On a new numerical analysis for the symmetric shortest queue problem

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Abstract—We consider a network of two M/M/1 parallel queues having the same poissonian arrival stream with rate $\lambda$. Upon his arrival to the system a customer heads to the shortest queue and stays until being served. If the two queues have the same length, an arriving customer chooses one of the two queues with the same probability. Each duration of service in the two queues is an exponential random variable with rate $\mu$ and no jockeying is permitted between the two queues. A new numerical method, based on linear programming and convex optimization, is performed for the computation of the steady state solution of the system.

Keywords—steady state solution, matrix formulation, convex set, shortest queue, linear programming.

I. INTRODUCTION

In this paper we consider two identical M/M/1 queues in parallel with a JSQ (joining the shortest queue) policy. The customers arrive to the system in accordance with a Poisson process of rate $\lambda$. Upon his arrival to the system, the customer selects the shortest queue and stays in the queue until being served. If the two queues have the same length, the customer chooses one of the two queues with the same probability. The duration of each service is an exponential random variable with rate $\mu$. No jockeying is permitted between the two queues. The goal of the present paper is the numerical computation of the steady state probability $p(i,j)$ where $i$ is the number of customers in queue 1 and $j$ the number of customers in queue 2.

The shortest queue problem was initially proposed by Haight ([7]). A successful model for the symmetric case was given when the jockeying between the two queues is allowed. Kingman ([7]) obtained some asymptotic results for the joint steady state distribution of the number of customers in the two queues. Flatto and McKean ([3]) used generating function techniques to give some limiting properties for the steady state probability. A numerical approach, using matrix geometric techniques to give some limiting properties for the steady state probability of the total number of customers in the system. Zhao and Grassmann ([13]) used the Flatto and Mckean ([3]) technique was developed by Gertsbakh ([4]). Cohen and Boxma ([2]) showed that the analysis of the shortest queue problem can be reduced to the solution of a boundary value problem.

II. STEADY STATE ANALYSIS OF THE SYSTEM

We consider the two dimensional stochastic process $(X_t, Y_t)$, where $X_t$ (resp. $Y_t$) is the number of customers in the queue 1 (resp. queue 2) at time $t$. The process $(X_t, Y_t)$ is a recurrent positive Markov process if and only if $\rho = \frac{\lambda}{\mu} < 2$ (Kingman 1961). For $(i,j) \in E = \mathbb{N} \times \mathbb{N}$, let $p(i,j) = \lim P(X_t = i, Y_t = j)$ the steady state probability of the process. The corresponding generator matrix is denoted by $Q = (q(e,e'))_{(e,e') \in \mathbb{N} \times \mathbb{N}}$.

Notation 1: For $k \geq 1$ we define the following $(k+1) \times 1$ vectors and $(k+1) \times (k+1)$ matrices:

\[
X_{2k} = \begin{pmatrix}
\frac{p(2k,0)}{p(2k-1,1)} \\
\vdots \\
\frac{p(k+1, k-1)}{p(k, k)}
\end{pmatrix}
\]

\[
X_{2k+1} = \begin{pmatrix}
\frac{p(2k+1,0)}{p(2k,1)} \\
\vdots \\
\frac{p(k+2, k-1)}{p(k+1, k)}
\end{pmatrix}
\]

\[
A_{2k} = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
\end{pmatrix}
\]

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\[ A_{2k+1} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} \]

**Notation 2:** The \((k + 1) \times 1\) vectors \(B_{2k}\) and \(B_{2k+1}\) are also defined as follows:

For \(k = 1\)

\[ B_2 = \begin{pmatrix} (1 + \rho) \frac{\rho}{2} - \frac{\rho}{2} & p(0,0) \end{pmatrix} \]

For \(k \geq 2\)

\[ B_{2k} = \begin{pmatrix} (1 + \rho) p(2k - 1, 0) \\ (2 + \rho) p(2k - 1, 1) - pp(2k - 2, 0) \\ (2 + \rho) p(2k - 2, 2) - pp(2k - 2, 1) \\ \vdots \\ (2 + \rho) p(k + 1, k - 2) - pp(k + 1, k - 3) \\ (2 + \rho) p(k, k - 1) - pp(k, k - 2) - \alpha_k \\ \end{pmatrix} \]

where \(\alpha_k = \frac{\rho}{2} p(k - 1, k - 1)\)

For \(k \geq 1\)

\[ B_{2k+1} = \begin{pmatrix} (1 + \rho) p(2k, 0) \\ (2 + \rho) p(2k - 1, 1) - pp(2k - 2, 0) \\ (2 + \rho) p(2k - 2, 2) - pp(2k - 2, 1) \\ \vdots \\ (2 + \rho) p(k + 1, k - 1) - pp(k + 1, k - 2) \\ (2 + \rho) p(k, k) - 2\rho p(k, k - 1) \end{pmatrix} \]

**Theorem 3:** The steady state system of balance equations can be written as follows:

\[ A_{2k} X_{2k} = B_{2k} \quad \text{and} \quad A_{2k+1} X_{2k+1} = B_{2k+1} \quad \text{for} \quad k \geq 1 \]

**Proof:** For all \(n \geq 1\), as showed in the transition diagram in figure 1; the system of balance equations expresses the probabilities \(\{p(i,j), i + j = n + 1\}\) in terms of the probabilities \(\{p(i,j), i + j = n\}\) and \(\{p(i,j), i + j = n - 1\}\). So we distinguish two cases:

**Case 1:** \(n = 2k \quad k \geq 2\)

Because of the symmetry, the system of balance equations is reduced to:

\((\lambda + \mu) p(2k - 1, 0) = \mu p(2k, 0) + \mu p(2k - 1, 1)\)

for \(0 < i < k - 1\):

\((\lambda + 2\mu) p(2k - 1 - i, i) = \lambda p(2k - 1 - i, i - 1) + \mu p(2k - 1 - i, i + 1)\)

\((\lambda + 2\mu) p(2k - 1 - i, i - 1) + \mu p(2k - i, i) + \mu p(2k - 1 - i, i + 1)\)

\((\lambda + \mu) p(2k, 0) + p(2k - 1, 1) = (1 + \rho) p(2k - 1, 0)\) (1)

**Case 2:** \(n = 2k + 1 \quad k \geq 1\)

Again, because of the symmetry, the system of balance equations can be written as:

\((\lambda + 2\mu) p(k, k) = 2\lambda p(k, k - 1) + 2\mu p(k + 1, k)\) (2)

for \(0 < i < k\):

\((\lambda + 2\mu) p(k + i, k - i) = \lambda p(k + i, k - i - 1) + \mu p(k + i, k - i + 1)\)

\((\lambda + \mu) p(2k, 0) = \mu p(2k + 1, 0) + \mu p(2k, 1)\) (3)

or:

\(2p(k + 1, k) = (2 + \rho) p(k, k) - 2\rho p(k, k - 1)\) (4)
for $0 < i < k$

$$p(k + i + 1, k - i) + p(k + i, k - i + 1) = (2 + \rho) p(k + i, k - i) - \rho p(k + i, k - i - 1)$$

and

$$p(2k - 1, 1) + p(2k, 0) = (1 + \rho) p(2k, 0)$$ (5)

Which is the formulation: $A_{2k+1}X_{2k+1} = B_{2k+1}$

**Proposition 4:** For $j = 1, 2, \ldots, k + 1$ the $j^{th}$ component of the vectors $X_{2k}$ (resp. $X_{2k+1}$) is of the form:

$$y_{2k}^{(n)} = \begin{pmatrix}
\frac{r_n(2k, 0)}{r_n(2k - 1, 1)} \\
\frac{r_n(2k - 1, 1)}{r_n(2k - 2, 0)} \\
\vdots \\
\frac{r_n(k + 1, k - 1)}{r_n(k, k)} \\
\frac{r_n(k, k)}{r_n(k + 2, k - 1)} \\
\end{pmatrix}$$

$$y_{2k+1}^{(n)} = \begin{pmatrix}
\frac{r_n(2k + 1, 0)}{r_n(2k, 1)} \\
\frac{r_n(2k, 1)}{r_n(2k - 1, 0)} \\
\vdots \\
\frac{r_n(k + 2, k - 1)}{r_n(k + 1, k)} \\
\end{pmatrix}$$

$$D_2^{(n)} = \left( \frac{(1 + \rho) \frac{r_n(2k + 1, 0)}{r_n(2k, 1)}}{r_n(1, 1)} \right)$$

For $k \geq 2$ and $2k \leq n$

$$D_{2k}^{(n)} = \begin{pmatrix}
(1 + \rho) r_n(2k - 1, 0) \\
(2 + \rho) r_n(2k - 2, 1) - \rho r_n(2k - 2, 0) \\
(2 + \rho) r_n(2k - 3, 2) - \rho r_n(2k - 3, 1) \\
\vdots \\
(2 + \rho) r_n(1, k - 1) - \rho r_n(1, k - 2) - \beta_k \\
\end{pmatrix}$$

where $\beta_k = \frac{\rho}{r_n} r_n(k - 1, k - 1)$

For $k \geq 1$ and $2k + 1 \leq n$

$$D_{2k+1}^{(n)} = \begin{pmatrix}
(1 + \rho) r_n(2k + 1, 0) \\
(2 + \rho) r_n(2k, 1) - \rho r_n(2k, 0) \\
(2 + \rho) r_n(2k - 1, 2) - \rho r_n(2k - 1, 1) \\
\vdots \\
(2 + \rho) r_n(1, k) - \rho r_n(1, k - 1) \end{pmatrix}$$

The vectors $Y_{2k}^{(n)}$ and $Y_{2k+1}^{(n)}$ are then defined by the recursive formulas:

$$A_{2k+1}Y_{2k+1}^{(n)} = D_{2k}^{(n)}$$

We also have: the $(k + 1)$ components of the vectors $Y_{2k}^{(n)}$ (resp $Y_{2k+1}^{(n)}$) are of the form:

$$\begin{pmatrix}
\sum_{i = 1}^{k} a_{i,j}^{(2k)} y_i^{(n)} + a_{0,j}^{(2k)} y_0^{(n)} \\
\sum_{i = 1}^{k} a_{i,j}^{(2k+1)} y_i^{(n)} + a_{0,j}^{(2k+1)} y_0^{(n)} \\
\end{pmatrix}$$

where $a_{i,j}^{(2k)}$, $a_{i,j}^{(2k+1)} \in \mathbb{R}$, $1 \leq j \leq (k + 1)$

**Proposition 7:** For $n \geq 2$, let the $\mathbb{R} \left[ \frac{1}{2} \right]$ sub set $C_n$ defined as follows:

$$C_n = \left\{ \left( y_1, y_2, \ldots, y_{\left[ \frac{n}{2} \right]} \right) \mid y_i > 0 \text{ for } i = 1, 2, \ldots, \left[ \frac{n}{2} \right] , r(i, j) > 0 \text{ if } i + j = n \right\}$$
then for \( Y^{(n)} = \left( y_1^{(n)}, y_2^{(n)}, \ldots, y_{\frac{n}{2}}^{(n)} \right) \in C_n \); the set of real numbers \( \{ r_{\frac{n}{2}}(i, j); i + j \leq n \} \) (obtained in terms of components of the vector \( Y^{(n)} \) in \( C_n \)) is a positive solution for the system of equilibrium equations (6).

**Proof:** \( C_n \) is nonempty while it contains the vector \( (p(1,1), p(2,2), \ldots, p(\left[ \frac{n}{2} \right], \left[ \frac{n}{2} \right])) \) up to a multiplicative factor and if \( y_{\frac{n}{2}}^{(n)} \) = \( \frac{p(i,i)}{p(0,0)} \) for \( 1 \leq i \leq \left[ \frac{n}{2} \right] \) and \( y_0^{(n)} = p(0,0) \) the corresponding \( r_{\frac{n}{2}}(i, j), i + j = n \) are exactly \( p(i,j); i + j = n \).

The condition \( r_{\frac{n}{2}}(i, j) > 0 \) if \( i + j = n \) gives for those real numbers the properties of the steady-state probabilities \( p(i,j); i + j = n \). Then \( \{ r_{\frac{n}{2}}(i, j); i + j \leq n - 1 \} \) (from which we build up \( \{ r_{\frac{n}{2}}(i, j); i + j = n \} \) over the system of equilibrium equations of (6); with \( y_{\frac{n}{2}}^{(n)} \) playing the role of \( p(i,i), 1 \leq i \leq \left[ \frac{n}{2} \right] \) have the same properties than \( \{ p(i,j); i + j \leq n - 1 \} \). So if \( y_0^{(n)} > 0 \) and \( (\left[ \frac{n}{2} \right], \left[ \frac{n}{2} \right], \ldots, y_{\left[ \frac{n}{2} \right]}^{(n)}) \in C_n \) the built system of real numbers \( \{ r_{\frac{n}{2}}(i, j); i + j \leq n \} \) is a positive solution for the equilibrium equations (6).

**Corollary 8:** If we note:

\[
S_n = \left\{ \left( y_1(n), y_2(n), \ldots, y_{\frac{n}{2}}(n) \right) \right\} \quad \text{so that } y_i > 0 \quad \forall 1 \leq i \leq \frac{n}{2}, \quad r(i,i) > 0 \quad \text{for } i + j = n
\]

then \( (S_n)_n \) is a decreasing sequence of sets \( (S_n \subset S_{n-1}) \) and the limit \( \cap S_n \) is so that \( \lim_{n \to \infty} \left( y_1(n) \right)_{i=1}^{\frac{n}{2}} \) is exactly the entire diagonal probabilities \( p(i,i) \geq 1 \).

**Proof:** Let \( \left( y_1^{(n)}, y_2^{(n)}, \ldots, y_{\frac{n}{2}}^{(n)} \right) \in S_n \), we note first that the components \( y_i^{(n)} \) for \( i > \left[ \frac{n}{2} \right] \) are free from the constraints \( r_{\frac{n}{2}}(i, j) > 0 \) for \( i + j = n \) then those components are identical for the two sets \( S_n \) and \( S_{n-1} \). While the components \( y_1^{(n)}, y_2^{(n)}, \ldots, y_{\left[ \frac{n}{2} \right]}^{(n)} \) have also to fulfill the constraints \( \{ r_{\frac{n}{2}}(i, j) > 0; i + j = n - 1 \} \) (previous proposition) then \( S_n \subset S_{n-1} \). So, when \( n \) goes to the infinity and due to the unicity of the positive solution for the infinite linear system of balance equations (the related Markov process is ergodic), the set \( S_n \) has for limit the single point of \( \mathbb{R}^n \) which is the diagonal probabilities \( p(i,i) \geq 1 \) up to a multiplicative factor.

**Remark 9:** All element of \( C_n \) can be viewed as an approximation of the vector \( (p(1,1), \ldots, p\left( \left[ \frac{n}{2} \right], \left[ \frac{n}{2} \right] \right)) \) up to a multiplicative factor.

**Remark 10:** While for all \( n \), \( p(i,i) \); \( i + j = n \) is expressed in terms of \( p(0,0), p(1,1), p(2,2), \ldots, p\left( \left[ \frac{n}{2} \right], \left[ \frac{n}{2} \right] \right) \), then the first step of computation is to get an approximation of those last probabilities by \( y_0^{(n)}, y_1^{(n)}, \ldots, y_{\frac{n}{2}}^{(n)} \) respectively, \( y_0^{(n)} \) is a multiplicative factor and \( y_1^{(n)}, \ldots, y_{\frac{n}{2}}^{(n)} \) are computed under the constraints given in the definition of \( C_n \). We then choose \( n \) sufficiently large in order to get a good approximation. In other words \( y_1^{(n)}, y_2^{(n)}, \ldots, y_{\frac{n}{2}}^{(n)} \) are almost the diagonal probabilities \( p(1,1), \ldots, p(n,n) \). In order to show the geometric behavior of the set \( C_n \) in the plane we choose \( \rho = 1.5 \) and we sketch \( C_4, C_5 \) and the projection of \( C_6 \) on the plane containing the first two coordinates \((y_1, y_2)\) (Figure 2).

**III. COMPUTATION METHODOLOGY**

In practice we need a finite number of the the probabilities \( \{ p(i,i) \} \). The sequence \( \{ p(i,i) \} \) is decreasing after some integer \( K \) (see [13]) and have the limit 0 (see Figure 3). The problem is then reduced to evaluate an integer \( N \) large enough so that the sum \( \sum_{n=N}^{\infty} \sum_{i+j=n} p(i,j) \) is very close to 1. To do this, we use the bounds for \( \sum_{n=N}^{\infty} \sum_{i+j=n} p(i,j) \) given in Halfin (6)). We then have:

\[
\sigma = \exp\frac{-\rho}{\rho^2+\rho+1} \quad \text{and:}
\]

\[
\sqrt{\sigma} \leq \rho < 2, \quad N \geq \sigma + 1 \quad \sum_{n=N}^{\infty} \pi_n \leq \left( \frac{\rho}{2} \right)^{N-\sigma}
\]

\[
1 \leq \rho < \sqrt{\sigma}, \quad N \geq 2 \quad \sum_{n=N}^{\infty} \pi_n \leq \left( \frac{\rho}{2} \right)^{N} \frac{2 + \rho}{1 + \rho}
\]

For \( 0 < \rho < 1 \), we have:

\[
\pi_n \leq \left( \frac{\rho}{2} \right)^n \frac{1}{2(1 + \rho)} \left( 2 + \rho - \rho^2 \frac{(1 + \rho)^n}{(1 + \rho)^n - 1} \right)
\]

\[
+ \frac{\rho^2}{2(1 + \rho)} \left( 1 + \rho \right)^{n-1}
\]
which implies that:
\[ \pi_n \leq \left( \frac{\rho}{2} \right)^n + \frac{1}{(1 + \rho)^n} \]

so \( \sum_{n=N}^{\infty} \pi_n \leq \left( \frac{\rho}{2} \right)^N + \frac{1}{(1 + \rho)^N} \frac{1}{\rho} \)

This last bound is interesting if \( \rho \) is close to 0. In this case, we consider the intensity flows for a Markov process in equilibrium as invoked in Halfin \[6\] and having the form:
\[ \lambda \pi_{n+1} = \mu \pi_n + 2 \mu \pi_n^+ \]

where \( \pi_n = 2p(n, 0) \) and \( \pi_n^+ = \sum_{i+j=n, j>0} p(i, j) \) (\( \pi_n = \pi_n + \pi_n^+ \)), we get:
\[ \pi_n = \frac{\lambda}{\mu \pi_n + 2 \mu \pi_n^+} \pi_{n-1} = \frac{\lambda}{\mu (1 + \frac{\pi_n}{\pi_n^+})} \pi_{n-1} \]

so, \( \pi_n \leq \rho^n \) which is more interesting if \( 0 < \rho < 1 \), with a special look for \( \rho \) close to 0. Thus:
\[ \sum_{n=N}^{\infty} \pi_n \leq \frac{\rho^N}{1 - \rho}. \]

So, for a given precision \( \epsilon \), the computation of the integer \( N \) satisfying \( \sum_{n=N+1}^{\infty} \pi_n \leq \epsilon \) can be done easily. As an example for \( \rho = 1 \) and \( \epsilon = 10^{-10} \) we find \( N = 34 \). We then need \( p(i, i) 1 \leq i \leq 17 \). The determination of \( N \) can also be regarded as the stopping rule for the computation algorithm described below. We note that the upper bound \( \epsilon \) of the error is made on the total sum of probabilities. We see further in the numerical results that \( \epsilon \) is around the upper bound of the error made between the computed value of \( p(i, i) \) and \( p(i, i) \) itself.

![Fig. 3. Evolution of the diagonal probabilities](image)

IV. Computation of the Steady State Probabilities and Numerical Results

A. Computation of \( y_i^{(N)} \) \( 1 \leq i \leq \left\lfloor \frac{N}{2} \right\rfloor \)

We recall that the integer \( N \) is so that all the probabilities \( \{p(i, j)/i + j \geq N + 1\} \), having a total sum smaller than a given \( \epsilon \), are neglected.

While the \( \left\lfloor \frac{N}{2} \right\rfloor + 1 \) components of the vector \( Y_0^{(N)} \) are of the form:
\[ \left( \sum_{i=1}^{\frac{N}{2}} \alpha_{i,j} y_i^{(N)} \right) y_0^{(N)}; 1 \leq j \leq \left\lfloor \frac{N}{2} \right\rfloor + 1 \]

then the unknowns \( \left\{ y_i^{(N)}; 1 \leq i \leq \left\lfloor \frac{N}{2} \right\rfloor \right\} \) are in the set solution of the system of linear inequalities:
\[ \sum_{i=1}^{\frac{N}{2}} \alpha_{i,j} y_i^{(N)} + \alpha_{0,j} > 0 \text{ and } y_j^{(N)} > 0 \]

for \( 1 \leq j \leq \left\lfloor \frac{N}{2} \right\rfloor + 1 \) and \( 1 \leq i \leq \left\lfloor \frac{N}{2} \right\rfloor \).

We use the simplex algorithm to obtain a lower and an upper bound for each \( y_i^{(N)} \) noted respectively \( y_{i,min}^{(N)} \) and \( y_{i,max}^{(N)} \). Because of the unicity due to the ergodicity of the related Markov process, those bounds are almost equal for a large value of \( N \), as to be seen in further computation (see table 1).

While \( y_i^{(N)} > 0 \), we can take the objective function \( \sum_{i=1}^{\frac{N}{2}} y_i^{(N)} \) (or any linear function of \( \left\{ y_i^{(N)}; 1 \leq i \leq \left\lfloor \frac{N}{2} \right\rfloor \right\} \) and the constraints (7). While \( \sum p(i, i) = \frac{1}{1 + \rho} \) (see [6]), we then compute \( y_0^{(N)} \) as a constant of normalisation by:
\[ y_0^{(N)} \left( 1 + \sum_{i=1}^{\frac{N}{2}} y_i^{(N)} \right) = \frac{1}{1 + \rho} \quad \text{or} \quad y_0^{(N)} = \frac{1}{\left( 1 + \rho \right) \left( 1 + \sum_{i=1}^{\frac{N}{2}} y_i^{(N)} \right)} \]

where \( \left( y_i^{(N)} \right)_{1 \leq i \leq \left\lfloor \frac{N}{2} \right\rfloor} \) is any solution for (7). We can take the arithmetic mean: \( y_i^{(N)} = \frac{1}{2} y_{i,min}^{(N)} + y_{i,max}^{(N)} \).

B. Algorithm description

Step 1: set up a value of \( \rho \).
Step 2: set up a precision \( \epsilon \) and compute the corresponding value of \( N \).
Step 3: put \( y_i^{(N)} y_0^{(N)} = r_N(i, i) 1 \leq i \leq \left\lfloor \frac{N}{2} \right\rfloor \) and get (by using a formal calculus) the \( \left\lfloor \frac{N}{2} \right\rfloor + 1 \) components of the vector \( Y_0^{(N)} \) denoted by \( \sum_{i=1}^{\frac{N}{2}} \alpha_{i,j} y_i^{(N)} + \alpha_{0,j} \) \( 1 \leq j \leq \left\lfloor \frac{N}{2} \right\rfloor + 1 \).
Step 4: use the simplex method with objective function

\[ y_i^{(N)} \geq 0 \quad 1 \leq i \leq \left\lceil \frac{N}{2} \right\rceil. \]

Step 5: get a lower and an upper bound for \( y_i^{(N)} \) denoted respectively \( y_{i,\text{min}}^{(N)} \) and \( y_{i,\text{max}}^{(N)} \) and put

\[ y_i = \frac{1}{2} \left( y_{i,\text{max}}^{(N)} + y_{i,\text{min}}^{(N)} \right). \]

So the set solution is a convex set denoted \( C_N' \).

Step 6: return to the matrix formulation for the computation of \( p(i,j) \) for \( i + j \leq N \).

### C. Numerical results and error analysis

**Proposition 11:** If we note \( \epsilon_0 \) and \( \epsilon_1 \) the error computation made on \( p(0,0) \) and \( p(i,j) \) respectively then:

\[ \epsilon_0 \leq \frac{1}{\rho} \left( 1 + \sum_{i=1}^{\left\lceil \frac{N}{2} \right\rceil} y_{i,\text{max}}^{(N)} \right)^2 = \frac{\epsilon (1 + \sum_{i=1}^{\left\lceil \frac{N}{2} \right\rceil} y_{i,\text{max}}^{(N)})^2}{\left( 1 + \sum_{i=1}^{\left\lceil \frac{N}{2} \right\rceil} y_{i,\text{max}}^{(N)} \right)} \]

\[ \epsilon_0 \text{ and } \epsilon_1 \leq \left( y_{i,\text{max}}^{(N)} - y_{i,\text{min}}^{(N)} \right) = \epsilon_1, \]

**Proof:** We know that \( p(0,0) = \lim_{N \to +\infty} \frac{1}{\rho} \left( 1 + \sum_{i=1}^{\left\lceil \frac{N}{2} \right\rceil} y_{i,0}^{(N)} \right) \) is a sequence of real positive numbers having the property: \( y_{i,\text{min}}^{(N)} \leq y_{i,0}^{(N)} \leq y_{i,\text{max}}^{(N)} \) for \( 1 \leq i \leq N \) and \( \left( y_{1,0}^{(N)}, y_{2,0}^{(N)}, \ldots, y_{\left\lceil \frac{N}{2} \right\rceil,0}^{(N)} \right) \in C_N \). Then

\[ p(0,0) = \frac{1}{1 + \rho} - \alpha_1 \]

where \( \alpha_1 \) which leads to:

\[ p(0,0) = \frac{1 - (1 + \rho) \alpha_1}{1 + \rho} \]

with \( 0 < \alpha_1 < \epsilon \) and

\[ \frac{1}{1 + \rho} \left( 1 + \sum_{i=1}^{\left\lceil \frac{N}{2} \right\rceil} y_{i,0}^{(N)} \right) \]

the integer \( N \) satisfying:

\[ \sum_{n=N+1}^{\infty} \pi_n \leq \epsilon. \]

So, the computed value of \( p(0,0) \) is \( y_0^{(N)} = \frac{1}{\rho} \left( 1 + \sum_{i=1}^{\left\lceil \frac{N}{2} \right\rceil} y_{i,0}^{(N)} \right) \) where

\[ \frac{1}{1 + \rho} \left( 1 + \sum_{i=1}^{\left\lceil \frac{N}{2} \right\rceil} y_{i,0}^{(N)} \right) \]

the sequence \( \left( y_i^{(N)} \right)_{1 \leq i \leq \left\lceil \frac{N}{2} \right\rceil} \) is chosen in the set \( C_N \).
We then have:

$$|p(0,0) - y_0| = \frac{1 - (1 + \rho)\alpha_1}{(1 + \rho)\left(1 + \sum_{i=1}^{N} y_{i,0}\right)} - \frac{1}{(1 + \rho)\left(1 + \sum_{i=1}^{N} y_{i,0}\right)}$$

$$= \frac{(1 - (1 + \rho)\alpha_1)}{(1 + \rho)\left(1 + \sum_{i=1}^{N} y_{i,0}\right)} - \frac{1}{(1 + \rho)\left(1 + \sum_{i=1}^{N} y_{i,0}\right)}$$

$$\leq \frac{\sum_{i=1}^{N} (y_{i,N} - y_{i,\min})}{\left[\sum_{i=1}^{N} y_{i,\min}\right]^2} + \epsilon \frac{\sum_{i=1}^{N} (y_{i,N} - y_{i,\max})}{\left[\sum_{i=1}^{N} y_{i,\max}\right]^2}$$

While the true value of $p(i,j)$ is given by: $p(i,j) = p(0,0)(\lim_{N \to \infty} y_{i,j}^{(N)})$ where: $y_{i,\min}^{(N)} \leq y_{i,j}^{(N)} \leq y_{i,\max}^{(N)}$ for any $i$ and $N$ such that $1 \leq i \leq \lfloor \frac{N}{2}\rfloor$, then $y_{i,\min}^{(N)} \leq \frac{p(i,j)}{p(0,0)} \leq y_{i,\max}^{(N)}$.

We denote further $\epsilon_1 = \max \epsilon_{i,j}$ the upper bound of the error committed on the probabilities $p(i,j)$.

It is well known that the convergence to the solution is fast for small values of $\rho$ ($\rho < 1$). More $\rho$ is close to 2, more the convergence is slow. This can be seen in the plots made in figure 3. We give then a sample of results obtained for different values of $\rho$. After each computation of a system of probabilities $p(i,j)$ we add a table in which we indicate the values of:

- $\epsilon$ of the error made on the total sum of probabilities
- $\epsilon_0$ the upper bound of the error made on the computation of $p(0,0)$
- $\epsilon_1$ the upper bound of the error made on the computation of the $\frac{p(i,j)}{p(0,0)}$ $i \geq 1$
- The integer $N$ having the property $\sum_{i+j \geq N+1} p(i,j) \leq \epsilon$
- The computed sum $\sum_{i+j \leq N} p(i,j)$

While the diagonal probabilities are of a special interest we also give them for $\rho = 0.2$ and $\rho = 1.9$. Some other cases of computation are made in further tables with values of $\rho$ close to 2.
### Table I: Some Sides of $\prod_{i=1}^{N} \left( \frac{y_{i}^{(N)}}{y_{i,\min}} \right)^{\epsilon_{i}}$ for $\rho = 1.5$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$y_{i,\max}$</th>
<th>$y_{i,\min}$</th>
<th>$\epsilon_{i}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.91576961624328916380</td>
<td>0.91576961624328916379</td>
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<tr>
<td>10</td>
<td>0.00581353037003270174</td>
<td>0.00581353037003270100</td>
<td>2.0</td>
</tr>
<tr>
<td>20</td>
<td>0.00001843593794715056</td>
<td>0.00001843593794713455</td>
<td>3.0</td>
</tr>
<tr>
<td>30</td>
<td>0.58464266523450426313 $10^{-7}$</td>
<td>0.58464266389157526824 $10^{-7}$</td>
<td>4.0</td>
</tr>
<tr>
<td>40</td>
<td>0.018540258000034771549 $10^{-9}$</td>
<td>0.18540258000034771549 $10^{-9}$</td>
<td>5.0</td>
</tr>
<tr>
<td>50</td>
<td>0.58795091077701311566 $10^{-12}$</td>
<td>0.5869540320783735787 $10^{-12}$</td>
<td>6.0</td>
</tr>
<tr>
<td>60</td>
<td>0.0192278298605164308290 $10^{-14}$</td>
<td>0.02068984149493612264 $10^{-15}$</td>
<td>7.0</td>
</tr>
</tbody>
</table>

### Table II: The $p(i, j)'s$ for $\rho = 0.2$

<table>
<thead>
<tr>
<th>$j$</th>
<th>$i$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
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<td>0</td>
<td>8.17</td>
<td>$10^{-1}$</td>
<td>1.5</td>
<td>$10^{-2}$</td>
<td>1.6</td>
<td>$10^{-4}$</td>
<td>1.6</td>
<td>$10^{-6}$</td>
<td>1.6</td>
<td>$10^{-8}$</td>
</tr>
<tr>
<td>1</td>
<td>8.17</td>
<td>$10^{-2}$</td>
<td>0.9</td>
<td>$10^{-4}$</td>
<td>0.8</td>
<td>$10^{-6}$</td>
<td>0.8</td>
<td>$10^{-8}$</td>
<td>0.8</td>
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</tr>
<tr>
<td>2</td>
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<td>$10^{-4}$</td>
<td>0.4</td>
<td>$10^{-6}$</td>
<td>0.3</td>
<td>$10^{-8}$</td>
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<td>$10^{-12}$</td>
</tr>
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<td>0.1</td>
<td>$10^{-10}$</td>
<td>0.1</td>
<td>$10^{-12}$</td>
<td>0.1</td>
<td>$10^{-14}$</td>
</tr>
<tr>
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<td>1.68</td>
<td>$10^{-8}$</td>
<td>0.1</td>
<td>$10^{-10}$</td>
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<td>$10^{-12}$</td>
<td>0.1</td>
<td>$10^{-14}$</td>
<td>0.1</td>
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<tr>
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<td>$10^{-10}$</td>
<td>0.1</td>
<td>$10^{-12}$</td>
<td>0.1</td>
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<tr>
<td>6</td>
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<td>0.1</td>
<td>$10^{-14}$</td>
<td>0.1</td>
<td>$10^{-16}$</td>
<td>0.1</td>
<td>$10^{-18}$</td>
<td>0.1</td>
<td>$10^{-20}$</td>
</tr>
<tr>
<td>7</td>
<td>0.13</td>
<td>$10^{-14}$</td>
<td>0.1</td>
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<td>$10^{-18}$</td>
<td>0.1</td>
<td>$10^{-20}$</td>
<td>0.1</td>
<td>$10^{-22}$</td>
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</table>

### Table III: The $p(i, j)'s$ for $\rho = 1.9$

<table>
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<th>$j$</th>
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<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.981</td>
<td>$10^{-2}$</td>
<td>0.195</td>
<td>$10^{-2}$</td>
<td>0.195</td>
<td>$10^{-2}$</td>
<td>0.195</td>
</tr>
<tr>
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<td>0.981</td>
<td>$10^{-2}$</td>
<td>0.981</td>
<td>$10^{-2}$</td>
<td>0.981</td>
</tr>
<tr>
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<td>7.56</td>
<td>$10^{-3}$</td>
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<td>$10^{-2}$</td>
<td>0.981</td>
<td>$10^{-2}$</td>
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</tr>
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<tr>
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</tbody>
</table>

$\epsilon = 0.01, \epsilon_{0} = 0.01, \epsilon_{1} = 0.01, N = 20$, the computed Sum = 1.0000000000000

$\epsilon = 0.01, \epsilon_{0} = 0.01, \epsilon_{1} = 0.01, N = 20$, the computed Sum = 1.0000000000000
V. Conclusion

This new numerical method is very simple for use and easily implemented for computation. The tools used are classical linear algebra and simplex method for which the software is available for solving big linear systems of inequalities and a great number of variables. Just a formal calculus (in terms of \( y_i^{(N)} \mid 1 \leq i \leq \left\lfloor \frac{N}{2} \right\rfloor \)) of \( \{r_N(i,j) \mid i+j = N\} \) and the use of the simplex algorithm (known as one of the most efficient algorithm regarding its complexity) for solving the system of inequalities \( \{r_N(i,j) \geq 0; i+j = N\} \). This will permit us to get the unknown steady state probabilities with a desired precision. For small values of \( \rho \) this method seems faster compared with the other ones. More \( \rho \) is close 0 more the number of operations is greater (see for example [13]).

An other point of interest is the adaptation of the method for other cases (due to the same structure of the steady state balance equations) of the shortest queue problem.

References