

Optimal Digital Pitch Aircraft Control

N. Popovich, P. Yan

Abstract—In this paper a controller for the pitch angle of an aircraft regarding to the elevator deflection angle is designed. The way how the elevator angle affects pitching motion of the aircraft is pointed out, as well as, how a pitch controller can be applied for the aircraft to reach certain pitch angle. In this digital optimal system, the elevator deflection angle and pitching angle of the plane are considered to be input and output respectively. A single input single output (SISO) system is presented. A digital pitch aircraft control is demonstrated. A simulation for the whole system has been performed. The optimal control weighting vectors, Q and R have been determined.

Keywords—Aircraft, control, digital, optimal, Q and R matrices.

I. INTRODUCTION

To design the digital pitch controller using an optimal method requires a State-Space model, which can be obtained through modeling the motion of an aircraft. The simulation is done by using Simulink. In optimal control, Q and R matrices are two weighting factors which will influence the Performance Index (PI). In order to minimize the PI, Q and R should be adjusted to proper values. In this paper, we will apply one method which can be used to tune Q and R so that we can get a relatively small value of PI. In order to do so, we should have a basic understanding of the system we are dealing with. The processes of determination of Q & R matrices for digital optimal pitch controller will be described in detail.

II. PLANE MODEL OF THE PITCH MOTION

A. Model

A simplified model of the pitch airplane control is given in Fig. 1.

If the elevator deflection angle has a small change over a short period, the input can be considered as a step input.

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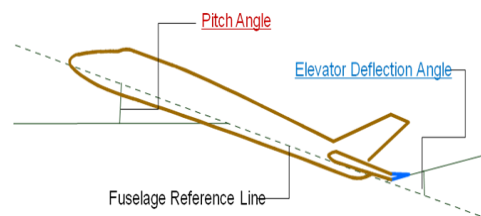


Fig. 1 Model of the pitching plane

B. Assumptions

The State-Space model obtained is based on these assumptions.

To begin with, the atmosphere condition is symmetric and the propulsive forces T are constant. Also, there is not suddenly strong wind effecting. Secondly, the plane is pitching at a constant velocity initially. Finally, the airplane is in steady flight condition with extremely small deviations. Therefore, all the changes in moments, inertia, velocity, angular velocity, elevator deflection angle, pitching angle, and rolling angle are assumed to be very small, so that there is not going to have huge influence in the overall motion.

C. Mathematical Description

The simulation of such a controller is based on a State-Space model. In order to obtain this model, we have to analyse the mechanical factors of motions of an airplane with the assumptions. Logically, the pitching motion consists of a horizontal motion along the x -axis, a vertical motion along the z -axis, and a rotational motion around y -axis. The resultant force is usually made up by aerodynamic force X , Z , propulsive force T , and gravitational force G .

Fig. 2 shows the horizontal motion along x -axis with forces components.

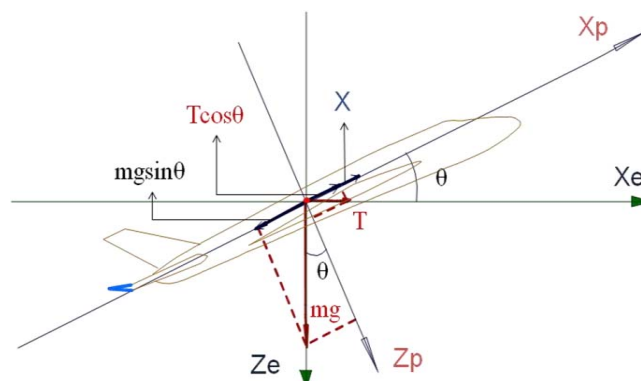


Fig. 2 Horizontal motion along x -axis

We rearrange a set of partial differential equations, which are developed from the basic principle in [2].

These equations include the aerodynamic forces in x-direction, as shown in Fig. 2. The final vision of expression for the resultant force along the x-axis is (1).

$$\Delta \dot{u} = \frac{\partial X_T}{\partial u} \Delta u + \frac{\partial X_T}{\partial w} \Delta w + (g \cos \theta_0) \Delta \theta + \frac{\partial X_T}{\partial \delta_e} \Delta \delta_e \quad (1)$$

Fig. 3 shows the vertical motion along z-axis with forces components.

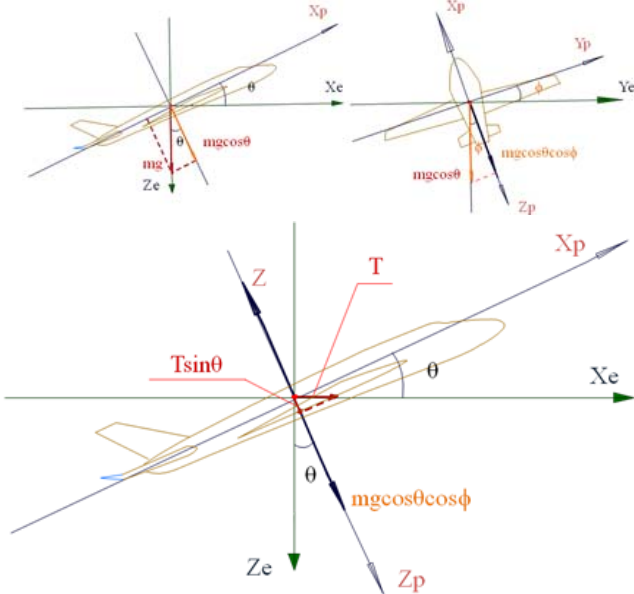


Fig. 3 Vertical motion along z-axis

Using the same procedures, the final version of expression for the resultant force along the z-axis is represented as (2).

$$\Delta \dot{w} = \frac{\partial Z_T}{\partial u} \Delta u + \frac{\partial Z_T}{\partial w} \Delta w + \frac{\partial Z_T}{\partial q} \Delta q + (g \sin \theta_0) \Delta \theta + \frac{\partial Z_T}{\partial \delta_e} \Delta \delta_e \quad (2)$$

Equation (3) is the final expression for the resultant moment about y-axis.

$$\Delta \dot{q} = \frac{\partial M}{\partial u} \Delta u + \frac{\partial M}{\partial w} \Delta w + \frac{\partial M}{\partial q} \Delta q + \frac{\partial M}{\partial \delta_e} \Delta \delta_e \quad (3)$$

Finally, the relationship of the pitch angle and pitch rate is defined with a simple differential equation, which is (4).

$$\Delta \dot{\theta} = \Delta q \quad (4)$$

As in [4], if all (1), (2), (3), and (4) are combined together in the compact form of matrix expression we can obtain the State-Space model of the plane, (5).

$$\begin{bmatrix} \Delta \dot{u} \\ \Delta \dot{w} \\ \Delta \dot{q} \\ \Delta \dot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{\partial X_T}{\partial u} & \frac{\partial X_T}{\partial w} & 0 & g \cos \theta_0 \\ \frac{\partial Z_T}{\partial u} & \frac{\partial Z_T}{\partial w} & \frac{\partial Z_T}{\partial q} & g \sin \theta_0 \\ \frac{\partial M}{\partial u} & \frac{\partial M}{\partial w} & \frac{\partial M}{\partial q} & 0 \\ \frac{\partial u}{\partial I_y} & \frac{\partial w}{\partial I_y} & \frac{\partial q}{\partial I_y} & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{bmatrix} + \begin{bmatrix} \frac{\partial X_T}{\partial \delta_e} \\ \frac{\partial Z_T}{\partial \delta_e} \\ \frac{\partial M}{\partial \delta_e} \\ \frac{\partial \delta_e I_y}{\partial \delta_e} \end{bmatrix} \Delta \delta_e \quad (5)$$

In [1] and [6], State-Space model of pitching motion, under certain prototype of Boeing 747 with 637000lb in weight at nominal speed $U_0=830\text{ft/sec}$ and 20000ft in height, gives us (6).

$$\begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -0.00643 & 0.0263 & 0 & -32.2 \\ -0.0941 & -0.624 & 820 & 0 \\ -0.000222 & -0.00153 & -0.668 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} 0 \\ -32.7 \\ -2.08 \\ 0 \end{bmatrix} \Delta \delta_e \quad (6)$$

In order to obtain the pitch angle as the output y, we have (7).

$$y = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} \quad (7)$$

III. DISCRETE TIME PITCH CONTROL MODEL

A. Discrete time State-Space model

We have obtained the continuous time State-Space model as (6) and (7). Since the controller is based on a discrete time model, we have to convert it to the form in (8), where $x(k)$ represents the state variables: $u(k)$, $w(k)$, $q(k)$, and $\theta(k)$.

$$\begin{cases} x(k+1) = Gx(k) + H\delta_e(k) \\ y(k) = Cx(k) \end{cases} \quad (8)$$

We did the conversion by using Matlab with a sampling time of 0.1 sec. The results are:

$$G = \begin{bmatrix} 0.8705 & -0.1574 & -9.4e-4 & -1.64e-5 \\ 0.09352 & 0.992 & -4.8e-5 & -8.38e-7 \\ 4.78e-3 & 0.0997 & 1 & -2.83e-8 \\ 1.61e-4 & 4.993e-3 & 0.1 & 1 \end{bmatrix}$$

$$H = \begin{bmatrix} 0.09352 \\ 4.784e-3 \\ 1.613e-4 \\ 4.059e-6 \end{bmatrix}$$

$$C = [0 \quad 2.08 \quad 1.261 \quad 0.01298]$$

B. Optimal Pitch Controller and Performance Index (PI)

A state feedback controller is simulated as Fig. 4.

$$u(k) = -Kx(k) \quad (9)$$

where in (9), K is a state feedback matrix.

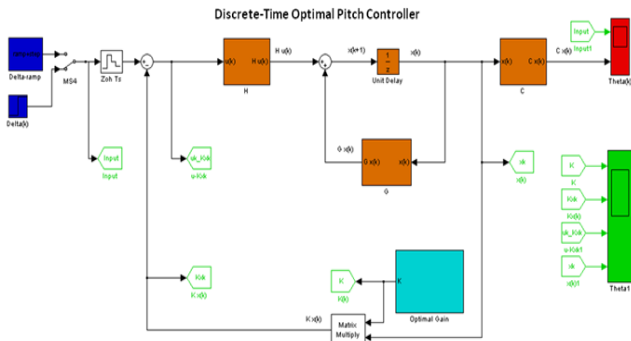


Fig. 4 Simulated model in Simulink

In order to find the optimal pitch controller parameters, we define a performance index (PI), in the form from [5].

$$J = \frac{1}{2} x'(N)Px(N) + \frac{1}{2} \sum_{k=0}^{N-1} [x'(k)Qx(k) + u'(k)Ru(k)] \quad (10)$$

To minimize this performance index, we apply the Euler-Lagrange equation. For the simplicity of the presentation, we will not go in details. The ending point N will be set to 20 sec during the observation. Solving discrete Riccati equation will give us a solution for Riccati matrix P .

$$P(k) = G'P(k+1)[I + HR^{-1}H'P(k+1)]^{-1}G + Q \quad (11)$$

Equation (11) involves two matrix inversions. Technically, this is likely to slow down a system. Thus, we try to rearrange it to the form shown as (12), which only contains one matrix inversion.

$$P(k) = G'\{P(k+1) - P(k+1)H[H'P(k+1)H + R]^{-1}H'P(k+1)\}G + Q \quad (12)$$

Through the mathematical manipulation of (9), (10), and (12), we can simply obtain (13), which is known in [3] and [7] as Kalman gain.

$$K = [H(k-1)'P(k)H(k-1) + R(k-1)]^{-1}H(k-1)'P(k)G(k-1) \quad (13)$$

Note that state feedback gain applying in this discrete time optimal pitch controller, in (9) is actually one step after the Kalman gain. However, it does not have negative impact on this system because the terms making up this gain do not contain time varying terms. As we can see, matrices H , R , and G are all constant matrices. No matter what step we take, these matrices will not change respect to time. As a result, the Kalman gain is able to be applied on this pitch controller, although it is one step forward to the original one.

When (13) is applied as a state feedback, the gains are not constant at the beginning. They are shown as dynamic feedback

gain in Fig. 5. After a short period, the gains converge to constant. This is known as steady state feedback gain.

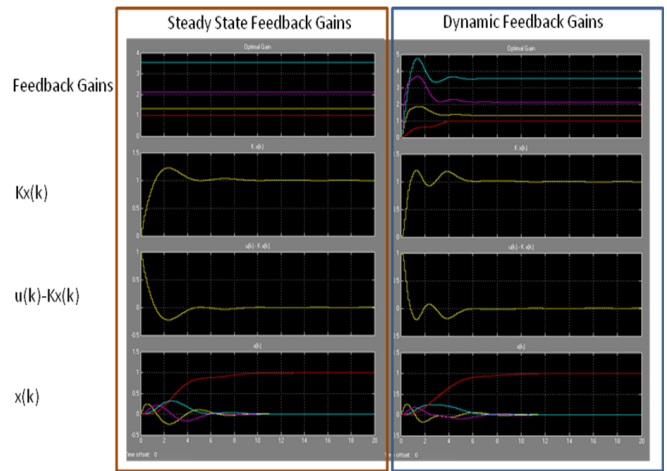


Fig. 5 Response of steady state feedback gain dynamic feedback gains

If these gains in Steady-State are directly connected to the feedback, the response of theta is shown in Fig. 5. By comparing it to the response of theta with dynamic feedback gains, we can see that both response patterns of theta are very similar. Without much effect on output theta, steady state feedback gains also provide less oscillation.

There is usually an advantage for a system to have less oscillation. In realistic, if the state feedback gain for a system is pre calculated, the overall system may response faster with assigning the steady state gain to it directly.

IV. DETERMINATION OF Q AND R MATRIX

A. Minimized value of PI

To begin with, we should keep the properties of Q and R matrices in mind. Q should strictly obey to the rule. It is that Q matrix has to be $n \times n$ symmetric, positive definite or positive semi-definite. R should be positive definite due to that it weights the input.

Simulation is done by using Simulink. The observation period is set to be 20 sec during simulation. Thus, there is a final Riccati value for P at time N . Taking Riccati matrix P into account, a simulation block which returns dynamic gain K and Riccati Matrix P is designed as Fig. 6. Simulation is done to calculate the value of PI as Fig. 7 presents.

These two figures allow us to calculate the value of PI.

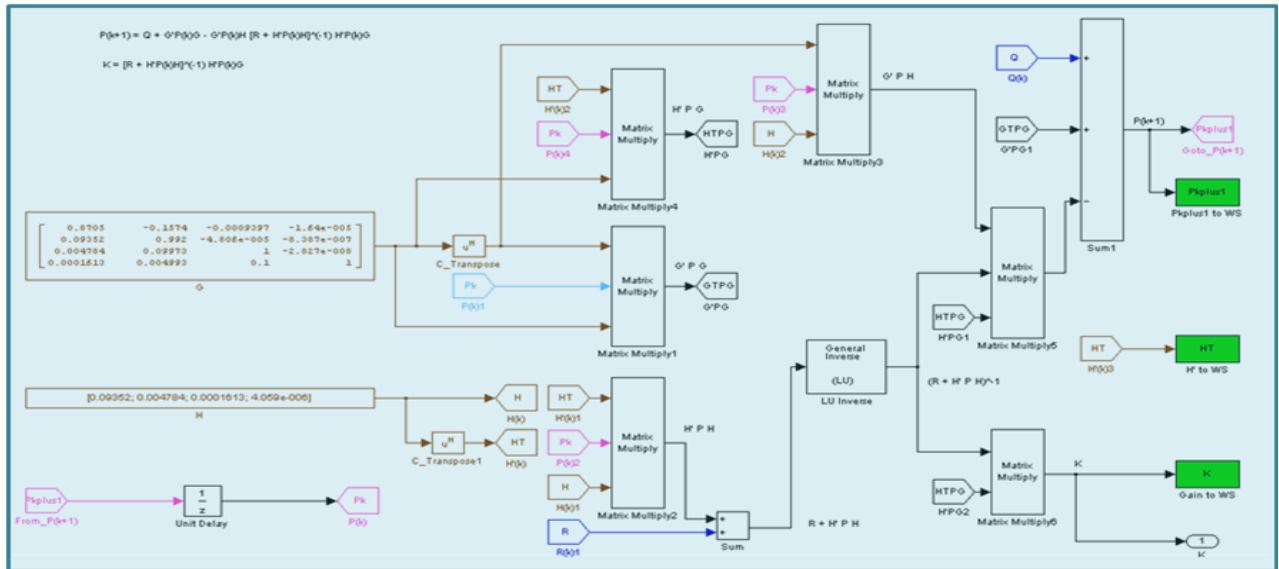
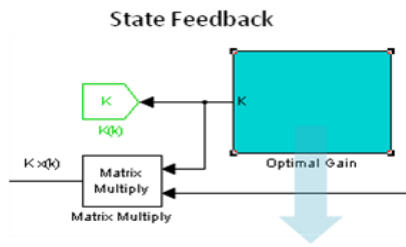


Fig. 6 Simulation of state feedback

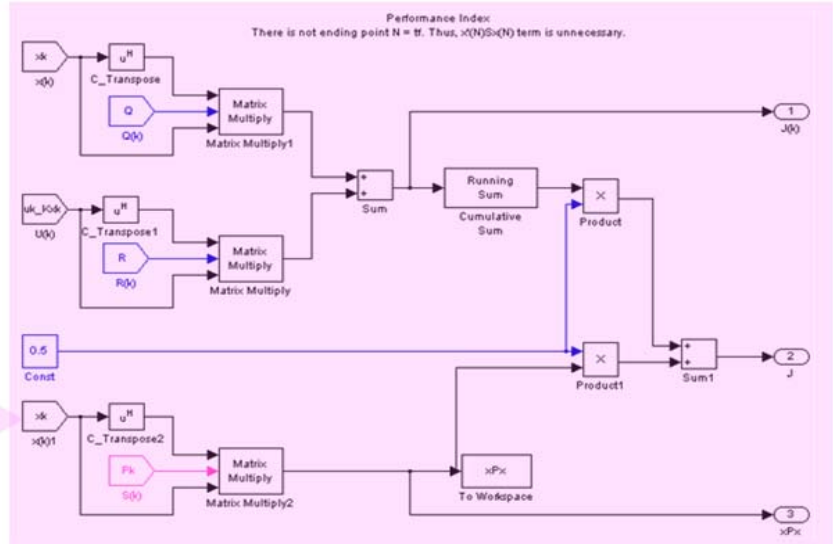
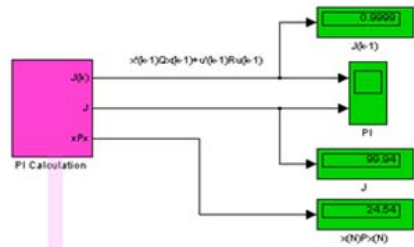


Fig. 7 PI calculation

B. Determination of Q

Firstly, we tune the main diagonal of this matrix. After setting it to be an ‘eye’ matrix, we run the simulation and see the response of the output. Due to the reason that the velocity components along x and z axis have less effect on the pitching motion, the pitch rate and the pitch angle become the major concern. Increase in the weighting factor of pitch angle, sitting on the 4th row and 4th column of Q matrix, raises the difference

between desired pitch angle and simulated pitch angle at the final time. Thus, it should be only adjusted at the range of 0.5 to 2. Furthermore, pitch rate is the dominant factor that eliminates the overshoot of pitch angle. If we increase the weighting factor for pitch rate, the overshoot of pitch is reduced significantly.

It comes to the next step after we tune the weighting factor Q on its main diagonal. Mathematically, there is a solid relationship between the pitch angle and pitch rate. The output curve will be smooth if the pitch rate and the pitch angle are

stressed among the state variables of velocity components along x and z axis. As an outcome, increase in the weight factor related to the velocity component along z axis makes contribution to the achievement of better output. Nevertheless, the output theta will start fluctuating, if the velocity component on x axis and pitch rate is over-stressed.

Overall, the weighting factors are dependent on the importance of that corresponding state variable. Beside the main diagonal of matrix Q, zero elements indicate there is little influence on pitch angle. The more important the state variable, the greater the value of weighting factor is.

$$Q = \begin{pmatrix} 1 & \times & \times & \times \\ & 1 & \times & \times \\ & & \times & \times \\ & & & 1 \end{pmatrix} \quad (14)$$

Associated with the techniques above, we can experimental test each vital element in (14) with symbol "cross", to observe the change in value of PI. Full data is shown in Appendix A.

The summary of change in PI referring to such properties, such as overshoot, is presented in TABLE I.

TABLE I
 SUMMARY OF CHANGES CAUSED BY INCREASE IN VALUES OF ELEMENTS IN Q MATRIX

Element Position	Output - Theta	Settling Time Ts	Overshoot	PI
4, 4	↓	↓	↓	↑
3, 3	~	↑	↓	↓
2, 2	~	~	✓	↑
1, 1	~	~	↑	↑
3, 4	~	~	~	✓
2, 4 ∈ (0, 3)	✓	↑	↑	✓
1, 4	~	~	↑	↑
2, 3	~	↑	↓	↓
1, 3 ∈ (0, 3)	~	↑	↓	↓
1, 2	~	~	↓	↓

~ - Almost no difference
 ↑ - Increase
 ↓ - Decrease
 ~ - Slightly Increase
 ~ - Slightly Decrease

C. Determination of R

There is only one dimension for R matrix. As we decrease the value of R, the final value of output is decreased significantly. TABLE II shows the trend of performance index, theta, overshoot and settling time once R increases. The results of obtaining such data are presented in Appendix B.

TABLE II
 SUMMARY OF CHANGES CAUSED BY INCREASE IN VALUE OF R

Weighting Factor	Output - Theta	Settling Time Ts	Overshoot	PI
R	↓	↑	↑	↑

When R is set to be zero, the value of performance index becomes zero. It is reasonable because the reference input is ignored by the system. Although it satisfies our criterion for optimal pitch controller, we will not adopt this value due to the reason that reference input is necessary and it generally has a value of 1.

D. Result

These are the optimal values of Q and R matrices, with a PI value of 99.65.

$$Q = \begin{bmatrix} 1 & 10 & 3 & 0 \\ 10 & 1 & 20 & 2.5 \\ 3 & 20 & 10 & 1.1 \\ 0 & 2.5 & 1.1 & 1 \end{bmatrix}$$

$$R = 0.87$$

The output Theta of controller respecting step input signal Delta is presented in Fig. 8.

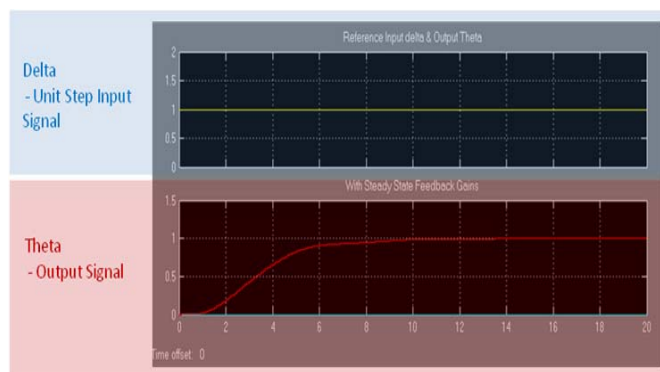


Fig. 8 Response to a unit step input

Obviously, this output (Theta) is non-oscillatory, so there is no overshoot. The rising time and setting time is about 3.5 sec and 7 sec, respectively.

VII. CONCLUSION

This paper has demonstrated the optimal control method for the digital pitch aircraft controller. The method used to determine Q and R matrices is considered to be the core of this paper. Weighting factors Q and R in optimal control system can be determined by fully understanding the target model. By tuning the relative elements in those matrices, we are able to obtain the one of the best values of Q and R matrices, which corresponds to a minimised PI value. Design of such pitch controller is one of the examples where this approach can be applied. A procedure for this method is suggested. A simulation of the optimal digital control for the aircraft has been performed.

APPENDIXES

Appendix A. Experimental results for elements of Q matrix

Element Position	Value	Theta	Ts	OS (%)	PI
4, 4	5	0.5	9	28	233.6
	10	0.35	8	/	241.8
3, 3	5	1.1	15	70	200.2
	10	1.1	17	45	197
2, 2	5	1.1	12	125	221.8
	10	1.1	13	110	227
	15	1.1	14	105	231.8
1, 1	5	1.1	13	150	262.8
	10	1.2	13	170	306.2
	15	1.25	15	180	340.8
3, 4	5	1.08	13	20	129.6
	10	1.08	12	20	123.6
	15	1.08	12	20	121.6
2, 4	5			Instable	
	2	1.02	20	90	139.4
	3			Instable	
1, 4	1.5	1.09	>20	83	149.1
	5	1.08	18	220	1.42E6
	10	1.08	15	160	1.81E8
2, 3	1.5	1.08	13	90	167.9
	5	1.1	9	67	149.2
	10	1.1	10	48	137.4
1, 3	5			Instable	
	2	1.1	>15	93	157.3
	3			Strong Oscillation	
1, 2	0	1.1	<15	110	195.3
	1	1.1	11	123	209.8
	5	1.1	11	109	192.6
	10	1.1	11	90	182.1

'4, 4' means the position at 4th row and 4th column of Q matrix. N=20 sec.

Appendix B. Experimental results for element of R matrix

Element Position	Value	Theta	Ts	OS (%)	PI
1, 1	0.5	0,76	12	100	114.5
	1.0	1.1	14	125	217.1
	2.0	1.5	16	175	417.3

'1, 1' means the position at 1st row and 1st column of R matrix. N=20 sec

REFERENCES

- [1] G. F. Franklin, D. J. Powell, and A. E. Naeini, *Feedback Control of Dynamic Systems*. New Jersey, The United States of America: Pearson Prentice Hall, Pearson Education, Inc, 2006.
- [2] J. L. Meriam, and L. G. Kraige, *Engineering Mechanics Dynamics*. The United States of America: John Wiley & Sons, Inc, 2003.
- [3] D. S. Naidu, *Optimal Control Systems*. The United States of America: CRC Press LLC, 2003.
- [4] R. C. Nelson, *Flight Stability and Automatic Control*. The McGraw-Hill Companies, Inc, 1998.
- [5] K. Ogata, *Discrete Time Control Systems*. New Jersey, The United States of America: Prentice-Hall, 1995.
- [6] K. Ogata, *Modern Control Engineering*. New Jersey: Prentice-Hall, Inc, 2002.
- [7] S. M. Shinnars, *Advanced Modern Control System Theory and Design*. John Wiley & Sons, Inc, 1998.