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On Generalized Exponential Fuzzy Entropy

Rajkumar Verma and Bhu Dev Sharma

Abstract—In the present communication, the existing measures of fuzzy entropy are reviewed. A generalized parametric exponential fuzzy entropy is defined.Our study of the four essential and some other properties of the proposed measure, clearly establishes the validity of the measure as an entropy.

Keywords—fuzzy sets, fuzzy entropy, exponential entropy, exponential fuzzy entropy.

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I. Introduction

The notion of fuzzy sets was proposed by Zadeh [13] in 1965 with a view to tackling problems in which indefiniteness arising from a sort of intrinsic ambiguity plays a fundamental role. Fuzziness, a feature of uncertainty, results from the lack of sharp distinction of the boundary of a set, i.e., an individual is neither definitely a member of the set nor definitely not a member of it. The first attempt to quantify the fuzziness was made in 1968 by Zadeh [14], who based on probabilistic framework introduced the entropy combining probability and membership function of a fuzzy event as weighted Shannon entropy [11]. De Luca and Termini [2] formulated axioms with which the fuzzy entropy measure should comply and defined a kind of entropy of a fuzzy set based on Shannon's function. Yager [12] defined an entropy measure of a fuzzy set in terms of a lack of distinction between fuzzy set and its negation based on norm. Pal and Pal [8, 9] proposed an entropy based on exponential function to measure the fuzziness called exponential fuzzy entropy. Hwang and Yang [4] defined fuzzy entropy by combining the concepts of Yager [12] and Pal and Pal [8, 9]. Some parametric generalizations of De Luca and Termini's entropy [2] have been studied by Kapur [7], Hooda [5], Bhandari and Pal [1], Fan and Ma [3].These parameters give certain flexibility in applications and their values have ultimately to be determined from the data itself. In this paper, new parametric generalized exponential entropy is proposed. This paper is organized as follows: In Section 2 some basic definitions related to probability and fuzzy set theory are briefly discussed. In Section 3 a new fuzzy entropy measure called, *exponential fuzzy entropy of order-α* is proposed and verifies the axiomatic requirements. In Section 4 some properties of the proposed measure are studied and limiting cases also discussed here and our conclusions are presented in Section 5.

II. Preliminaries

In this section we present some basic concepts related to probability theory and fuzzy sets which will be needed in the following analysis. First, let us cover probabilistic part of the preliminaries.

Let $\Delta_n = \{P = (p_1, \ldots, p_n) : p_i \geq 0, \sum_{i=1}^{n} p_i = 1\}$, $n \geq 2$ be a set of n-complete probability distributions.

For any probability distribution $P = (p_1, \ldots, p_n) \in \Delta_n$, Shannon's entropy [11], is defined as

$$H(P) = -\sum_{i=1}^{n} p(x_i) \log p(x_i) \tag{1}$$

Various generalized entropies have been introduced in the literature, taking the Shannon entropy as basic and have found applications in various disciplines such as economics, statistics, information processing and computing etc.

Generalizations of Shannon's entropy started with Rènyi's entropy [10] of order-α, given by

$$H_\alpha(P) = \frac{1}{(1-\alpha)} \log \Big[\sum_{i=1}^{n} (p(x_i))^\alpha \Big], \quad \alpha \neq 1, \alpha > 0 \tag{2}$$

Pal and Pal [8, 9] analyzed the classical Shannon information entropy and proposed a information entropy called exponential entropy given by

$$E(P) = \sum_{i=1}^{n} p(x_i)(e^{(1-p(x_i))} - 1) \tag{3}$$

These authors point out that, the exponential entropy has an advantage over Shannon's entropy. For the uniform probability distribution $P = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$ exponential entropy has a fixed upper bound

$$\lim_{n\to\infty} E\Big(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\Big) = (e-1) \tag{4}$$

which is not the case for Shannon's entropy.

Corresponding to (2), Kvålseth [6] introduced and studied generalized exponential entropy of order-α, given by

$$E_\alpha(P) = \frac{\sum_{i=1}^{n} p(x_i)(e^{(1-p^\alpha(x_i))} - 1)}{\alpha}, \quad \alpha > 0 \tag{5}$$

Definition 1: Let $X = (x_1, \ldots, x_n)$ be a discrete universe of discourse. A fuzzy set A on X is characterized by a membership function $\mu_A(x) : X \to [0,1]$.The value $\mu_A(x)$ of A at $x \in X$ stands for the degree of membership of x in A.

Definition 2: A fuzzy set A^* is called a sharpened version of fuzzy set A if the following conditions are satisfied:

$$\mu_{A^*}(x) \leq \mu_A(x), \qquad \text{if } \mu_A(x) \leq 0.5;\ \forall i$$

and

$$\mu_{A^*}(x) \geq \mu_A(x), \qquad \text{if } \mu_A(x) \geq 0.5;\ \forall i$$

Rajkumar Verma, Department of Mathematics, Jaypee Institute of Information Technology, Noida, U.P-201307, INDIA, e-mail: rkver83@gmail.com.
Bhu Dev Sharma,Department of Mathematics,Jaypee Institute of Information Technology,Noida,U.P-201307,INDIA, e-mail: bhudev.sharma@jiit.ac.in.
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Definition 3: Let $FS(X)$ denote the family of all FSs of universe X, assume $A, B \in FS(X)$ given as

$$A = \{\langle x, \mu_A(x)\rangle | x \in X\};$$
$$B = \{\langle x, \mu_B(x)\rangle | x \in X\};$$

then some set operations can be defined as follows:

i. $A^C = \{\langle x, 1-\mu_A(x), \mu_A(x)\rangle | x \in X\}$,
ii. $A \cap B = \{\langle x, min(\mu_A(x), \mu_B(x))\rangle | x \in X\}$,
iii. $A \cup B = \{\langle x, max(\mu_A(x), \mu_B(x))\rangle | x \in X\}$.

First attempt to quantify the uncertainty associated with a fuzzy event in the context of discrete probabilistic framework appears to have been made by Zadeh [14], who defined the (weighted) entropy of a fuzzy set A with respect to (X, P) as

$$H(A,P) = -\sum_{i=1}^{n} \mu_A(x_i) p(x_i) \log p(x_i) \tag{6}$$

De Luca and Termini [2] introduced a set of four axioms and these axioms are widely accepted as a criterion for defining any fuzzy entropy. In fuzzy set theory, the fuzzy entropy is a measure of fuzziness which expresses the amount of average ambiguity or difficulty in making a decision whether an element belongs to a set or not. A measure of fuzziness in a fuzzy set should have at least the following axioms:

P1 (***Sharpness***): $H(A)$ is minimum if and only if A is a crisp set, i.e. $\mu_A(x_i) = 0 \,\text{or}\, 1 \;\forall i$.
P2 (***Maximality***): $H(A)$ is maximum if and only if A is a most fuzzy set, i.e. $\mu_A(x_i) = \frac{1}{2} \;\forall i$.
P3 (***Resolution***): $H(A^*) \leq H(A)$, where A^* is a sharpened version of A.
P4 (***Symmetry***): $H(A) = H(A^C)$, where A^C is the complement set of A.

Since $\mu_A(x_i)$ and $(1-\mu_A(x_i))$ gives the same degree of fuzziness, therefore, De Luca and Termini [2] defined fuzzy entropy for a fuzzy set A corresponding to(1) as

$$H(A) = -\frac{1}{n}\sum_{i=1}^{n}\Big[\mu_A(x_i)\log\mu_A(x_i) + (1-\mu_A(x_i))\log(1-\mu_A(x_i))\Big] \tag{7}$$

Later on Bhandari and Pal [1] made a survey on information measures on fuzzy sets and gave some new measures of fuzzy entropy. Corresponding to (2) they have suggested the following measure:

$$H_\alpha(A) = \frac{1}{(1-\alpha)}\sum_{i=1}^{n}\log\Big[\mu_A^\alpha(x_i) + (1-\mu_A(x_i))^\alpha\Big] \quad \text{where } \alpha \neq 1, \alpha > 0, \tag{8}$$

Pal and Pal [8, 9] defined exponential fuzzy entropy for a fuzzy set corresponding (3)as

$$E(A) = \frac{1}{n(\sqrt{e}-1)}\sum_{i=1}^{n}\Big[\mu_A(x_i)e^{(1-\mu_A(x_i))} + (1-\mu_A(x_i))e^{(\mu_A(x_i))} - 1\Big] \tag{9}$$

Throughout this paper,we denote the set of all fuzzy sets on X by $FS(X)$.

In the next section we propose generalized fuzzy entropy measure corresponding to (4),called *exponential fuzzy entropy of order-*α and verify the axiomatic requirements.

III. Exponential fuzzy entropy of order-α

We proceed with the following formal definition:
Definition 4: Let A be the fuzzy set A fuzzy set defined on discrete universe of discourse $X = (x_1, \ldots, x_n)$ having the membership values $\mu_A(x_i), i = 1, 2, \ldots, n$.
We define the exponential fuzzy entropy of order-α corresponding to (5), as

$$E_\alpha(A) = \frac{1}{n(e^{(1-0.5^\alpha)}-1)}\sum_{i=1}^{n}\Big[\mu_A(x_i)e^{(1-\mu_A^\alpha(x_i))} + (1-\mu_A(x_i))e^{(1-(1-\mu_A(x_i))^\alpha)} - 1\Big], \quad \alpha > 0 \tag{10}$$

Theorem 1: The measure (10) satisfies measure of fuzzy entropy.
Proof: Symmetry follows from the definition.We prove the properties (1) to (3) are satisfied by (10).
P1 (Sharpness): First let $E_\alpha(A) = 0$, then

$$\Big[\mu_A(x_i)e^{(1-\mu_A^\alpha(x_i))} + (1-\mu_A(x_i))e^{(1-(1-\mu_A(x_i))^\alpha)}\Big] = 1 \tag{11}$$

Now $\alpha > 0$, (11) will hold when either $\mu_A(x_i) = 0$ or $\mu_A(x_i) = 1$ for all $i = 1, 2, \ldots, n$.

Next, conversely, if A is a crisp set, then either $\mu_A(x_i) = 0$ or $\mu_A(x_i) = 1$ for all $i = 1, 2, \ldots, n$.
It gives

$$\Big[\mu_A(x_i)e^{(1-\mu_A^\alpha(x_i))} + (1-\mu_A(x_i))e^{(1-(1-\mu_A(x_i))^\alpha)}\Big] = 1 \quad \text{for all}\,\alpha > 0 \tag{12}$$

that is

$$E_\alpha(A) = 0.$$

Hence $E_\alpha(A)$ if and only if A a crisp set.

P2 (Maximality): Let

$$E_\alpha(A) = \sum_{i=1}^{n} f(\mu_A(x_i)) \tag{13}$$

where

$$f(\mu_A(x_i)) = \frac{1}{n(e^{(1-0.5^\alpha)}-1)}\Big[\mu_A(x_i)e^{(1-\mu_A^\alpha(x_i))} + (1-\mu_A(x_i))e^{(1-(1-\mu_A(x_i))^\alpha)} - 1\Big], \quad \alpha > 0 \tag{14}$$

Now differentiating (14) with respect to $\mu_A(x_i)$,we get

$$\frac{\partial f(\mu_A(x_i))}{\partial \mu_A(x_i)} = \frac{1}{n(e^{(1-0.5^\alpha)}-1)}\Big[\Big(e^{(1-\mu_A^\alpha(x_i))} - e^{(1-(1-\mu_A(x_i))^\alpha)}\Big) - \alpha\Big(\mu_A^\alpha(x_i)e^{(1-\mu_A^\alpha(x_i))} - (1-\mu_A(x_i))^\alpha e^{(1-(1-\mu_A(x_i))^\alpha)}\Big)\Big] \tag{15}$$

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Open Science Index, Mathematical and Computational Sciences Vol:5, No:12, 2011 publications.waset.org/9417/pdf

Let $0 \leq \mu_A(x_i) < 0.5$,then

$$\frac{\partial f(\mu_A(x_i))}{\partial \mu_A(x_i)} > 0, \quad 0 < \alpha < 1 \text{ as also for } \alpha > 1 \tag{16}$$

Similarly, for $0.5 < \mu_A(x_i) \leq 1$, we have

$$\frac{\partial f(\mu_A(x_i))}{\partial \mu_A(x_i)} < 0, \quad 0 < \alpha < 1 \text{ as also for } \alpha > 1 \tag{17}$$

and for $\mu_A(x_i) = 0.5$,

$$\frac{\partial f(\mu_A(x_i))}{\partial \mu_A(x_i)} = 0, \quad 0 < \alpha < 1 \text{ as also for } \alpha > 1 \tag{18}$$

Thus $f(\mu_A(x_i))$ is a concave function which has a global maximum at $\mu_A(x_i) = 0.5$. Hence $E_\alpha(A)$ is maximum if and only if A is the most fuzzy set, i.e. $\mu_A(x_i) = 0.5$ for all $i = 1, 2, \ldots, n$.

P3 (Resolution): Since $H_\alpha(A)$ is increasing function of $\mu_A(x_i)$ in the range [0,0.5) and is decreasing function of $\mu_A(x_i)$ in the range (0.5,1], therefore

$$\mu_{A^*}(x_i) \leq \mu_A(x_i) \Rightarrow E_\alpha(A^*) \leq E_\alpha(A), \quad \text{in } [0, 0.5),$$
$$\text{and}$$
$$\mu_{A^*}(x_i) \geq \mu_A(x_i) \Rightarrow E_\alpha(A^*) \geq E_\alpha(A), \quad \text{in } (0.5, 1].$$

Hence

$$E_\alpha(A^*) \leq E_\alpha(A).$$

P4 (Symmetry): It is obvious from the definition,

$$E_\alpha(A) = E_\alpha(A^C).$$

This proves the theorem.

In the next section, we study some properties of $E_\alpha(A)$, the exponential fuzzy entropy of order-α.

IV. Properties of exponential fuzzy entropy of order-α

The measure of exponential fuzzy entropy of order-α has the following properties:

Theorem 2: For $A, B \in FS(X)$,

$$E_\alpha(A \cup B) + E_\alpha(A \cap B) = E_\alpha(A) + E_\alpha(B).$$

Proof: Let

$$X_+ = \{x \,|\, x \in X, \mu_A(x_i) \geq \mu_B(x_i)\} \tag{19}$$
$$X_- = \{x \,|\, x \in X, \mu_A(x_i) < \mu_B(x_i)\} \tag{20}$$

where $\mu_A(x)$ and $\mu_B(x)$ be the fuzzy membership functions of A and B respectively.

$$E_\alpha(A \cup B) = \frac{1}{(e^{(1-0.5^\alpha)} - 1)} \sum_{i=1}^{n} \Big[\mu_{A\cup B}(x_i) e^{(1-\mu^\alpha_{A\cup B}(x_i))} + (1 - \mu_{A\cup B}(x_i)) e^{(1-(1-\mu_{A\cup B}(x_i))^\alpha)} - 1\Big] \tag{21}$$

$$= \frac{1}{n(e^{(1-0.5^\alpha)} - 1)} \Big[\sum_{x_i \in X_+} [\mu_A(x_i) e^{(1-\mu^\alpha_A(x_i))} + (1 - \mu_A(x_i)) e^{(1-(1-\mu_A(x_i))^\alpha)} - 1] + \sum_{x_i \in X_-} [\mu_B(x_i) e^{(1-\mu^\alpha_B(x_i))} + (1 - \mu_B(x_i)) e^{(1-(1-\mu_B(x_i))^\alpha)} - 1]\Big] \tag{22}$$

and

$$E_\alpha(A \cap B) = \frac{1}{n(e^{(1-0.5^\alpha)} - 1)} \sum_{i=1}^{n} \Big[\mu_{A\cap B}(x_i) e^{(1-\mu^\alpha_{A\cap B}(x_i))} + (1 - \mu_{A\cap B}(x_i)) e^{(1-(1-\mu_{A\cap B}(x_i))^\alpha)} - 1\Big] \tag{23}$$

$$= \frac{1}{n(e^{(1-0.5^\alpha)} - 1)} \Big[\sum_{x_i \in X_+} [\mu_B(x_i) e^{(1-\mu^\alpha_B(x_i))} + (1 - \mu_B(x_i)) e^{(1-(1-\mu_B(x_i))^\alpha)} - 1] + \sum_{x_i \in X_-} [\mu_A(x_i) e^{(1-\mu^\alpha_A(x_i))} + (1 - \mu_A(x_i)) e^{(1-(1-\mu_A(x_i))^\alpha)} - 1]\Big] \tag{24}$$

Adding (22) and (24) we obtain,

$$E_\alpha(A \cup B) + E_\alpha(A \cap B) = E_\alpha(A) + E_\alpha(B).$$

This proves the theorem.

Corollary 1: For any $A \in FS(X)$, and A^C the complement of fuzzy set A,

$$E_\alpha(A) = E_\alpha(A^C) = E_\alpha(A \cup A^C) = E_\alpha(A \cap A^C).$$

Theorem 3: $E_\alpha(A)$ attains the maximum value when the set is most fuzzy and the minimum value when the set is least fuzzy, and independent of α.

Proof: It had already been proved that $E_\alpha(A)$ is maximum if and only if A is most fuzzy set and minimum when A is least fuzzy i.e.,crisp set.So,it is enough to prove that the maximum and minimum values are independent on α.

Let A be a most fuzzy set, i.e., $\mu_A(x_i) = 0.5, \forall\, i$. Then,

$$= \frac{\sum_{i=1}^{n} \Big[(0.5) e^{(1-0.5^\alpha)} + (0.5) e^{(1-(0.5)^\alpha)} - 1\Big]}{n(e^{(1-0.5^\alpha)} - 1)}, \quad \alpha > 0,$$
$$= \frac{\sum_{i=1}^{n} \Big[e^{(1-0.5^\alpha)} - 1\Big]}{n(e^{(1-0.5^\alpha)} - 1)}, \quad \alpha > 0,$$
$$= 1.$$

which is independent of α.
On the other hand, if A is a least fuzzy set i.e., $\mu_A(x_i) = 0$ or $1\ \forall\, i$, then $E_\alpha(A) = 0$ for any α.
This proves the theorem.

Limiting and Particular cases:

It is interesting to note that exponential fuzzy entropy of order-α proposed by us,reduces to Pal and Pal[9] exponential entropy and De-Luca and Termini [2] logarithmic entropy for different values of α, as follows:

(a) In case $\alpha = 1$ in (10), it reduces to

$$E(A) = \frac{1}{n(\sqrt{e}-1)} \sum_{i=1}^{n} \Big[\mu_A(x_i) e^{(1-\mu_A(x_i))} + (1-\mu_A(x_i)) e^{(\mu_A(x_i))} - 1\Big] \quad (25)$$

which is Pal and Pal exponential entropy.

(b) When $\alpha \to 0$, then (10) gives,

$$\lim_{\alpha \to 0} E_\alpha(A) = H(A) = \frac{1}{n} \sum_{i=1}^{n} \Big[\mu_A(x_i) \log \mu_A(x_i) + (1-\mu_A(x_i)) \log(1-\mu_A(x_i))\Big] \quad (26)$$

which is De-Luca and Termini logarithmic entropy.

V. Conclusions

This work introduces a new entropy measure called exponential fuzzy entropy of order-α in the setting of fuzzy set theory.Some properties of this measure have been also studied.This measure generalizes Pal and Pal [9] exponential entropy and De-Luca and Termini [2] logarithmic entropy. Introduction of parameter-α provides new flexibility and wider application of exponential fuzzy entropy to different situations.

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Rajkumar Verma is a Ph.D. student in Department of Mathematics, Jaypee Institute of Information Technology (Deemed University), Noida, (INDIA).He received the B.Sc.and M.Sc.degree from Chaudhary Charan Singh University (formerly, Meerut University), Meerut in 2004 and 2006 respectively. His research interests include Information Theory, Fuzzy information measures, Pattern recognition, Multicriteria Decision-making analysis, Soft set theory.

Bhu Dev Sharma currently Professor of Mathematics at JIIT-University, Noida, India did his Ph.D. from University of Delhi. He was a faculty at the University of Delhi (1966–1979) and earlier at other places. He spent about 29 years abroad 20 years (1988-2007) in USA and 9 years (1979-1988) in the West Indies, as Professor of Mathematics and in between as chair of the department.

Professor Sharma has published over 120 research papers in the areas covering Coding Theory, Information Theory, Functional Equations and Statistics. He has successfully guided 23 persons for Ph.D. in India and abroad. He has also authored/co-authored 23 published books in Mathematics and some others on Indian studies. He is the Chief Editor of ***Jr. of Combinatorics, Information and System Sciences***–JCISS (1976-present), and is on the editorial boards of several other journals.

He is member of several professional academic organizations. Past President of Forum for Interdisciplinary Mathematics (FIM), Vice President of Calcutta Mathematical Society, Ramanujan Math Society, Vice President of Academy of Discrete Mathematics and Applications (ADMA). After returning to India, he is working on establishing a Center of Excellence in Interdisciplinary Mathematics (CEIM) in Delhi, India.

Professor Sharma has received several awards and recognitions in India and abroad. He is widely traveled that would include Germany and several other countries of Europe, Brazil, Russia, and China. He has organized several International Conferences on Math/Stat in India, USA, Europe, Australia and China. Currently five persons are registered for Ph.D. under his supervision.

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