned} J_{r-m-2l+1,r-m-2l+1} &= J_{r-m-2l+2,r-m-2l+2} \\ &= \cdots = J_{n-r+2m+4l,n-r+2m+4l} = J_1. \end{aligned}$$

Write

$$D = (u_1, u_2, \cdots, u_n) \text{ and } D^{-1} = (v_1, v_2, \cdots, v_n)^T,$$

where $u_1, u_2, \cdots, u_n$ are column vectors of D and $v_1, v_2, \cdots, v_n$ are column vectors of D^{-1}. Clearly, $v_i^T u_i = 1$, for $i = 1, 2, \cdots, n$, and $v_j^T u_i = 0$, for $i \ne j$. Let

$$\alpha_{3i-2} = u_{4i-3},\ \alpha_{3i-1} = u_{4i-2},\ \alpha_{3i} = u_{4i-1} \text{ for } i = 1, 2, \cdots, l,$$

$$\alpha_{3l+2j-1} = u_{4l+3j-2},\ \alpha_{3l+2j} = u_{4l+3j-1}, \text{ for } j = 1, 2, \cdots, m,$$

$$\alpha_{3l+2m+s} = u_{4l+3m+2s-1}, \text{ for } s = 1, 2, \cdots, r-3l-2m,$$

$$\beta_{3i-2} = v_{4i-3},\ \beta_{3i-1} = v_{4i-2},\ \beta_{3i} = v_{4i-1} \text{ for } i = 1, 2, \cdots, l,$$

$$\beta_{3l+2j-1} = v_{4l+3j-2},\ \beta_{3l+2j} = v_{4l+3j-1}, \text{ for } j = 1, 2, \cdots, m,$$

$$\beta_{3l+2m+s} = v_{4l+3m+2s-1}, \text{ for } s = 1, 2, \cdots, r-3l-2m.$$

It is easy to see that α_i and β_i satisfy the condition (1). By (3), we have

$$B = DJD^{-1} = \alpha_1\beta_1^T + \alpha_2\beta_2^T + \cdots + \alpha_r\beta_r^T.$$

The conclusion follows. □

Next, we generalize the above result to any $B^k = 0$, that is, N_k.

Theorem 3. *Let B be a real matrix of order n with rank$(B) = r$. Then $B^k = 0$ if and only if there exist nonnegative integers $l_1, l_2, \cdots, l_k$ and nonzero real column vectors $\alpha_1, \alpha_2, \cdots \alpha_r$ and $\beta_1, \beta_2, \cdots \beta_r$ of order n with $\sum_{i=1}^{k} il_i = n$, $\sum_{i=1}^{k} (i-1)l_i = r$, and*

$$\beta_j^T \alpha_i = \begin{cases} 1, & j \equiv s(\mathrm{mod} k-1), s = 1, 2, \cdots, k-2, \\ & 1 \le j \le (k-1)l_k - 1,\ i = j+1, \\ 0, & \textit{otherwise} \end{cases} \quad (4)$$

such that

$$B = \sum_{1 \le i \le r} \alpha_i \beta_i^T. \quad (5)$$

Proof. *Sufficiency*. Let $B = \alpha_1\beta_1^T + \alpha_2\beta_2^T + \cdots + \alpha_r\beta_r^T$. By (4), we have

$$\begin{aligned} B^2 &= (\alpha_1\beta_1^T + \alpha_2\beta_2^T + \cdots + \alpha_r\beta_r^T) \\ &\quad (\alpha_1\beta_1^T + \alpha_2\beta_2^T + \cdots + \alpha_r\beta_r^T) \\ &= (\alpha_1\beta_2^T + \alpha_2\beta_3^T + \cdots + \alpha_{k-2}\beta_{k-1}^T) \\ &\quad + (\alpha_k\beta_{k+1}^T + \cdots + \alpha_{2k-3}\beta_{2k-2}^T) + \cdots + \\ &\quad (\alpha_{(k-1)(l_k-1)+1}\beta_{(k-1)(l_k-1)+2}^T + \cdots + \\ &\quad \alpha_{(k-1)l_k-1}\beta_{(k-1)l_k}^T), \\ B^{k-1} &= (\alpha_1\beta_1^T + \alpha_2\beta_2^T + \cdots + \alpha_r\beta_r^T)[(\alpha_1\beta_{k-2}^T \\ &\quad + \alpha_2\beta_{k-1}^T) + (\alpha_k\beta_{2k-3}^T + \alpha_{k+1}\beta_{2k-2}^T) + \cdots + \\ &\quad (\alpha_{(k-2)l_k+1}\beta_{(k-1)l_k-1}^T + \alpha_{(k-2)l_k+2}\beta_{(k-1)l_k}^T)] \\ &= \alpha_1\beta_{k-1}^T + \alpha_k\beta_{k+1}^T + \cdots + \alpha_{(k-2)l_k+1}\beta_{(k-1)l_k}^T \end{aligned}$$

and

$$\begin{aligned} B^k &= BB^{k-1} \\ &= (\alpha_1\beta_1^T + \alpha_2\beta_2^T + \cdots + \alpha_r\beta_r^T)(\alpha_1\beta_{k-1}^T \\ &\quad + \alpha_k\beta_{k+1}^T + \cdots + \alpha_{(k-2)l_k+1}\beta_{(k-1)l_k}^T) \\ &= 0. \end{aligned}$$

Necessity. Let $B^k = 0$ with rank$(B) = r$. By Lemma 3, the Jordan form J of B is a direct sum of l_i copies of J_i, where $\sum_{i=1}^{k} il_i = n, \sum_{i=1}^{k} (i-1)l_i = r$ and it implies that there exists a nonsingular real matrix D of order n such that

$$D^{-1}BD = J = \begin{bmatrix} J_{11} & & & & \\ & J_{22} & & & \\ & & \ddots & & \\ & & & J_{n-r-1,n-r-1} & \\ & & & & J_{n-r,n-r} \end{bmatrix}, \quad (6)$$

where

$$J_{11} = \cdots = J_{l_k l_k} = J_k,$$

$$J_{l_k+1,l_k+1} = \cdots = J_{l_k+l_{k-1},l_k+l_{k-1}} = J_{k-1},$$

$$\cdots,$$

$$J_{1+\sum_{i=2}^{k} l_i, 1+\sum_{i=2}^{k} l_i} = \cdots = J_{n-r,n-r} = J_1.$$

Write

$$D = (u_1, u_2, \cdots, u_n) \text{ and } D^{-1} = (v_1, v_2, \cdots, v_n)^T,$$

where $u_1, u_2, \cdots, u_n$ are column vectors of D and $v_1, v_2, \cdots v_n$ are column vectors of D^{-1}, $v_i^T u_i = 1$, for $i = 1, 2, \cdots, n$, and $v_j^T u_i = 0$, for $i \neq j$. Let

$$\alpha_{(s-1)i+m+\sum_{j=s+1}^{k} jl_j} = u_{si+m+\sum_{j=s+1}^{k} jl_j},$$

$$\alpha_{(s-1)i+m+\sum_{j=s+1}^{k} jl_j} = u_{si+m+\sum_{j=s+1}^{k} jl_j}$$

for $s = 1, 2, \cdots, k$, $i = 1, 2, \cdots, l_s - 1$, and $m = 1, 2, \cdots, s-1$. It is easy to see that α_i and β_i satisfy the condition (4). By (6), we have that

$$B = DJD^{-1} = \alpha_1\beta_1^T + \alpha_2\beta_2^T + \cdots + \alpha_r\beta_r^T.$$

The proof is completed. □

III. Main results

Based on the above analysis, one can obtain the following construction methods to find a sign pattern in N_k.

A. Construction Method 1—Jordan Method

By Lemma 3, we may obtain the Jordan form method to construct a sign patterns in N_k. For example, let

$$J = \begin{bmatrix} J_4 & \\ & J_2 \end{bmatrix}, \; D = \begin{bmatrix} 1 & 2 & 2 & 0 & 1 & 3 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 2 \\ 1 & 0 & 2 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$D^{-1} = \begin{bmatrix} -1 & -1 & -2 & 2 & 2 & 3 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 1 & 1 & 1 & -1 & -2 & -2 \\ -1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 0 & 1 & 0 \end{bmatrix},$$

Note that

$$B = DJD^{-1} = \begin{bmatrix} 0 & 2 & 2 & 0 & -2 & -1 \\ 1 & 0 & 1 & -1 & -1 & -2 \\ 0 & 1 & 1 & 0 & -1 & -1 \\ -2 & 1 & 0 & 2 & 1 & 3 \\ 1 & 0 & 1 & -1 & -1 & -2 \\ 1 & 1 & 1 & -1 & -1 & -2 \end{bmatrix}, \quad B^4 = 0,$$

Then

$$A = \text{sgn}\,(B) = \begin{bmatrix} 0 & 1 & 1 & 0 & -1 & -1 \\ 1 & 0 & 1 & -1 & -1 & -1 \\ 0 & 1 & 1 & 0 & -1 & -1 \\ -1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 \end{bmatrix}.$$

B. Construction Method 2—vectors spanning method

Let $l_1, l_2, \cdots, l_k$ be nonnegative integers with $\sum_{i=1}^{k} il_i = n, \sum_{i=1}^{k} (i-1)l_i = r$. Let real column vectors $\alpha_1, \alpha_2, \cdots \alpha_r$ and $\beta_1, \beta_2, \cdots \beta_r$ of order n satisfy the condition

$$\beta_j^T \alpha_i = \begin{cases} 1 & j \equiv s(\bmod k-1), s = 1, 2, \cdots, k-2, \\ & 1 \leq j \leq (k-1)l_k - 1, i = j+1, \\ 0 & \text{otherwise}. \end{cases} \quad (7)$$

By Theorem 3, the real matrix

$$B = \sum_{1 \leq i \leq r} \alpha_i \beta_i^T \quad (8)$$

is nilpotent of index at most k, and its sign pattern is in N_k. For example, let $n = 8$, $r = 6$, $l = m = 1$,

$$\alpha_1 = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \\ 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \; \alpha_2 = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \; \alpha_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

$$\alpha_4 = \begin{bmatrix} 1\\2\\0\\1\\1\\1\\1\\1 \end{bmatrix},\ \alpha_5 = \begin{bmatrix} 1\\1\\0\\0\\1\\1\\1\\1 \end{bmatrix},\ \alpha_6 = \begin{bmatrix} 2\\2\\1\\0\\0\\2\\2\\2 \end{bmatrix},$$

$$\beta_1 = (1,1,0,-1,0,0,-1,-1),\ \beta_2 = (-1,1,2,-1,1,-1,-1,1),$$

$$\beta_3 = (-1,0,0,0,0,0,1,0),\ \beta_4 = (2,0,0,-1,1,0,-1,-1),$$

$$\beta_5 = (1,-1,-1,1,-1,1,-1,1),\ \beta_6 = (-1,0,0,0,0,0,0,1),$$

$$B = \sum_{1\le i\le 6} \alpha_i\beta_i = \begin{bmatrix} 1 & 2 & 1 & -3 & 1 & 0 & -4 & 1\\ 2 & 5 & 5 & -7 & 4 & -2 & -8 & 1\\ -2 & 0 & 0 & 0 & 0 & 0 & 1 & 1\\ 2 & 2 & 2 & -3 & 2 & -1 & -3 & -1\\ 3 & 1 & 1 & -3 & 1 & 0 & -4 & 0\\ -1 & 2 & 3 & -3 & 2 & -1 & -4 & 3\\ 1 & 2 & 1 & -3 & 1 & 0 & -4 & 1\\ 1 & 2 & 1 & -3 & 1 & 0 & -4 & 1 \end{bmatrix},$$

$$D^4 = 0.$$

Then

$$A = \mathrm{sgn}\,(B) \in N_4.$$

C. Construction Method 3—block method

Theorem 4. *Suppose $A = \begin{bmatrix} A_1 & A_2\\ A_3 & A_4 \end{bmatrix} \in N_k$, where A_1 and A_4 are square, then for any positive integer m, we have*

$$\widetilde{A} = \begin{bmatrix} A_1 & A_2 & \cdots & A_2\\ A_3 & A_4 & \cdots & A_4\\ \vdots & \vdots & \ddots & \vdots\\ A_3 & A_4 & \cdots & A_4 \end{bmatrix} \in N_k,$$

where $\widetilde{A}$ has $(m+1)^2$ blocks.

Proof. Note that, if $A \in N_k$, then there is a real matrix

$$B = \begin{bmatrix} B_1 & B_2\\ B_3 & B_4 \end{bmatrix} \in Q(A),$$

where $B_i \in Q(A_i)(i = 1,2,3,4)$, such that $B^k = 0$.

Let

$$B^j = \begin{bmatrix} f_{1j}(B_1,B_2,B_3,B_4) & f_{2j}(B_1,B_2,B_3,B_4)\\ f_{3j}(B_1,B_2,B_3,B_4) & f_{4j}(B_1,B_2,B_3,B_4) \end{bmatrix}$$

for $j = 1,2,\cdots,k-1$.

For short, we denote

$$B^j = \begin{bmatrix} f_{1j} & f_{2j}\\ f_{3j} & f_{4j} \end{bmatrix}$$

for $j = 1,2,\cdots,k-1$. And

$$\widetilde{B} = \begin{bmatrix} B_1 & \frac{1}{m}B_2 & \cdots & \frac{1}{m}B_2\\ B_3 & \frac{1}{m}B_4 & \cdots & \frac{1}{m}B_4\\ \vdots & \vdots & \ddots & \vdots\\ B_3 & \frac{1}{m}B_4 & \cdots & \frac{1}{m}B_4 \end{bmatrix}.$$

When $k = 2$, it follows that

$$B^2 = \begin{bmatrix} B_1^2 + B_2B_3 & B_1B_2 + B_2B_4\\ B_3B_1 + B_4B_3 & B_3B_2 + B_4^2 \end{bmatrix},$$

$$\widetilde{B}^2 = \begin{bmatrix} B_1^2+B_2B_3 & \frac{1}{m}(B_1B_2+B_2B_4)\cdots & \frac{1}{m}(B_1B_2+B_2B_4)\\ B_3B_1+B_4B_3 & \frac{1}{m}(B_1B_2+B_2B_4)\cdots & \frac{1}{m}(B_1B_2+B_2B_4)\\ \vdots & \vdots \quad \ddots & \vdots\\ B_3B_1+B_4B_3 & \frac{1}{m}(B_1B_2+B_2B_4)\cdots & \frac{1}{m}(B_1B_2+B_2B_4) \end{bmatrix}$$
$$= \begin{bmatrix} f_{12} & \frac{1}{m}f_{22} & \cdots & \frac{1}{m}f_{22}\\ f_{32} & \frac{1}{m}f_{42} & \cdots & \frac{1}{m}f_{42}\\ \vdots & \vdots & \ddots & \vdots\\ f_{32} & \frac{1}{m}f_{42} & \cdots & \frac{1}{m}f_{42} \end{bmatrix}.$$

So $\widetilde{B}^2 = 0$. Thus $\widetilde{A} \in N_k$.

Suppose that we have

$$\widetilde{B}^s = \begin{bmatrix} f_{1s} & \frac{1}{m}f_{2s} & \cdots & \frac{1}{m}f_{2s}\\ f_{3s} & \frac{1}{m}f_{4s} & \cdots & \frac{1}{m}f_{4s}\\ \vdots & \vdots & \ddots & \vdots\\ f_{3s} & \frac{1}{m}f_{4s} & \cdots & \frac{1}{m}f_{4s} \end{bmatrix} \text{ for } 2 \le s < k,$$

then

$$B^{s+1} = B^sB = \begin{bmatrix} f_{1s} & f_{2s}\\ f_{3s} & f_{4s} \end{bmatrix} \begin{bmatrix} B_1 & B_2\\ B_3 & B_4 \end{bmatrix}$$
$$= \begin{bmatrix} f_{1s}B_1 + f_{2s}B_3 & f_{1s}B_2 + f_{2s}B_4\\ f_{3s}B_1 + f_{4s}B_3 & f_{3s}B_2 + f_{4s}B_4 \end{bmatrix}$$

and

$$\widetilde{B}^{s+1} = \begin{bmatrix} f_{1s} & \frac{1}{m}f_{2s} & \cdots & \frac{1}{m}f_{2s}\\ f_{3s} & \frac{1}{m}f_{4s} & \cdots & \frac{1}{m}f_{4s}\\ \vdots & \vdots & \ddots & \vdots\\ f_{3s} & \frac{1}{m}f_{4s} & \cdots & \frac{1}{m}f_{4s} \end{bmatrix} \begin{bmatrix} B_1 & \frac{1}{m}B_2 & \cdots & \frac{1}{m}B_2\\ B_3 & \frac{1}{m}B_4 & \cdots & \frac{1}{m}B_4\\ \vdots & \vdots & \ddots & \vdots\\ B_3 & \frac{1}{m}B_4 & \cdots & \frac{1}{m}B_4 \end{bmatrix}$$
$$= \begin{bmatrix} f_{1s}B_1+f_{2s}B_3 & \frac{1}{m}(f_{1s}B_2+f_{2s}B_4) & \cdots & \frac{1}{m}(f_{1s}B_2+f_{2s}B_4)\\ f_{3s}B_1+f_{4s}B_3 & \frac{1}{m}(f_{3s}B_2+f_{4s}B_4) & \cdots & \frac{1}{m}(f_{3s}B_2+f_{4s}B_4)\\ \vdots & \vdots & \ddots & \vdots\\ f_{3s}B_1+f_{4s}B_3 & \frac{1}{m}(f_{3s}B_2+f_{4s}B_4) & \cdots & \frac{1}{m}(f_{3s}B_2+f_{4s}B_4) \end{bmatrix}.$$

So

$$B^k = \begin{bmatrix} f_{1(k-1)}B_1 + f_{2(k-1)}B_3 & f_{1(k-1)}B_2 + f_{2(k-1)}B_4\\ f_{3(k-1)}B_1 + f_{4(k-1)}B_3 & f_{3(k-1)}B_2 + f_{4(k-1)}B_4 \end{bmatrix} \in N_k,$$

$$\widetilde{B}^k = 0.$$

By the principle of mathematical induction, we have $\widetilde{A} \in N_k$.□

D. Construction Method 4—null space method

Theorem 5. *Let B and C be nilpotent real matrices of indices of k with order n_1 and n_2, respectively. Let p be a positive integer. The kernel of a matrix B, denoted by $Ker(B)$, also called the null space, is the kernel of the linear map defined by the matrix B. Suppose that the following conditions hold:*

$$u_1, u_2, \cdots u_p \in Ker(B^i),\ v_1, v_2, \cdots, v_p \in Ker((C^j)^T), \tag{9}$$

where $1 \le i < k$, $1 \le j < k$, *and* $i + j \le k$. *Then the following partitioned block real matrix of order* $n_1 + n_2$

$$D = \begin{bmatrix} B & X \\ 0 & C \end{bmatrix}$$

is nilpotent of index at most k *and* $A = sgn(D) \in N_k$, *where* $X = u_1 v_1^T + u_2 v_2^T + \cdots + u_p v_p^T$.

Proof. In fact, let $D = \begin{bmatrix} B & X \\ 0 & C \end{bmatrix}$, where B and C are square. Then

$$D^k = \begin{bmatrix} B^k & B^{k-1}X + B^{k-2}XC + \cdots + XC^{k-1} \\ 0 & C^k \end{bmatrix}.$$

Thus $D^k = 0$ if and only if $B^k = 0, C^k = 0$ and

$$B^{k-1}X + B^{k-2}XC + \cdots + XC^{k-1} = 0.$$

It is obvious that $B^k = 0$ and $C^k = 0$. In addition, we observe that

$$\begin{aligned} & B^{k-1}X + B^{k-2}XC + \cdots + XC^{k-1} \\ = & B^{k-2}(Bu_1v_1^T + Bu_2v_2^T + \cdots + Bu_pv_p^T) \\ & + B^{k-3}(u_1, \cdots, u_p) \begin{pmatrix} v_1^T \\ \vdots \\ v_p^T \end{pmatrix} C \\ & + \cdots + (u_1, \cdots, u_p) \begin{pmatrix} v_1^T \\ \vdots \\ v_p^T \end{pmatrix} C^{k-1}. \end{aligned}$$

Therefore, we get the desired result with the above condition (9). □

IV. Conclusion

In this paper, sign patterns allowing nilpotence of index at most k are researched and four methods to construct sign patterns under the condition that allows nilpotence of index at most k are obtained, which generalizes some recent results in [1], [4] and has a certain theoretical and practical value.

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