World Academy of Science, Engineering and Technology
International Journal of Computer and Information Engineering
Vol:6, No:12, 2012

Fuzzy Numbers and MCDM Methods for Portfolio Optimization

Thi T. Nguyen, and Lee N. Gordon-Brown

Abstract**—**A new deployment of the multiple criteria decision making (MCDM) techniques: the Simple Additive Weighting (SAW), and the Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) for portfolio allocation, is demonstrated in this paper. Rather than exclusive reference to mean and variance as in the traditional mean-variance method, the criteria used in this demonstration are the first four moments of the portfolio distribution. Each asset is evaluated based on its marginal impacts to portfolio higher moments that are characterized by trapezoidal fuzzy numbers. Then centroid-based defuzzification is applied to convert fuzzy numbers to the crisp numbers by which SAW and TOPSIS can be deployed. Experimental results suggest the similar efficiency of these MCDM approaches to selecting dominant assets for an optimal portfolio under higher moments. The proposed approaches allow investors flexibly adjust their risk preferences regarding higher moments via different schemes adapting to various (from conservative to risky) kinds of investors. The other significant advantage is that, compared to the mean-variance analysis, the portfolio weights obtained by SAW and TOPSIS are consistently well-diversified.

Keywords**—**Fuzzy numbers, SAW, TOPSIS, portfolio optimization, higher moments, risk management.

I. Introduction

PORTFOLIO selection theory is focused upon analysis of the performance of particular assets such that the risk of loss in holding or selling can be identified, so enabling an investor to implement a preferred strategy (e.g. lower risk, high return). Among the methods devised for this kind of performance analysis that of Markowitz [1] has been much used. Known as the standard mean variance optimization (MVO), it utilizes the mean and (co)variance of asset returns. Thus an investor/portfolio manager can take care not only of the realized returns, but also of the risk represented by the standard deviation of portfolio returns. However, given the non-Gaussian nature of the distribution of asset returns, researchers have questioned the adequacy of the mean-variance criteria. Thus emerges the impetus for using the same data to investigate higher moments.

Regardless of what criteria are chosen to evaluate the asset's performance, portfolio selection must be solved by optimization models. Numerous decision support approaches have been proposed to deal with optimization problems in multiple criteria circumstances. Typical methods that are widely used in multiple criteria decision making (MCDM) situations are Simple Additive Weighting (SAW) and Technique for Order Preference by Similarity to Ideal Solution (TOPSIS). The SAW and TOPSIS methods introduced by Hwang & Yoon [2] are probably the best known and very widely used methods of classical MCDM. The significant advantage in terms of rank reversals of SAW compared to other MCDM methods is highlighted in [3]. Chang and Yeh [4] in evaluating airline competitiveness using MCDM methods also confirmed the superiority of SAW due to its simplicity. Recent extensions of SAW have been developed under the fuzzy environment as reported in [5,6]. On the other hand, TOPSIS seems to be applied more widely in literature. Its applications involve a broad range of decision making such as planning, resource allocation or alternative selection. For example, Bottani and Rizzi [7] presented a framework based on TOPSIS and fuzzy set theory to evaluate and select the most appropriate third party logistics service provider. Shih et al. [8] extended TOPSIS for group decision making which is rather simple to use and meaningful for aggregation. A TOPSIS approach to solving large-scale multiple objective programming problems involving fuzzy parameters was deployed in Abo-Sinna and Abou-El-Enien [9]. Jahanshahloo et al. [10] suggested an advanced TOPSIS method to deal with interval data and then applied it to find the best place for creating a factory. Based on direct definitions in consequence space, Chen et al. [11] proposed a new approach to setting ideal and anti-ideal points that can be integrated into the TOPSIS method. The method is also able to handle non-monotonic as well as monotonic criteria in a unified framework. Additionally, fuzzy TOPSIS is utilized in [12] to aggregate scores for all potential locations in order to select the best place for implementing an urban distribution centre.

As yet, very few applications of the MCDM methods in the portfolio allocation have been available in the literature. The major challenge is in evaluating alternatives where expert judgments BASED on qualitative criteria might fluctuate. In a different approach to portfolio allocation, this paper presents an exploitation of quantitative factors, i.e. higher order moments, in terms of marginal impacts (contributions) of individual assets on the whole portfolio. These marginal impacts are modelled by trapezoidal fuzzy numbers and their centroids are applied in evaluating individual assets under implementation of MCMD methods, i.e. SAW and TOPSIS, for optimal portfolios. The following section presents motivation and formulas of higher order moment theory in portfolio selection. Details of modelling marginal impacts by fuzzy numbers and the associated evaluating methods are addressed in Section III. Section IV outlines the SAW and

T. T. Nguyen and L. N. Gordon-Brown are with the Department of Econometrics and Business Statistics, Faculty of Business and Economics, Monash University, Victoria, Australia (e-mails: thanh.thi@monash.edu and lee.gordon-brown@monash.edu).

TOPSIS methods and their applications for portfolio allocation. Illustrative examples are presented in Section V with comparisons to the standard MVO, followed by the conclusion.

II. Higher Moment Preferences and Marginal Impacts via Expected Utility

To measure the effect of higher moments on the asset allocation, it is necessary to consider a standard expected utility $U(W)$ of an investor over the terminal wealth W. Let $R = (R_1, \ldots, R_n)'$ be the vector of rates of return of n risky assets, $\mu = E[R] = (\mu_1, \ldots, \mu_n)'$ the expected return vector, $w = (w_1, \ldots, w_n)'$ the weight vector representing proportion of wealth allocated to various assets, and thus the terminal wealth is given by $W = (1 + r_p)$ where $r_p = w'R$. In order to approximate the expected utility, the infinite Taylor series of the utility function is deployed

$$U(W) = \sum_{k=0}^{\infty} \frac{U^{(k)}(E(W))}{k!}(W - E(W))^k \quad (1)$$

The expected utility is derived by applying the expectation operator to the above equation

$$E[U(W)] = E\left[\sum_{k=0}^{\infty} \frac{U^{(k)}(E(W))}{k!}(W - E(W))^k\right] = \sum_{k=0}^{\infty} \frac{U^{(k)}(E(W))}{k!} E[(W - E(W))^k] \quad (2)$$

Clearly, the expected utility from an investment in risky assets depends on all central moments of the distribution of the terminal wealth. The infinite Taylor expansion is a solution for the expected utility but, is not possible for numerical implementations. A truncation on the first k orders of the infinite Taylor series is a reasonable approximation of the expected utility. We set out the fourth-order Taylor expansion ($k = 4$) that extends the conventional mean-variance method by including skewness and kurtosis aiming at a better approximation of the expected utility.

$$E[U(W)] \approx U(E(W)) + \frac{1}{2!}U^{(2)}(E(W))\mu^{(2)} + \frac{1}{3!}U^{(3)}(E(W))\mu^{(3)} + \frac{1}{4!}U^{(4)}(E(W))\mu^{(4)} \quad (3)$$

where $\mu^{(n)}$ is the nth-order centered moment $\mu^{(n)} = E\left((W - E(W))^n\right)$

Formulae of portfolio expected return, variance, skewness and kurtosis are defined:

$$\mu_p = E[r_p] = w'\mu \quad (4)$$

$$\sigma_p^2 = E[(r_p - \mu_p)^2] = E[(W - E(W))^2] \quad (5)$$

$$s_p^3 = E[(r_p - \mu_p)^3] = E[(W - E(W))^3] \quad (6)$$

$$\kappa_p^4 = E[(r_p - \mu_p)^4] = E[(W - E(W))^4] \quad (7)$$

Then,

$$E[U(W)] \approx U(E(W)) + \frac{1}{2!}U^{(2)}(E(W))\sigma_p^2 + \frac{1}{3!}U^{(3)}(E(W))s_p^3 + \frac{1}{4!}U^{(4)}(E(W))\kappa_p^4 \quad (8)$$

The brief co-moment matrix-based presentation of the portfolio return, variance, skewness and kurtosis introduced in [13,14] are adopted herein.
Define the (n, n) co-variance matrix as

$$M_2 = E[(R - \mu)(R - \mu)'] = \{\sigma_{ij}\} \quad (9)$$

The (n, n^2) co-skewness matrix

$$M_3 = E[(R - \mu)(R - \mu)' \otimes (R - \mu)'] = \{s_{ijk}\} \quad (10)$$

The (n, n^3) co-kurtosis matrix

$$M_4 = E[(R - \mu)(R - \mu)' \otimes (R - \mu)' \otimes (R - \mu)'] = \{\kappa_{ijkl}\} \quad (11)$$

where $\otimes$ stands for the Kronecker product, σ_{ij}, s_{ijk} and κ_{ijkl} are elements of the co-variance, co-skewness and co-kurtosis matrices respectively.
In detail:

$$\sigma_{ij} = E[(R_i - \mu_i)(R_j - \mu_j)] \text{ where } i, j = 1, \ldots, n \quad (12)$$

$$s_{ijk} = E[(R_i - \mu_i)(R_j - \mu_j)(R_k - \mu_k)] \text{ where } i, j, k = 1, \ldots, n \quad (13)$$

$$\kappa_{ijkl} = E[(R_i - \mu_i)(R_j - \mu_j)(R_k - \mu_k)(R_l - \mu_l)] \text{ where } i, j, k, l = 1, \ldots, n \quad (14)$$

Because of the symmetric property, not all of the elements of these matrices need to be computed. For instance, in case of the (n, n) co-variance matrix, only $n(n+1)/2$ elements have to be computed. Similarly the (n, n^2) co-skewness matrix requires $n(n+1)(n+2)/6$ different elements, and the (n, n^3) co-kurtosis matrix needs $n(n+1)(n+2)(n+3)/24$ different elements.

Given a portfolio weight vector w, the moments of the portfolio return: expected return, variance, skewness and kurtosis of the portfolio are now respectively computed as follows:

$$\mu_p = \sum_{i=1}^{n} w_i \mu_i = w'\mu \tag{15}$$

$$\sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij} = w' M_2 w \tag{16}$$

$$s_p^3 = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} w_i w_j w_k s_{ijk} = w' M_3 (w \otimes w) \tag{17}$$

$$\kappa_p^4 = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} w_i w_j w_k w_l \kappa_{ijkl} = w' M_4 (w \otimes w \otimes w) \tag{18}$$

The partial derivatives with respect to the weight vector w are:

$$\frac{\partial \mu_p}{\partial w} = \mu \tag{19}$$

$$\frac{\partial \mu_p^2}{\partial w} = 2 M_2 w \tag{20}$$

$$\frac{\partial s_p^3}{\partial w} = 3 M_3 (w \otimes w) \tag{21}$$

$$\frac{\partial \kappa_p^4}{\partial w} = 4 M_4 (w \otimes w \otimes w) \tag{22}$$

Note that expressions on the right hand side of the above partial derivatives are $n \times 1$ vectors where their n elements are correspondent to the n asset classes. Conventionally, marginal impact of an asset is measured by the partial derivative of the portfolio higher moment with respect to the asset holding. Accordingly, the marginal contribution of asset i to the portfolio return, variance, skewness, and kurtosis is the i-th element of these partial derivative vectors, (19)-(22), respectively. Marginal contribution expresses how much the portfolio higher moments will change with respect to a small change of the weight of an asset. The asset with higher marginal impact will have more influence to the overall portfolio compared to the others.

With regard to the return criterion, the contribution of a given asset i to the whole portfolio return is obviously its expected return μ_i. However it is not straightforward with the portfolio variance, skewness or kurtosis: the marginal contribution of a given asset is not decreased (increased) to its own variance, kurtosis (skewness) but also takes account of its diversification potential in terms of co-variances, co-kurtosis (co-skewness) to other assets. This is an explanation against the possible argument that the evaluations of assets can be solely based on their own higher moments. Take an example of the portfolio's variance:

$$\sigma_p^2 = w' M_2 w = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij} = \sum_{i=1}^{n} w_i^2 \sigma_i^2 + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} w_i w_j \sigma_{ij} \tag{23}$$

The latter component of the above two-component decomposition relation represents the diversification effect of the overall portfolio variance. Manifestly, the portfolio variance encompasses not only the variance of individual assets but also takes into account the co-variance between assets. The partial derivative with respect to weight w_i of asset i is:

$$\frac{\partial \mu_p^2}{\partial w_i} = 2 \sum_{j=1}^{n} w_j \sigma_{ij} \tag{24}$$

The above equation shows the variance marginal contribution of the asset i to the whole portfolio variance. The σ_{ij} factor can be realized from historical or simulation data but the w_j factor is still unknown in this stage of the allocation process. The same situation exists for skewness and kurtosis since their marginal impacts also involve the unknown portfolio weight vector w (21-22). For that reason estimates of the marginal impacts are required in evaluating performance of different assets.

Recall the two constraints imposed on the portfolio weights $\sum_{i=1}^{n} w_i = 1$ and $w_i \geq 0, \forall i$, it is uncomplicated to find the minimum and maximum portfolio variance, skewness and kurtosis using constrained optimization solvers. The functions of the portfolio variance, skewness and kurtosis, presented in (16)-(18), are derivative continuous multivariate functions with the second, third, and fourth orders respectively. The number of variables in these functions is in proportion to the number of assets in the portfolio. From minimum and maximum solutions, the portfolio weight vectors $w = (w_1, \ldots, w_n)'$ will be exposed. For each of the criteria, i.e. variance, skewness and kurtosis, two weight vectors related to minimum and maximum circumstances will be obtained. Whenever a weight vector w is known, marginal contributions of assets on the portfolio will be calculated by expressions on the right hand side of formulae (19)-(22). As a result, given a criterion, we will obtain marginal impacts of any assets in both minimum and maximum extremes. The marginal impacts of each asset in extreme cases can be obtained but the exact contribution of an asset on the portfolio higher moments is uncertain before the allocation process is accomplished (before choosing w). This problem is addressed using fuzzy numbers. The following section presents relevant fuzzy concepts and how to model these marginal contributions by fuzzy numbers.

III. Fuzzy Modelling Marginal Contributions

A. Relevant Fuzzy Set Concepts and Notions

Standard fuzzy sets along with their basic characteristics were first coined by Zadeh [15]. Accordingly, a standard fuzzy set A is defined by a membership function $f_A(x)$ mapping from a universal set of concern X to a range from 0 to 1: $f_A(x): X \to [0, 1]$. For each $x \in X$, the value of $f_A(x)$ expresses the degree (or grade) of membership of the element x of X in standard fuzzy set A.

α-cut is one of the most important concepts of fuzzy sets. Given a particular number $\alpha \in [0,1]$ and a fuzzy set A defined on X, the α-cut of A, denoted by ${}^{\alpha}A$, is a crisp set encompassing elements of X satisfying: ${}^{\alpha}A = \{x | f_A(x) \geq \alpha\}$.

Support and core of A are crisp sets respectively defined by ${}^{0+}A$ and ${}^{1}A$. A is called normal fuzzy set when its core is not empty, otherwise it is called sub-normal.

A fuzzy interval or fuzzy number is a special fuzzy set defined on the set of real numbers, $\mathbb{R}$. A fuzzy number A is characterized for each $x \in \mathbb{R}$ by the canonical form:

$$f_A(x) = \begin{cases} f_A^L(x), & a \leq x \leq b, \\ \omega, & b \leq x \leq c, \\ f_A^R(x), & c \leq x \leq d, \\ 0 & \text{otherwise} \end{cases} \tag{25}$$

where $\omega \in (0,1]$ is a constant, $a, b, c, d \in \mathbb{R}$, $a \leq b \leq c \leq d$, $f_A^L: [a,b] \to [0,\omega]$ is a increasing real-valued function whereas $f_A^R: [c,d] \to [0,\omega]$ is a real-valued decreasing function. A is a normal fuzzy number if $\omega = 1$, otherwise it is a non-normal fuzzy number. The most widely used are normal trapezoidal fuzzy numbers denoted by $A = (a, b, c, d)$ whose membership functions are piecewise linear

$$f_A(x) = \begin{cases} (x-a)/(b-a), & a \leq x \leq b, \\ 1, & b \leq x \leq c, \\ (d-x)/(d-c), & c \leq x \leq d, \\ 0 & \text{otherwise} \end{cases} \tag{26}$$

B. Modelling Marginal Impacts by Fuzzy Numbers

As with Scott and Horvath [16], investors' expected utility depends positively on return and skewness and negatively on variance and kurtosis. The rational investor accordingly would prefer higher portfolio skewness but lower variance and kurtosis. We try to allocate the portfolio wealth to satisfy or at least nearly satisfy the investor's preference or utility. The portfolio weight vector should bring the investor an acceptable result in accordance with his/her expectation: maximum (as high as possible) skewness and minimum (as low as possible) variance and kurtosis. Hence, it is logical to design fuzzy numbers representing variance contributions of assets based mainly on their marginal contributions when the portfolio variance attains minimum. Similarly, contributions of assets with respect to the skewness (kurtosis) criterion will be characterized by fuzzy numbers that are designed towards skewness (kurtosis) marginal contributions in the maximum (minimum) context. In other words, maximum (minimum) is the preferred extreme regarding skewness (variance or kurtosis). With a preferred extreme (e.g. variance minimum), fuzzy numbers are designed that the possibility to achieve (or at least nearly achieve) this extreme is higher than the possibility to get the other extreme (variance maximum). The proportion parameter $\rho \in [0,1]$ in the following equations represents the bias level towards the preferred extremes.

In the MCDM applications, investors can flexibly demonstrate a preference weighting scheme concerning their attitude with regard to different criteria by the preference ratio series $s = (s_r : s_v : s_s : s_k)$ where s_r, s_v, s_s and s_k are in that order the importance level of return, variance, skewness and kurtosis. For example, the scheme (1:1:2:1) indicates that the investor will focus more on portfolio skewness rather than return, variance and kurtosis. The scheme (3:0:1:3) says the investor greatly favours return and kurtosis, slightly favours skewness and pays no attention to variance. We explore some typical schemes exhibited in Table I. The scheme (2:1:2:1) would be appropriate for young investors and contrastingly the scheme (1:2:1:2) is for conservative investors. The scheme (1:1:0:0) is equivalent to the conventional mean-variance approach.

TABLE I
INVESTOR PREFERENCE WEIGHTING SCHEMES USED FOR EXPERIMENTS

Weighting schemes $s = (s_r : s_v : s_s : s_k)$	(2:1:2:1)	(1:2:1:2)	(4:3:2:1)	(1:2:3:4)	(1:1:0:0)	(0:0:1:1)
Normalized ratios $\bar{s} = (\bar{s_r}, \bar{s_v}, \bar{s_s}, \bar{s_k})$	(1/3, 1/6, 1/3, 1/6)	(1/6, 1/3, 1/6, 1/3)	(2/5, 3/10, 1/5, 1/10)	(1/10, 1/5, 3/10, 2/5)	(1/2, 1/2, 0, 0)	(0, 0, 1/2, 1/2)
$\boldsymbol{\rho}$ **in fuzzy number designs**	(1, 1/2, 1, 1/2)	(1/2, 1, 1/2, 1)	(1, 3/4, 1/2, 1/4)	(1/4, 1/2, 3/4, 1)	(1, 1, 0, 0)	(0, 0, 1, 1)

Based on intended weighting schemes from investors, the parameters ρ in designing trapezoidal fuzzy numbers will be specified for each criterion. Let us define $s_{max} = \max(s_r, s_v, s_s, s_k)$. For any scheme $(s_r : s_v : s_s : s_k)$, ρ in cases of variance (denoted by ρ_v), skewness (ρ_s) and kurtosis (ρ_k) are: $\rho_v = s_v / s_{max}$, $\rho_s = s_s / s_{max}$ and $\rho_k = s_k / s_{max}$. So $\rho_v = 1$ if $s_v = s_{max} = \max(s_r, s_v, s_s, s_k)$ implies most concern about variance, or $\rho_v = 0$ if $s_v = 0$ implies no concern about variance, and alike interpretations for ρ_s and ρ_k. When $\rho = 1$, the trapezoidal fuzzy number is of special triangular shape.

In Nguyen and Gordon-Brown [17] we used triangular fuzzy numbers to model marginal impacts of stocks. In this paper, we use the more general fuzzy numbers that have the trapezoidal shape. Assume a normal trapezoidal fuzzy number $A_i(a_i, b_i, c_i, d_i)$ is to be constructed to stand for marginal contributions of asset i with min_mc_i and max_mc_i are its marginal contributions in the minimum and maximum circumstances, so the support of A_i is a set $\{x | x \in (a_i, d_i)\}$

and the core of A_i is a set $\{x|x \in (b_i, c_i)\}$. Hence, $a_i = min\ (min\_mc_i,\ max\_mc_i)$ and $d_i = max\ (min\_mc_i,\ max\_mc_i)$

a) Variance or kurtosis fuzzy numbers – illustrated by Fig.s 1,2

$$b_i = \min\left(\frac{min\_mc_i + max\_mc_i}{2} + \rho\left(min\_mc_i - \frac{min\_mc_i + max\_mc_i}{2}\right), min\_mc_i\right) \quad (27)$$

$$c_i = \max\left(\frac{min\_mc_i + max\_mc_i}{2} + \rho\left(min\_mc_i - \frac{min\_mc_i + max\_mc_i}{2}\right), min\_mc_i\right) \quad (28)$$

i) If $a_i = min\ (min\_mc_i,\ max\_mc_i) = min\_mc_i$ and $d_i = max\ (min\_mc_i,\ max\_mc_i) = max\_mc_i$ then:

$$b_i = min\_mc_i \quad (29a)$$

$$c_i = \frac{min\_mc_i + max\_mc_i}{2} + \rho\left(min\_mc_i - \frac{min\_mc_i + max\_mc_i}{2}\right) \quad (29b)$$

With (29^{II}), we examine two cases:

$$\rho \to 0 \ \leftrightarrow \ c_i \to \frac{min\_mc_i + max\_mc_i}{2} \quad (29c)$$

$$\rho \to 1 \ \leftrightarrow \ c_i \to min\_mc_i \quad (29d)$$

This circumstance is illustrated by Fig. 1.

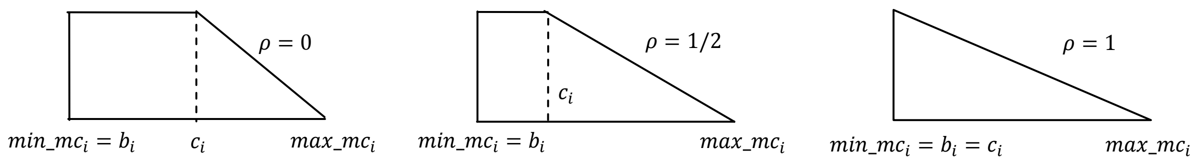


Fig. 1 Variance and kurtosis fuzzy numbers when $a_i = min\_mc_i$ and $d_i = max\_mc_i$

ii) If $a_i = min\ (min\_mc_i,\ max\_mc_i) = max\_mc_i$ and $d_i = max\ (min\_mc_i,\ max\_mc_i) = min\_mc_i$ then (Fig. 2).

$$b_i = \frac{min\_mc_i + max\_mc_i}{2} + \rho\left(min\_mc_i - \frac{min\_mc_i + max\_mc_i}{2}\right) \quad (30a)$$

$$c_i = min\_mc_i \quad (30b)$$

$$\rho \to 0 \ \leftrightarrow \ b_i \to \frac{min\_mc_i + max\_mc_i}{2} \quad (30c)$$

$$\rho \to 1 \ \leftrightarrow \ b_i \to min\_mc_i \quad (30d)$$

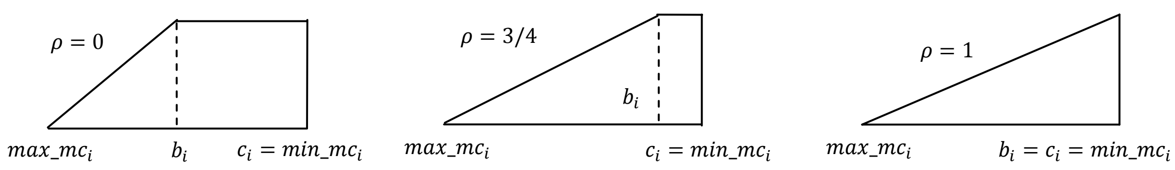


Fig. 2 Variance and kurtosis fuzzy numbers when $a_i = max\_mc_i$ and $d_i = min\_mc_i$

b) Skewness fuzzy numbers – illustrated by Figs. 3,4

$$b_i = \min\left(\frac{min\_mc_i + max\_mc_i}{2} + \rho\left(max\_mc_i - \frac{min\_mc_i + max\_mc_i}{2}\right), max\_mc_i\right) \quad (31)$$

$$c_i = \max\left(\frac{min\_mc_i + max\_mc_i}{2} + \rho\left(max\_mc_i - \frac{min\_mc_i + max\_mc_i}{2}\right), max\_mc_i\right) \quad (32)$$

i) If $a_i = min\ (min\_mc_i,\ max\_mc_i) = min\_mc_i$ *and* $d_i = max\ (min\_mc_i,\ max\_mc_i) = max\_mc_i$ *then (Fig. 3).*

$$b_i = \frac{min\_mc_i + max\_mc_i}{2} + \rho\left(max\_mc_i - \frac{min\_mc_i + max\_mc_i}{2}\right) \quad (33a)$$

$$c_i = max\_mc_i \quad (33b)$$

$$\rho \to 0 \ \leftrightarrow \ b_i \to \frac{min\_mc_i + max\_mc_i}{2} \quad (33c)$$

$$\rho \to 1 \ \leftrightarrow \ b_i \to max\_mc_i \quad (33d$$

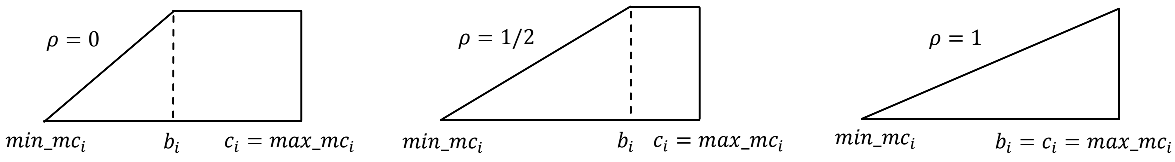


Fig. 3 Skewness fuzzy numbers when $a_i = min\_mc_i$ and $d_i = max\_mc_i$

ii) If $a_i = min\ (min\_mc_i,\ max\_mc_i) = max\_mc_i$ *and* $d_i = max\ (min\_mc_i,\ max\_mc_i) = min\_mc_i$ *then (Fig. 4).*

$$b_i = max\_mc_i \quad (34a)$$

$$c_i = \frac{min\_mc_i + max\_mc_i}{2} + \rho\left(max\_mc_i - \frac{min\_mc_i + max\_mc_i}{2}\right) \quad (34b)$$

$$\rho \to 0 \ \leftrightarrow \ c_i \to \frac{min\_mc_i + max\_mc_i}{2} \quad (34c)$$

$$\rho \to 1 \ \leftrightarrow \ c_i \to max\_mc_i \quad (34d)$$

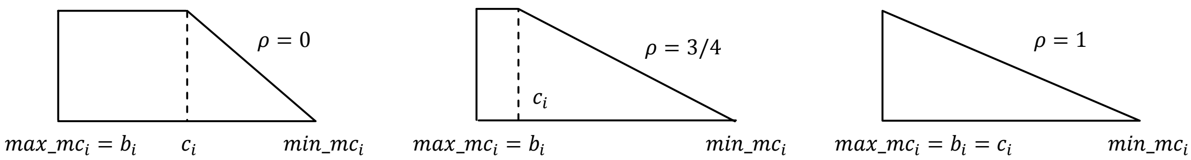


Fig. 4 Skewness fuzzy numbers when $a_i = max\_mc_i$ and $d_i = min\_mc_i$

The shapes of fuzzy numbers in case (i) and case (ii) look similar, but the position of b_i, c_i and the differences at the pole values of these fuzzy numbers are worth noting.

Given that marginal contributions of assets on the portfolio have been modelled by fuzzy numbers, during deployment of MCDM methods these numbers need to be evaluated or compared with others. The next subsection is devoted to a presentation of methods for evaluating fuzzy numbers and the method selected in this research.

C. Evaluating Fuzzy Numbers

In order to compare a fuzzy number with others, we suggest using its representative crisp number obtained via the centroid-based defuzzification method. Denote $g_A^L(y): [0, \omega] \to [a, b]$ and $g_A^R(y): [0, \omega] \to [a, d]$ are the inverse functions of $f_A^L(x)$ and $f_A^R(x)$, respectively. In the case of normal trapezoidal fuzzy number the functions $g_A^L(y)$ and $g_A^R(y)$ can be analytically expressed as $g_A^L(y) = a + (b - a)y$ and $g_A^R(y) = d + (d - c)y$ where $0 \le y \le 1$. The Wang et al. [18] centroid formulae based on the general canonical form of a normal trapezoidal fuzzy number A are as follows:

$$\bar{x}_0(A) = \frac{\int_{-\infty}^{+\infty} x f_A(x)dx}{\int_{-\infty}^{+\infty} f_A(x)dx} = \frac{\int_a^b x f_A^L(x)dx + \int_b^c x dx + \int_c^d x f_A^R(x)dx}{\int_a^b f_A^L(x)dx + \int_b^c dx + \int_c^d f_A^R(x)dx}, \quad (35)$$

$$\bar{y}_0(A) = \frac{\int_0^1 y(g_A^R(y) - g_A^L(y))dy}{\int_0^1 (g_A^R(y) - g_A^L(y))dy}, \quad (36)$$

where the numerator $\int_0^1 y(g_A^R(y) - g_A^L(y))dy$ represents the weighted average of the area, while the denominator $\int_0^1 (g_A^R(y) - g_A^L(y))dy$ is the area of the trapezoid.

For normal trapezoidal fuzzy numbers $A = [a, b, c, d]$, formulae of Wang et al. lead to:

$$\bar{x}_0(A) = \frac{1}{3}\left[a + b + c + d - \frac{dc - ab}{(d+c) - (a+b)}\right] \quad (37)$$

$$\bar{y}_0(A) = \frac{1}{3}\left[1 + \frac{c - b}{(d+c) - (a+b)}\right] \quad (38)$$

Centroids on the horizontal axis are used as a basis to evaluate assets. If horizontal coordinates $\bar{x}_0$ of all assets in the portfolio are completely equal then the vertical centroid coordinates $\bar{y}_0$ will be applied, though this situation seldom occurs in practice where the numbers of assets is large enough.

This also means that the representative location on the horizontal axis is more important than the average height in comparing fuzzy numbers [19]. The following applies this fuzzy number evaluating paradigm in MCDM applications for portfolio selection.

IV. MCDM Approaches to Portfolio Selection

In applications of the MCDM methods to portfolio allocation with higher moments, we build the decision matrix by assigning its elements with x-centroids of fuzzy numbers representing marginal impacts of assets with respect to each criterion. The decision matrix, $D = [x_{ij}]_{n\times m}$, of $n \times m$ dimension is defined as

$$D = \begin{matrix} & C_1 & C_2 & \dots & C_m \\ \begin{matrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{matrix} & \multicolumn{4}{l}{\begin{bmatrix} x_{11} & x_{12} & \dots & x_{1m} \\ x_{21} & x_{22} & \dots & x_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nm} \end{bmatrix}} \end{matrix} \tag{39}$$

where A_1, A_2, …, A_n are possible assets among which investors have to allocate their initial wealth, C_1, C_2, …, C_m are criteria with which asset performance is measured, x_{ij} is the centroid of a trapezoidal fuzzy number representing marginal impact of asset A_i with respect to criterion C_j. In the application herein, $m = 4$ with C_1, C_2, C_3, C_4 are respectively return, variance, skewness and kurtosis criteria. Note that with the return criterion, the marginal impact of an asset is its crisp expected return so that we assign asset expected return into the return column correspondingly.

A. SAW Method

SAW requires a comparable scale for all elements in the decision matrix. Let min_j and max_j be the minimum and maximum values in the jth column. The comparable scale is obtained using the following formulas to result in the normalized matrix $R = [r_{ij}]_{n\times m}$.

For benefit criteria (the larger the rating, the greater the preference), r_{ij} is

$$r_{ij} = \frac{x_{ij} - min_j}{max_j - min_j} \tag{40a}$$

For cost criteria (the smaller the rating, the greater the preference):

$$r_{ij} = \frac{max_j - x_{ij}}{max_j - min_j} \tag{40b}$$

The weight of each criterion is extracted from the normalization of the investor's preference vector $\bar{s} = (\overline{s_r}, \overline{s_v}, \overline{s_s}, \overline{s_k}) = (\overline{s_1}, \overline{s_2}, \overline{s_3}, \overline{s_4}) = \{\overline{s_j}\}$ (see Table I). The performance score p_i of the ith asset will be calculated

$$p_i = \sum_{j=1}^{m} \overline{s_j}\, r_{ij} \tag{41}$$

Normalize the vector $p = (p_1, p_2, \dots, p_n)$ consisting of preference scores of all assets to obtain the final portfolio weight: $w = \bar{c} = (\overline{p_1}, \overline{p_2}, \dots, \overline{p_n})$.

B. TOPSIS Method

The portfolio allocation problem is solved under the TOPSIS procedure via the following steps.

Step 1: Construct the normalized decision matrix denoted by $R = [r_{ij}]_{n\times m}$ based on elements of the decision matrix D (39). This process allows comparison across the attributes by transforming the various attribute dimensions into nondimensional attributes.

$$r_{ij} = \frac{x_{ij}}{\sqrt{\sum_{i=1}^{n} x_{ij}^2}} \tag{42}$$

This method works well if the data is positive or zero. If data in a column contains negative numbers, there are still negative numbers in the column after normalizing. A solution to deal with this is to shift data by adding all numbers in that column with the absolute of the most negative number (minimum value of the column) such that the most negative one will become zero and all other numbers become positive. Then we can apply the above method for normalization.

Step 2: Construct the weighted normalized fuzzy decision matrix $V = [v_{ij}]_{n\times m}$. The weight of each criterion is also extracted from the normalization of the investor's preference as in the SAW method. The element v_{ij} can be defined

$$v_{ij} = r_{ij}.\overline{s_j}, \qquad j = 1,2,\dots,m \tag{43}$$

Step 3: Identify the Positive Ideal Solution (PIS), (A^+) and Negative Ideal Solution (NIS), (A^-).

$$A^+ = \{(\max_i v_{ij} \,|j \in B), (\min_i v_{ij} \,|j \in C) | i = 1,2,\dots,n\} = \{v_1^+, v_2^+, \dots, v_m^+\} \tag{44a}$$

$$A^- = \{(\min_i v_{ij} \,|j \in B), (\max_i v_{ij} \,|j \in C) | i = 1,2,\dots,n\} = \{v_1^-, v_2^-, \dots, v_m^-\} \tag{44b}$$

where B and C are the set of benefit criteria and cost criteria respectively. Clearly, B includes $j = 1,3$ that respectively are return and skewness. In contrast, C includes $j = 2,4$ corresponding to variance and kurtosis criteria.

Step 4: Calculate the separation measure of each asset from ideal solutions A^+ and A^- by the m-dimensional Euclidean distance.

$$d_i^+ = \sqrt{\sum_{j=1}^{m} (v_{ij} - v_j^+)^2}, \qquad i = 1,2,\dots,n \tag{45a}$$

$$d_i^- = \sqrt{\sum_{j=1}^{m} (v_{ij} - v_j^-)^2}, \qquad i = 1,2,\dots,n \tag{45c}$$

Step 5: Calculate the relative closeness of each asset to the ideal solutions. The crisp closeness coefficient c_i of the asset A_i with respect to A^+ is defined as

$$c_i = \frac{d_i^-}{d_i^+ + d_i^-}, \qquad i = 1,2,\dots,n \tag{46}$$

It is obvious that $c_i = 1$ if $A_i = A^+$ and $c_i = 0$ if $A_i = A^-$. In other words, an asset A_i is closer to the PIS (A^+) and farther from NIS (A^-) as c_i approaches 1. Based on the closeness coefficients, we can determine the ranking orders of all assets and evaluate quantitatively what the proportion of an asset is, relative to other assets.

Step 6: Normalize the vector $c = (c_1, c_2, \dots, c_n)$ consisting of closeness coefficients of all assets to obtain the portfolio weight: $w = \bar{c} = (\overline{c_1}, \overline{c_2}, \dots, \overline{c_n})$.

V. Illustrative Examples

We use historical return data introduced by Markowitz [20] for experiments to assess performance of proposed approaches. This data set consists of nine stocks and has been widely used for portfolio optimization experiments. The data span the period from 1937 to 1954 with 18 yearly observations for each stock. Table II represents moments of individual stocks where the return row is the arithmetic average rate of return whereas variance, skewness and kurtosis are respectively calculated using (12)-(14) with $i = j = k = l = 1, \dots, 9$.

TABLE II
Historical Moments of Individual Stocks

Moments	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9
Return	0.0659	0.0616	0.1461	0.1734	0.1981	0.0551	0.1276	0.1903	0.1156
Variance	0.0534	0.0147	0.0855	0.0955	0.1279	0.0413	0.0288	0.1467	0.0793
Skewness	0.0051	-0.0005	0.0278	-0.0066	-0.0031	0.0003	-0.0035	0.0413	-0.0030
Kurtosis	0.0066	0.0006	0.0291	0.0249	0.0370	0.0028	0.0030	0.0603	0.0154

Prior to implementing the MCDM methods, co-variance matrix, co-skewness matrix and co-kurtosis matrix are constructed using (9), (10) and (11) respectively. Maximum and minimum extremes of portfolio higher moments are achieved using a constrained optimization solver with objective functions (16-18) and two constraints $\sum_{i=1}^{n} w_i = 1$ and $w_i \geq 0, \forall i = 1, \dots, 9$. Marginal impacts of assets on portfolio moments at the extremes are then calculated using (19)-(22).

Under each determined scheme, the parameter ρ for each scheme will be indicated (see Table I) and then fuzzy numbers representing marginal impacts are designed for each stock with respect to each criterion. Tables III, IV, and V respectively represent fuzzy numbers of marginal impacts regarding variance, skewness and kurtosis criteria.

With scheme (2:1:2:1), we have $\rho_v = 1/2$, $\rho_s = 1$ and $\rho_k = 1/2$ so that for the variance and kurtosis fuzzy numbers: $a = b$, and for skewness: $b = c = d$ (trapezoidal fuzzy numbers are reduced to triangular fuzzy numbers).

TABLE III
Variance Marginal Impact Fuzzy Numbers - Scheme 2:1:2:1, $\rho_v = 1/2$

A	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9
a	0.0450	0.0279	0.0380	0.0516	0.0248	0.0288	0.0319	0.0518	0.0410
b	0.0450	0.0279	0.0380	0.0516	0.0248	0.0288	0.0319	0.0518	0.0410
c	0.0537	0.0336	0.0628	0.0837	0.0694	0.0364	0.0385	0.1122	0.0572
d	0.0800	0.0507	0.1373	0.1800	0.2031	0.0592	0.0583	0.2935	0.1057

TABLE IV
Skewness Marginal Impact Fuzzy Numbers - Scheme 2:1:2:1, $\rho_s = 1$

A	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9
a	-0.0176	-0.0022	-0.0258	-0.0287	-0.0401	-0.0055	-0.0063	-0.0112	-0.0401
b	0.0154	-0.0014	-0.0035	-0.0039	-0.0047	0.0128	-0.0036	-0.0071	0.0052
c	0.0154	-0.0014	-0.0035	-0.0039	-0.0047	0.0128	-0.0036	-0.0071	0.0052
d	0.0154	-0.0014	-0.0035	-0.0039	-0.0047	0.0128	-0.0036	-0.0071	0.0052

TABLE V
KURTOSIS MARGINAL IMPACT FUZZY NUMBERS - SCHEME 2:1:2:1, $\rho_k = 1/2$

A	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9
a	0.0036	0.0022	0.0028	0.0037	0.0022	0.0022	0.0025	0.0043	0.0045
b	0.0036	0.0022	0.0028	0.0037	0.0022	0.0022	0.0025	0.0043	0.0045
c	0.0072	0.0076	0.0125	0.0327	0.0449	0.0090	0.0134	0.0636	0.0152
d	0.0182	0.0236	0.0415	0.1198	0.1728	0.0295	0.0463	0.2414	0.0473

For different schemes, parameters of fuzzy numbers in Tables III-V are adjusted so that their centroids are different. The decision matrix D is constructed following the format of formula (39) for the scheme (2:1:2:1) as presented in the following table (Table VI).

TABLE VI
THE DECISION MATRIX - SCHEME 2:1:2:1

D	C_1 Return	C_2 Variance	C_3 Skewness	C_4 Kurtosis
S_1	0.0659	0.0572	0.0044	0.0087
S_2	0.0616	0.0359	-0.0017	0.0097
S_3	0.1461	0.0728	-0.0109	0.0163
S_4	0.1734	0.0965	-0.0122	0.0443
S_5	0.1981	0.0872	-0.0165	0.0619
S_6	0.0551	0.0395	0.0067	0.0118
S_7	0.1276	0.0411	-0.0045	0.0178
S_8	0.1903	0.1364	-0.0085	0.0873
S_9	0.1156	0.0636	-0.0099	0.0195

A. SAW Portfolios

The normalized matrix R in SAW application is obtained using (40a) and (40b) and presented in Table VII.

Performance scores of assets for the scheme (2:1:2:1) is calculated using (41) where preference vector s is normalized $\bar{s} = (\overline{s_r}, \overline{s_v}, \overline{s_s}, \overline{s_k}) = (\bar{2}, \bar{1}, \bar{2}, \bar{1}) = (1/3, 1/6, 1/3, 1/6)$ (see Table I). In Table VIII, p is the performance scores of assets and $w = \bar{c}$ is the final portfolio weights.

TABLE VII
SAW NORMALIZED DECISION MATRIX - SCHEME 2:1:2:1

R	C_1 Return	C_2 Variance	C_3 Skewness	C_4 Kurtosis
S_1	0.0758	0.7877	0.9010	1
S_2	0.0451	1	0.6396	0.9868
S_3	0.6360	0.6332	0.2399	0.9026
S_4	0.8275	0.3966	0.1851	0.5469
S_5	1	0.4893	0	0.3228
S_6	0	0.9645	1	0.9609
S_7	0.5070	0.9481	0.5176	0.8840
S_8	0.9456	0	0.3469	0
S_9	0.4231	0.7242	0.2854	0.8627

TABLE VIII
SAW PREFERENCE SCORES AND PORTFOLIO WEIGHTS - SCHEME 2:1:2:1

2:1:2:1	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9
p	0.6235	0.5594	0.5479	0.4948	0.4687	0.6542	0.6469	0.4308	0.5006
$w = \bar{c}$	0.1266	0.1135	0.1112	0.1004	0.0951	0.1328	0.1313	0.0874	0.1016

TABLE IX
SAW PORTFOLIO WEIGHTS FOR OTHER INVESTIGATED SCHEMES

Schemes	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9
1:2:1:2	0.1355	0.1399	0.1183	0.0866	0.0790	0.1451	0.1407	0.0392	0.1156
4:3:2:1	0.1078	0.1086	0.1157	0.1074	0.1151	0.1161	0.1353	0.0895	0.1045
1:2:3:4	0.1528	0.1452	0.1143	0.0800	0.0600	0.1607	0.1375	0.0366	0.1129
1:1:0:0	0.0825	0.1005	0.1220	0.1173	0.1440	0.0928	0.1398	0.0910	0.1101
0:0:1:1	0.1796	0.1537	0.1080	0.0692	0.0306	0.1853	0.1325	0.0328	0.1084

The whole calculation procedure will be re-performed from the step of fuzzy number design for other schemes and results are assembled in Table IX.

B. TOPSIS Portfolios

Same as the SAW method, the decision matrix D in TOPSIS application will be different for different schemes. For the first investigated scheme (2:1:2:1), the matrix D will be the same as in Table VI. The normalized decision matrix R (Table X) for scheme (2:1:2:1) is constructed using (42).

TABLE X
TOPSIS NORMALIZED DECISION MATRIX - SCHEME 2:1:2:1

R	C_1 Return	C_2 Variance	C_3 Skewness	C_4 Kurtosis
S_1	0.1610	0.2494	0.5401	0.0717
S_2	0.1503	0.1564	0.3834	0.0802
S_3	0.3566	0.3170	0.1438	0.1348
S_4	0.4235	0.4206	0.1110	0.3656
S_5	0.4838	0.3800	0.0000	0.5109
S_6	0.1346	0.1720	0.5995	0.0970
S_7	0.3116	0.1792	0.3103	0.1469
S_8	0.4648	0.5942	0.2080	0.7202
S_9	0.2823	0.2772	0.1711	0.1608

The weighted normalized matrix V is constructed using (43). For scheme (2:1:2:1) the matrix V is shown in Table XI.

TABLE XI
TOPSIS WEIGHTED NORMALIZED MATRIX - SCHEME 2:1:2:1

V	C_1 Return	C_2 Variance	C_3 Skewness	C_4 Kurtosis
S_1	0.0537	0.0416	0.1800	0.0119
S_2	0.0501	0.0261	0.1278	0.0134
S_3	0.1189	0.0528	0.0479	0.0225
S_4	0.1412	0.0701	0.0370	0.0609
S_5	0.1613	0.0633	0.0000	0.0851
S_6	0.0449	0.0287	0.1998	0.0162
S_7	0.1039	0.0299	0.1034	0.0245
S_8	0.1549	0.0990	0.0693	0.1200
S_9	0.0941	0.0462	0.0570	0.0268

The PIS and NIS, A^+ and A^-, are identified using (44^I) and (44^{II}) and represented in Table XII.

TABLE XII
PIS AND NIS POINTS - SCHEME 2:1:2:1

2:1:2:1	C_1 Return	C_2 Variance	C_3 Skewness	C_4 Kurtosis
A^+	0.1613	0.0261	0.1998	0.0119
A^-	0.0449	0.0990	0.0000	0.1200

The separation measures to ideal points, A^+ and A^-, and closeness coefficients c_i are calculated using (45^I), (45^{II}), (46) respectively and tabulated in Table XIII. The final portfolio weights w for the scheme (2:1:2:1) is shown in the last row by normalizing vector c.

TABLE XIII
SEPARATION MEASURES, CLOSENESS COEFFICIENT AND PORTFOLIO WEIGHT – SCHEME 2:1:2:1

2:1:2:1	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9
d_i^+	0.1105	0.1324	0.1603	0.1768	0.2161	0.1165	0.1129	0.1846	0.1598
d_i^-	0.2179	0.1818	0.1394	0.1224	0.1266	0.2360	0.1676	0.1301	0.1310
c_i	0.6636	0.5786	0.4651	0.4090	0.3696	0.6695	0.5974	0.4134	0.4505
$w = \bar{c}$	0.1437	0.1253	0.1008	0.0886	0.0800	0.1450	0.1294	0.0895	0.0976

TABLE XIV
TOPSIS PORTFOLIO WEIGHTS FOR OTHER INVESTIGATED SCHEMES

Schemes	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9
1:2:1:2	0.1417	0.1415	0.1225	0.0841	0.0666	0.1446	0.1412	0.0349	0.1229
4:3:2:1	0.1131	0.1117	0.1128	0.1042	0.1073	0.1172	0.1342	0.0945	0.1050
1:2:3:4	0.1604	0.1459	0.1148	0.0798	0.0514	0.1631	0.1342	0.0352	0.1152
1:1:0:0	0.0966	0.1086	0.1219	0.1065	0.1256	0.1046	0.1384	0.0832	0.1144
0:0:1:1	0.1751	0.1453	0.1062	0.0739	0.0413	0.1822	0.1285	0.0404	0.1071

For the other schemes, the TOPSIS calculation steps are re-performed and portfolio weights are reported in Table XIV.

C. Comparisons between SAW and TOPSIS Portfolios

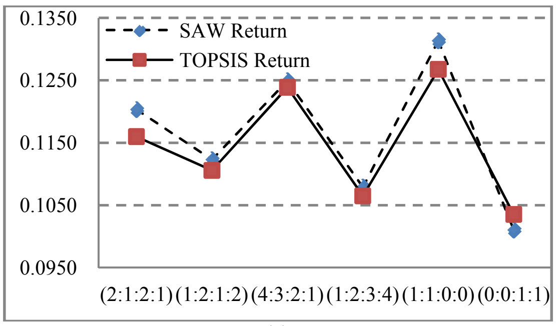
Throughout the investigated schemes, SAW and TOPSIS select similar stocks for optimal portfolios. For instance, in scheme (2:1:2:1), stocks S_1, S_6, and S_7 are dominant whereas stocks S_5, and S_8 are inferior in both methods (Tables VIII, XIII). Or else in schemes (1:2:3:4) and (0:0:1:1), both methods select S_1, S_2, S_6 and S_7 while stocks S_4, S_5, and S_8 are unfavourable. Most stocks are not very sensitive to the scheme changing but there are three highly changeable stocks S_1, S_5, and S_6 occurring in both SAW and TOPSIS. The most difference in the portfolio weights of the investigated schemes happens between the schemes (1:1:0:0) and (0:0:1:1). This is understandable since the scheme (1:1:0:0) only considers return and variance whereas in contrast the scheme (0:0:1:1) only takes skewness and kurtosis into account. It is evident that the investor's preference scheme affects the strategy by which her or his portfolio will be allocated. The scheme (1:1:0:0) is considered equivalent to the conventional mean-variance approach since it does not account for higher moments, i.e. skewness, kurtosis. The difference in portfolio weights of this scheme with those of other schemes, especially the scheme (0:0:1:1), reinforces the importance of portfolio higher moments in portfolio selection since ignoring them may lead to significantly different risky investment strategies.

Both methods result in stock weights varying similarly up and down with change of schemes. This is recognized more obviously with the changeable stocks between schemes (1:1:0:0) and (0:0:1:1). For example, stock S_1 increases from 8% in scheme (1:1:0:0) to 18% in scheme (0:0:1:1) in the SAW method and similarly in the TOPSIS method, S_1 weight increases from 10% (1:1:0:0) to 18% (0:0:1:1). Else, the stock S_5 weight decreases from scheme (1:1:0:0) to scheme (0:0:1:1) in both methods: from 14% down to 3% in SAW and from 13% to 4% in TOPSIS. Or stock S_6 proportion increases 9% to 19% in SAW and 10% to 18% in TOPSIS.

TABLE XV
SAW AND TOPSIS PORTFOLIO HIGHER MOMENTS OF DIFFERENT SCHEMES

Schemes	Return		Variance		Skewness		Kurtosis	
	SAW	TOPSIS	SAW	TOPSIS	SAW	TOPSIS	SAW	TOPSIS
2:1:2:1	0.1203	0.1160	0.0373	0.0359	-0.0027	-0.0024	0.0029	0.0027
1:2:1:2	0.1123	0.1106	0.0327	0.0323	-0.0022	-0.0021	0.0025	0.0024
4:3:2:1	0.1249	0.1239	0.0388	0.0386	-0.0029	-0.0029	0.0032	0.0032
1:2:3:4	0.1079	0.1065	0.0317	0.0316	-0.0019	-0.0018	0.0023	0.0023
1:1:0:0	0.1313	0.1267	0.0412	0.0391	-0.0033	-0.0031	0.0038	0.0034
0:0:1:1	0.1010	0.1035	0.0305	0.0314	-0.0014	-0.0016	0.0021	0.0022

Table XV reports the higher moments of SAW and TOPSIS portfolios. Clearly from both methods, when investors favour a particular criterion more, the values of that criterion become more optimal in its direction. For instance, the scheme (0:0:1:1) most concerns skewness and kurtosis, and the portfolio derived from this scheme has the highest skewness at -0.0014 in SAW (or -0.0016 in TOPSIS) and lowest kurtosis at 0.0021 in both SAW and TOPSIS compared to other schemes. In contrast, the scheme (1:1:0:0) implies no concern about skewness and kurtosis and as a result the corresponding portfolio has the lowest skewness: -0.0033 in SAW and -0.0033 in TOPSIS and highest kurtosis: 0.0038 in SAW and 0.0034 in TOPSIS. Since scheme (4:3:2:1) concerns more about return than scheme (1:2:3:4) then the return of SAW portfolio derived from scheme (4:3:2:1) at 0.1249 is higher than that of scheme (1:2:3:4) at 0.1079. The same situation happens in TOPSIS portfolios: returns decrease from scheme (4:3:2:1) to scheme (1:2:3:4) with respective values: 0.1239 and 0.1065. Likewise, portfolio kurtosis of scheme (1:2:3:4) is lower than that of scheme (4:3:2:1): 0.0023 and 0.0032 respectively in both SAW and TOPSIS because investors pay more attention to kurtosis in scheme (1:2:3:4) than in scheme (4:3:2:1). Generally, the changes of portfolio higher moments by scheme changing demonstrate the comparable performance of SAW and TOPSIS concerning the ability to handle investor's higher moment risk preferences.



(a)

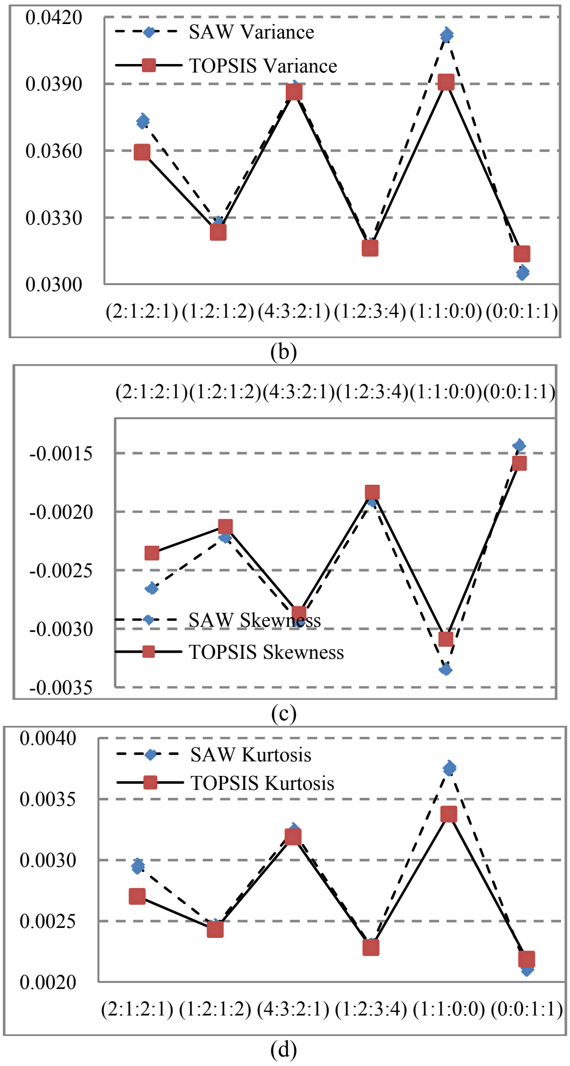


Fig. 5 SAW and TOPSIS portfolio return (a), variance (b), skewness (c) and kurtosis (d)

Fig. 5 graphically depicts comparable performance between SAW and TOPSIS. Within each graph, values of portfolio higher moments by scheme changing in SAW and TOPSIS change (up and down) very closely. Variations of return, variance and kurtosis are analogous and contrasting with the changes of skewness. There are, as always, trade-offs within optimal criteria between two methods. Returns of SAW portfolios are higher (better) than that of TOPSIS portfolios in most schemes except scheme (0:0:1:1). However, SAW portfolio variance and kurtosis are higher (worse) than TOPSIS portfolio variance and kurtosis. Correspondingly SAW portfolio skewness values are worse (lower) than that of TOPSIS except in scheme (0:0:1:1).

D. MVO Portfolios

To obtain the MVO efficient portfolios, we solve the following quadratic optimization problems:

$\min_w \sigma_p^2$, subject to

$$\sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij}, \qquad \mu_p = \sum_{i=1}^{n} w_i \mu_i = \mu^*, \quad (47)$$

$$\sum_{i=1}^{n} w_i = 1 \text{ and } w_i \geq 0,\ \forall i = 1, \dots, n$$

where μ^* is the expected portfolio return. An alternative formulation, maximize return for fixed variances, could result in the same interpretation. Table XVI details some typical efficient portfolio weights corresponding to the increasing return and variance of the portfolio.

The similarity between MVO and MCDM approaches, i.e., SAW and TOPSIS, concerning dominance of some stocks in optimal portfolios is recognized. For instance, stocks S_3 and S_7 are superior in MVO and similarly in these two MCDM methods. Likewise, stock S_8 is the most inferior in both MVO and MCDM approaches. This implies the analogous efficiency of the MCDM applications and MVO in selecting dominant assets in a large set for portfolio optimization.

TABLE XVI
SOME EFFICIENT MVO PORTFOLIO WEIGHTS

Return	Variance	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9
0.1000	0.0171	0	0.5488	0	0	0.1225	0	0.3287	0	0
0.1242	0.0224	0	0.2254	0.0630	0	0.1456	0	0.5660	0	0
0.1483	0.0315	0	0	0.0914	0	0.2698	0	0.6388	0	0
0.1900	0.0925	0	0	0.0010	0	0.6061	0	0.0798	0.3132	0
0.1981	0.1279	0	0	0	0	1	0	0	0	0

Stock S_5 is highly selected in MVO but modestly chosen in SAW and TOPSIS. However, with the MVO equivalent scheme (1:1:0:0) in both SAW and TOPSIS, stock S_5 is the most selected with around 13-14% compared to its weights in other schemes (Tables IX, XIV). Stock S_1 and S_6 are absolutely rejected in MVO but selected in SAW and TOPSIS. Proportions of these two stocks in the MVO equivalent scheme (1:1:0:0) are the least with around 8-10% compared to their weights in other schemes. This result emphasises the analogy of MVO and MCDM methods regarding the capacity of selecting and rejecting assets.

Except for the aforementioned similarity, the difference between MCDM methods and MVO and also the advantage of MCDM methods is the availability of the well diversified characteristic in its optimal portfolio weights. The diversification effect is inherent in applications of MCDM methods although they are deployed in various preference schemes. Conversely, the MVO optimal portfolios are poorly diversified since they are systematically concentrated on a few assets in contradiction with the ideal of diversification. Practitioners are aware that it is unwise to discard a large number of assets because covariance and average return are unstable over time, especially in a dynamic economic environment. This poorly diversified feature of MVO is demonstrated in this experiment: MVO selects only three to four out of nine experimental stocks (see Table XVI). MCDM portfolio selection results in the advantage of a well-diversified selection.

VI. Conclusion

Applying fuzzy numbers to represent marginal impacts is an appealing idea since it is impossible to indicate which stocks are more important than the others under the higher moment context at the time of conducting the portfolio allocation. Investor's preference regarding portfolio higher moment risks has been handled well by trapezoidal fuzzy number modelling of marginal impacts. Fuzzy numbers in particular or fuzzy logic in general provide an extremely helpful means to represent uncertainty and human knowledge. Besides, centroid based defuzzification method applied for fuzzy factors (variance, skewness and kurtosis) facilitates the integration between crisp factor (return) and fuzzy factors into MCDM calculation frameworks where none of the factors (originally in different scales) may distort the others.

SAW and TOPSIS result in similar performance regarding portfolio higher moment optimization. They are also analogous to MVO in terms of ability to select superior stocks over various risk preference schemes. On the other hand, both SAW and TOPSIS have the advantage of well-diversified feature in portfolio weights compared to MVO.

Along with higher moments, other quantitative risk measures, e.g. liquidity risk, value at risk, expected shortfall, etc. are also acknowledged as important in risky investment management. Further interesting research would extend to incorporate these kinds of risks pertaining to investigation of other MCDM methods such as ELECTRE, VIKOR or AHP.

References

[1] H. M. Markowitz, "Portfolio selection," *Journal of Finance*, vol. 7, pp. 77-91, 1952.
[2] C. L. Hwang and K. Yoon, *Multiple attribute decision making: Methods and* applications. Heidelberg: Springer-Verlag, 1981.
[3] S. H. Zanakis, A. Solomon, N. Wishart, and S. Dublish, "Multi-attribute decision making: A simulation comparison of select methods," *European Journal of Operational Research*, vol. 107, pp. 507-529, 1998.
[4] Y. H. Chang and C. H. Yeh, "Evaluating airline competitiveness using multiattribute decision making," *Omega* , vol. 29, pp. 405-415, 2001.
[5] M. Modarres and S. Sadi-Nezhad, "Fuzzy simple additive weighting method by preference ratio" *Intelligent Automation and Soft Computing*, vol. 11, pp. 235-244, 2005.
[6] S. Y. Chou, Y. H. Chang, and C. Y. Shen, "A fuzzy simple additive weighting system under group decision-making for facility location selection with objective/subjective attributes," *European Journal of Operational Research*, vol. 189, pp. 132-145, 2008.
[7] E. Bottani and A. Rizzi, "A fuzzy TOPSIS methodology to support outsourcing of logistics services," *Supply Chain Management*, vol. 11, pp. 294-308, 2006.
[8] H. S. Shih, H. J. Shyur, and E. S. Lee, "An extension of TOPSIS for group decision making," *Mathematical and Computer Modelling*, vol. 45, pp. 801-813, 2007.
[9] M. A. Abo-Sinna and T. H. Abou-El-Enien, "An interactive algorithm for large scale multiple objective programming problems with fuzzy parameters through TOPSIS approach," *Applied Mathematics and Computation*, vol. 177, pp. 515-527, 2006.
[10] G. R. Jahanshahloo, F. H. Lotfi, and R. A. Davoodi, "Extension of TOPSIS for decision-making problems with interval data: Interval efficiency," *Mathematical and Computer Modelling*, vol. 49, pp. 1137-1142, 2009.
[11] Y. Chen, D. M. Kilgour, and K. W. Hipel, "An extreme-distance approach to multiple criteria ranking," *Mathematical and Computer Modelling*, vol. 53, pp. 646-658, 2011.
[12] A. Awasthi, S. S. Chauhan, and S. K. Goyal, "A multi-criteria decision making approach for location planning for urban distribution centers under uncertainty," *Mathematical and Computer Modelling*, vol. 53, pp. 98-109, 2011.
[13] E. Jondeau and M. Rockinger, "Optimal portfolio allocation under higher moments," *European Financial Mangement*, vol. 12, pp. 29-55, 2006.
[14] L. Martellini and V. Ziemann, "Improved estimates of higher-order comoments and implications for portfolio selection," *Review of Financial Studies*, vol. 23, pp. 1467-1502, 2010.
[15] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, pp. 338-353, 1965.
[16] R. C. Scott and P. A. Horvath, "On the direction of preference for moments of higher order than the variance," *Journal of Finance*, vol. 35, pp. 915-919, 1980.
[17] T. T. Nguyen and L. N. Gordon-Brown, "Constrained fuzzy hierarchical analysis for portfolio selection under higher moments," *IEEE Transactions on Fuzzy Systems*, vol. 20, no. 4, pp. 666-682, 2012.
[18] Y. M. Wang, J. B.Yang, D. L. Xu, and K. S. Chin, "On the centroids of fuzzy numbers," *Fuzzy Sets and Systems*, vol. 157, pp. 919-926, 2006.
[19] Y. J. Wang and H. S. Lee, "The revised method of ranking fuzzy numbers with an area between the centroid and original points," *Computers and Mathematics with Applications*, vol. 55, pp. 2033-2042, 2008.
[20] H. M. Markowitz, *Portfolio selection: Efficient diversification of investments,* New York: Wiley, 1959.