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On a new numerical analysis for the symmetric shortest queue problem

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Abstract—We consider a network of two M/M/1 parallel queues having the same poisonnian arrival stream with rate λ. Upon his arrival to the system a customer heads to the shortest queue and stays until being served. If the two queues have the same length, an arriving customer chooses one of the two queues with the same probability. Each duration of service in the two queues is an exponential random variable with rate μ and no jockeying is permitted between the two queues. A new numerical method, based on linear programming and convex optimization, is performed for the computation of the steady state solution of the system.

Keywords—steady state solution, matrix formulation, convex set, shortest queue, linear programming.

I. Introduction

In this paper we consider two identical M/M/1 queues in parallel with a JSQ (joining the shortest queue) policy. The customers arrive to the system in accordance with a Poisson process of rate λ. Upon his arrival to the system, the customer joins the shortest queue and stays in the queue until being served. If the two queues have the same length, the customer joins one of the two queues with the same probability. The duration of each service is an exponential random variable with rate μ. No jockeying is permitted between the two queues. The goal of the present paper is the numerical computation of the steady state probability $p(i, j)$ where i is the number of customers in queue 1 and j the number of customers in queue 2.

The shortest queue problem was initially proposed by Haight ([**?**]). A successful model for the symmetric case was given when the jockeying between the two queues is allowed. Kingman ([7]) obtained some asymptotic results for the joint steady state distribution of the number of customers in the two queues. Flatto and Mckean ([3]) used generating function techniques to give some limiting properties for the steady state probability. A numerical approach, using matrix geometric ([9]) technique was developed by Gertsbakh ([4]). Cohen and Boxma ([2]) showed that the analysis of the shortest queue problem can be reduced to the solution of a boundary value problem.

Halfin ([6]) performed a linear programming method for the problem by giving a lower and an upper bound for the probability distribution of the total number of customers in the system. Zhao and Grassmann ([13]) used the Flatto and McKean ([3]) results and analysis tools in order to develop a numerical solution. Adan, Wessels and Zijm ([1]) showed that the steady state distribution of the queue length is a mixture of product form distributions. Wang and Locker ([11]) presented a model where the state space of the related Markov process was truncated into banded arrays, so they derive the probability of queue length and the customer sojourn time. Yao and Knessl ([12]) considered the JSQ problem with two M/M/∞ queues; they perform an analytical and a numerical computation of the steady state solution . Recently Tarabia ([10]) presented a solution for the problem when jockeying is permitted and the capacity of each queue is finite. In this paper, we present an easy computation method adapted for this type of problems, based on a new formulation of the steady state balance equations. The probabilities $p(i, j)$ are linear expressions of the diagonal probabilities $p(i, i)$. An algorithm based on a linear program will permit us to get a convex set which contains the last probabilities. This new method was succesfully tested for two infinite server parallel queues in [8]. The software available for the simplex method allows us to make a computation with a high precision for a large range of values of the intensity traffic $\rho = \dfrac{\lambda}{\mu}$.

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II. Steady state analysis of the system

We consider the two dimensional stochastic process $(X_t, Y_t)_t$ where X_t (resp. Y_t) is the number of customers in the queue 1 (resp. queue 2) at time t. The process $(X_t, Y_t)_t$ is a recurrent positive Markov process if and only if $\rho = \dfrac{\lambda}{\mu} < 2$ (Kingman 1961). For $(i, j) \in E = \mathbb{N} \times \mathbb{N}$; let $p(i, j) = \lim_{t\to\infty} P(X_t = i, Y_t = j)$ the steady state probability for the process. The corresponding generator matrice is denoted by $Q = (q(e, e'))_{(e,e')\in\mathbb{N}^2\times\mathbb{N}^2}$

Notation 1: For $k \geq 1$ we define the following $(k + 1) \times 1$ vectors and $(k + 1) \times (k + 1)$ matrices:

$$X_{2k} = \begin{pmatrix} p(2k, 0) \\ p(2k-1, 1) \\ . \\ . \\ p(k+1, k-1) \\ p(k, k) \end{pmatrix} \quad X_{2k+1} = \begin{pmatrix} p(2k+1, 0) \\ p(2k, 1) \\ . \\ . \\ p(k+2, k-1) \\ p(k+1, k) \end{pmatrix}$$

$$A_{2k} = \begin{pmatrix} 1 & 1 & 0 & 0. & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A_{2k+1} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

Notation 2: The $(k+1)\times 1$ vectors B_{2k} and B_{2k+1} are also defined as follows:
For $k=1$

$$B_2 = \begin{pmatrix} \left((1+\rho)\frac{\rho}{2} - \frac{\rho}{2}\right) p(0,0) \\ p(1,1) \end{pmatrix}$$

For $k \geq 2$

$$B_{2k} = \begin{pmatrix} (1+\rho)\,p(2k-1,0) \\ (2+\rho)\,p(2k-2,1) - \rho p(2k-2,0) \\ (2+\rho)\,p(2k-3,2) - \rho p(2k-3,1) \\ . \\ . \\ (2+\rho)\,p(k+1,k-2) - \rho p(k+1,k-3) \\ (2+\rho)\,p(k,k-1) - \rho p(k,k-2) - \alpha_k \\ p(k,k) \end{pmatrix}$$

where $\alpha_k = \dfrac{\rho}{2} p(k-1,k-1)$
For $k \geq 1$

$$B_{2k+1} = \begin{pmatrix} (1+\rho)\,p(2k,0) \\ (2+\rho)\,p(2k-1,1) - \rho p(2k-1,0) \\ (2+\rho)\,p(2k-2,2) - \rho p(2k-2,1) \\ . \\ . \\ (2+\rho)\,p(k+1,k-1) - \rho p(k+1,k-2) \\ (2+\rho)\,p(k,k) - 2\rho p(k,k-1) \end{pmatrix}$$

Theorem 3: The steady state system of balance equations can be written as follows:

$$A_{2k}X_{2k} = B_{2k} \quad \text{and} \quad A_{2k+1}X_{2k+1} = B_{2k+1} \qquad \text{for} \quad k \geq 1$$

Proof: For all $n \geq 1$, as showed in the transition diagram in figure 1; the system of balance equations expresses the probabilities $\{p(i,j),\ i+j=n+1\}$ in terms of the probabilities $\{p(i,j),\ i+j=n\}$ and $\{p(i,j),\ \ i+j=n-1\}$. So we distinguish two cases:

Case 1: $n = 2k \quad k \geq 2$
Because of the symmetry, the system of balance equations is reduced to:

$$(\lambda+\mu)\,p(2k-1,0) = \mu p(2k,0) + \mu p(2k-1,1)$$

for $0 < i < k-1$:

$$(\lambda+2\mu)\,p(2k-1-i,i) = \lambda p(2k-1-i,i-1) + \mu p(2k-i,i) + \mu p(2k-1-i,i+1)$$

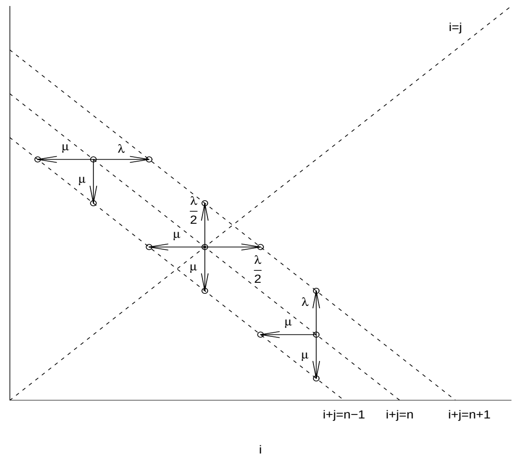

Fig. 1. The Transition diagram

and

$$(\lambda+2\mu)\,p(k,k-1) = \frac{\lambda}{2} p(k-1,k-1) + \lambda p(k,k-2) + \mu p(k,k) + \mu p(k+1,k-1)$$

Which is equivalent to:

$$p(2k,0) + p(2k-1,1) = (1+\rho)\,p(2k-1,0) \tag{1}$$

for $0 < i < k-1$:

$$p(2k-i,i) + p(2k-1-i,i+1) = (2+\rho)\,p(2k-1-i,i) - \rho p(2k-1-i,i-1)$$

and

$$p(k,k) + p(k+1,k-1) = (2+\rho)\,p(k,k-1) - \frac{\rho}{2} p(k-1,k-1) - \rho p(k,k-2)$$

This last system leads to the formulation: $A_{2k}X_{2k} = B_{2k}$

Case 2: $n = 2k+1 \quad k \geq 1$
Again, because of the symmetry, the system of balance equations can be written as:

$$(\lambda+2\mu)\,p(k,k) = 2\lambda p(k,k-1) + 2\mu p(k+1,k) \tag{2}$$

for $0 < i < k$

$$(\lambda+2\mu)p(k+i,k-i) = \lambda p(k+i,k-i-1) + \mu p(k+i+1,k-i) + \mu p(k+i,k-i+1)$$

and

$$(\lambda+\mu)\,p(2k,0) = \mu p(2k+1,0) + \mu p(2k,1) \tag{3}$$

or:

$$2p(k+1,k) = (2+\rho)\,p(k,k) - 2\rho p(k,k-1) \tag{4}$$

for $0 < i < k$

$$p(k+i+1,k-i)+p(k+i,k-i+1) = (2+\rho)\,p(k+i,k-i)-\rho p(k+i,k-i-1)$$

and

$$p(2k,1)+p(2k+1,0)=(1+\rho)\,p(2k,0) \qquad (5)$$

Which is the formulation: $A_{2k+1}X_{2k+1}=B_{2k+1}$ ∎

Proposition 4: For $j=1,2,..,k+1$ the j^{th} component of the vectors X_{2k} (resp. X_{2k+1}) is of the form:

$$\left(\sum_{i=1}^{i=k}\alpha_{i,j}^{(2k)}x_i+\alpha_{0,j}^{(2k)}\right)x_0$$

$$(\text{resp. } \left(\sum_{i=1}^{i=k}\alpha_{i,j}^{(2k+1)}x_i+\alpha_{0,j}^{(2k+1)}\right)x_0)$$

where $\alpha_{i,j}^{(2k)},\ \alpha_{i,j}^{(2k+1)}\in\mathbb{R}$, $x_ix_0=p(i,i)$ for $i\geq 1$ and $p(0,0)=x_0>0$

Proof: We use a recurrence argument. For $k=1$, we get from the formulas of theorem 1:

$$X_2=\begin{pmatrix}p(2,0)\\p(1,1)\end{pmatrix}=\begin{pmatrix}1&-1\\0&1\end{pmatrix}\begin{pmatrix}\left((1+\rho)\frac{\rho}{2}-\frac{\rho}{2}\right)x_0\\x_1x_0\end{pmatrix} =\begin{pmatrix}\frac{1}{2}x_0\left(\rho^2-2x_1\right)\\x_1x_0\end{pmatrix}$$

$$X_3=\begin{pmatrix}p(3,0)\\p(2,1)\end{pmatrix}=\begin{pmatrix}1&-\frac{1}{2}\\0&\frac{1}{2}\end{pmatrix}\begin{pmatrix}\frac{1}{2}\rho^2\left(\rho+1\right)x_0\\x_0\left(2x_1+\rho x_1-\rho^2\right)\end{pmatrix} =\begin{pmatrix}\frac{1}{2}x_0\left(\rho+2\right)\left(\rho^2-x_1\right)\\\left(-\frac{1}{2}\right)\left(\rho^2-\rho x_1-2x_1\right)x_0\end{pmatrix}$$

with $x_1x_0=p(1,1)$. We note that the expressions of X_2 and X_3 gives the coefficients $\alpha_{1,j}^{(2)},\ \alpha_{1,j}^{(3)}$ for $j=1,2$.

Now, we assume that X_{2k-1} satisfies the proposition, then because of the expressions giving the vectors X_{2k} and X_{2k+1} (B_{2k} is given in terms of the X_{2k-2} and X_{2k-1} components, B_{2k+1} is given in terms of the X_{2k} and X_{2k-1} components), the needed epressions of X_{2k} and X_{2k+1} are obtained. So the coefficients $\alpha_{i,j}^{(n)}$ are computed by a recursion. ∎

Remark 5: This last proposition shows that if $i+j=n$ then $p(i,j)$ is expressed in terms of $p(0,0),p(1,1),p(2,2),......p\left(\left[\frac{n}{2}\right],\left[\frac{n}{2}\right]\right)$ where $[x]$ is an integer having the property: $[x]\leq x<[x]+1$

Notation 6: In the following sections, for each integer n, we introduce the real numbers denoted $\{r_n(i,j),\ i+j\leq n\}$ satisfying the system of equilibrium equations

$$r_n(e)\sum_{e'\neq e}q(e,e')=\sum_{e'\neq e}r_n(e')q(e',e) \qquad (6)$$

where $(e)=(i,j);i+j\leq n-1$; so we put:
$r_n(0,0)=y_0^{(n)}$, $r_n(0,1)=r_n(1,0)=\dfrac{\rho}{2}r_n(0,0)$, $r_n(i,i)=y_i^{(n)}y_0^{(n)}$ for $i\geq 1$

$$Y_{2k}^{(n)}=\begin{pmatrix}r_n(2k,0)\\r_n(2k-1,1)\\.\\.\\r_n(k+1,k-1)\\r_n(k,k)\end{pmatrix}\quad Y_{2k+1}^{(n)}=\begin{pmatrix}r_n(2k+1,0)\\r_n(2k,1)\\.\\.\\r_n(k+2,k-1)\\r_n(k+1,k)\end{pmatrix}$$

$$D_2^{(n)}=\begin{pmatrix}\left((1+\rho)\frac{\rho}{2}-\frac{\rho}{2}\right)r_n(0,0)\\r_n(1,1)\end{pmatrix}$$

For $k\geq 2$ and $2k\leq n$

$$D_{2k}^{(n)}=\begin{pmatrix}(1+\rho)\,r_n(2k-1,0)\\(2+\rho)\,r_n(2k-2,1)-\rho r_n(2k-2,0)\\(2+\rho)\,r_n(2k-3,2)-\rho r_n(2k-3,1)\\.\\.\\(2+\rho)\,r_n(k+1,k-2)-\rho r_n(k+1,k-3)\\(2+\rho)\,r_n(k,k-1)-\rho r_n(k,k-2)-\beta_k\\r_n(k,k)\end{pmatrix}$$

where $\beta_k=\dfrac{\rho}{2}r_n(k-1,k-1)$

For $k\geq 1$ and $2k+1\leq n$

$$D_{2k+1}^{(n)}=\begin{pmatrix}(1+\rho)\,r_n(2k,0)\\(2+\rho)\,r_n(2k-1,1)-\rho r_n(2k-1,0)\\(2+\rho)\,r_n(2k-2,2)-\rho r_n(2k-2,1)\\.\\.\\(2+\rho)\,r_n(k+1,k-1)-\rho r_n(k+1,k-2)\\(2+\rho)\,r_n(k,k)-2\rho r_n(k,k-1)\end{pmatrix}$$

The vectors $Y_{2k}^{(n)}$ and $Y_{2k+1}^{(n)}$ are then defined by the recursive formulas:

$$A_{2k}Y_{2k}^{(n)}=D_{2k}^{(n)}\quad\text{and}\quad A_{2k+1}Y_{2k+1}^{(n)}=D_{2k+1}^{(n)}\quad\text{for } k\geq 1$$

We also have: the $(k+1)$ components of the vectors $Y_{2k}^{(n)}$ (resp $Y_{2k+1}^{(n)}$) are of the form:

$$\left(\sum_{i=1}^{i=k}\alpha_{i,j}^{(2k)}y_i^{(n)}+\alpha_{0,j}^{(2k)}\right)y_0^{(n)}$$

$$(\text{resp. } \left(\sum_{i=1}^{i=k}\alpha_{i,j}^{(2k+1)}y_i^{(n)}+\alpha_{0,j}^{(2k+1)}\right)y_0^{(n)})$$

where $\alpha_{i,j}^{(2k)},\ \alpha_{i,j}^{(2k+1)}\in\mathbb{R}$, $1\leq j\leq(k+1)$

Proposition 7: For $n\geq 2$, let the $\mathbb{R}^{\left[\frac{n}{2}\right]}$ sub set C_n defined as follows:

$$C_n=\left\{\left(y_1,y_2,......,y_{\left[\frac{n}{2}\right]}\right)/\ y_l>0\text{ for }l=1,2,...\left[\tfrac{n}{2}\right],\ r(i,j)>0\text{ if }i+j=n\ \right\}$$

then for $Y^{(n)} = \left(y_1^{(n)}, y_2^{(n)}, ..., y^{(n)}_{\left[\frac{n}{2}\right]} \right) \in C_n$; the set of real numbers $\{r_n(i,j); i+j \le n\}$ (obtained in terms of components of the vector $Y^{(n)}$ in C_n) is a positive solution for the system of equilibrium equations (6)

Proof: C_n is nonempty while it contains the vector $\left(p(1,1), p(2,2), ...p\left(\left[\frac{n}{2}\right], \left[\frac{n}{2}\right] \right) \right)$ up to a multiplicative factor and if $y_i^{(n)} = \frac{p(i,i)}{p(0,0)}$ for $1 \le i \le \left[\frac{n}{2}\right]$ and $y_0^{(n)} = p(0,0)$ the corresponding $r_n(i,j), i+j=n$ are exactly $p(i,j)$; $i+j=n$.

The condition $r_n(i,j) > 0$ if $i+j=n$ gives for those real numbers the properties of the steady state probabilities $p(i,j); i+j=n$. then $\{r_n(i,j); i+j \le n-1\}$ (from which we build up $\{r_n(i,j); i+j=n\}$ over the system of equilibrium equations (6) ; with $y_i^{(n)}$ playing the role of $p(i,i)$, $1 \le i \le \left[\frac{n}{2}\right]$) have the same properties than $\{p(i,j); i+j \le n-1\}$. So if $y_0^{(n)} > 0$ and $\left(y_1^{(n)}, y_2^{(n)}, ..., y^{(n)}_{\left[\frac{n}{2}\right]} \right) \in C_n$ the builded system of real numbers $\{r_n(i,j); i+j \le n\}$ is a positive solution for the equilibrium equations (6). ■

Corollary 8: If we note:

$$S_n = \left\{ \left(y_1(n), y_2(n), ..., y_{\left[\frac{n}{2}\right]}(n), ... \right) \text{ so that } y_l > 0 \;\; \forall l \ge 1 \;,\; r(i,j) > 0 \text{ for } i+j=n \right\}$$

then $(S_n)_n$ is a decreasing sequence of sets ($S_n \subset S_{n-1}$) and the limit $\cap S_n$ is so that $\lim_{n \to +\infty} \left(y_i^{(n)} y_0^{(n)} \right)_{i \ge 1}$ is exactly the entire diagonal probabilities $p(i,i)$ $i \ge 1$.

Proof: Let $\left(y_1^{(n)}, y_2^{(n)}, ..., y^{(n)}_{\left[\frac{n}{2}\right]}, ... \right) \in S_n$, we note first that the components $y_l^{(n)}$ for $l > \left[\frac{n}{2}\right]$ are free from the constraints $r_n(i,j) > 0$ for $i+j=n$ then those components are identic for the two sets S_n and S_{n-1}. While the components $\left(y_1^{(n)}, y_2^{(n)}, ..., y^{(n)}_{\left[\frac{n}{2}\right]} \right)$ have also to fulfil the constraints $\{r_n(i,j) > 0; i+j=n-1\}$ (previous proposition) then $S_n \subset S_{n-1}$. So, when n goes to the infinity and due to the unicity of the positive solution for the infinite linear system of balance equations (the related Markov process is ergodic), the set S_n has for limit the single point of $\mathbb{R}^{\mathbb{N}}$ which is the diagonal probabilities $(p(i,i))_{i \ge 1}$ up to a multiplicative factor. ■

Remark 9: All element of C_n can be viewed as an approximation of the vector $(p(1,1), ...$
$....., p\left(\left[\frac{n}{2}\right], \left[\frac{n}{2}\right]\right))$ up to a multiplicative factor.

Remark 10: While for all n , $p(i,j)$; $i+j=n$ is expressed in terms of $p(0,0), p(1,1), p(2,2),$
$......, p\left(\left[\frac{n}{2}\right], \left[\frac{n}{2}\right]\right)$, then the first step of computation is

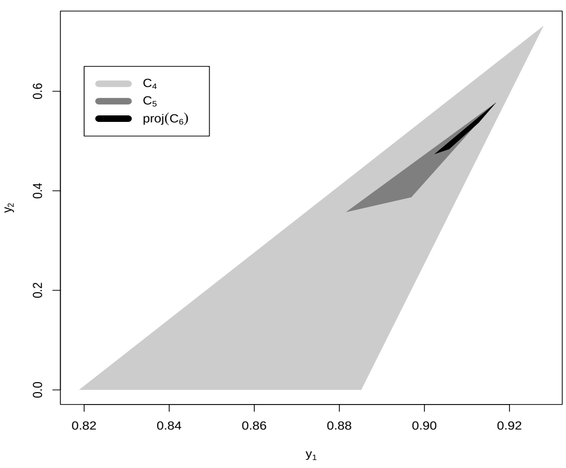

Fig. 2. Evolution of the feasible regions

to get an approximation of those last probabilities by $y_0^{(n)}, y_0^{(n)} y_1^{(n)}, ...y_0^{(n)} y^{(n)}_{\left[\frac{n}{2}\right]}$ respectively, $y_0^{(n)}$ is a multiplicative factor and $y_1^{(n)}, ...y^{(n)}_{\left[\frac{n}{2}\right]}$ are computed under the constraints given in the definition of C_n. We then choose n suficiently large in order to get a good approximation. In other words $y_1^{(n)} y_0^{(n)}, y_2^{(n)} y_0^{(n)}, ..., y_n^{(n)} y_0^{(n)}$ are almost the diagonal probabilities: $p(1,1), ..., p(n,n)$. In order to show the geometric behavior of the set C_n in the plane we choose $\rho = 1.5$ and we sketch C_4, C_5 and the projection of C_6 on the plane containing the first two coordinates (y_1, y_2) (Figure 2)

III. Computation methodology

In practice we need a finite number of the the probabilities $\{p(i,i)\}$.The sequence $\{p(i,i)\}_i$ is decreasing after some integer K (see [13]) and have the limit 0 (see Figure 3). The problem is then reduced to evaluate an integer N large enough so that the sum $\sum_{\{(i,j)/i+j \le N\}} p(i,j)$ is very close to 1. To do this, we use the bounds for $\sum_{n=N}^{\infty} \pi_n$ where $\pi_n = \sum_{\{(i;j)/i+j=n\}} p(i,j)$ given in Halfin ([6]). We then have :

For $\sigma = \frac{\ln \frac{2+\rho}{(2-\rho)(1+\rho)}}{\ln(1+\rho)}$ and:

$$\sqrt{2} \le \rho < 2, \qquad N \ge \sigma + 1 \qquad \sum_{n=N}^{\infty} \pi_n \le \left(\frac{\rho}{2}\right)^{N-\sigma}$$

$$1 \le \rho < \sqrt{2}, \qquad N \ge 2 \qquad \sum_{n=N}^{\infty} \pi_n \le \left(\frac{\rho}{2}\right)^{N} \frac{2+\rho}{1+\rho}$$

For $0 < \rho < 1$, we have:

$$\pi_n \le \left(\frac{\rho}{2}\right)^n \frac{1}{2(1+\rho)} \left(2 + \rho - \frac{\rho^2 (1+\rho)^n}{(1+\rho)^n - 1} \right) + \frac{\rho^2}{2(1+\rho)} \frac{1}{(1+\rho)^n - 1}$$

which implies that:

$$\pi_n \leq \left(\frac{\rho}{2}\right)^n + \frac{1}{(1+\rho)^n}$$

$$\text{so } \sum_{n=N}^{\infty} \pi_n \leq \left(\frac{\rho}{2}\right)^N \frac{1}{1-\frac{\rho}{2}} + \left(\frac{1}{1+\rho}\right)^{N-1} \frac{1}{\rho}$$

This last bound is not intersting if ρ is close to 0. In this case, we considere the intensity flows for a Markov process in equilibrium as invoked in Halfin [6] and having the form:

$\lambda\pi_{n-1} = \mu\pi_n^- + 2\mu\pi_n^+$ where $\pi_n^- = 2p(n,0)$ and $\pi_n^+ = \sum_{\{i+j=n, i>0, j>0\}} p(i,j) \quad (\pi_n = \pi_n^- + \pi_n^+)$, we get:

$\pi_n = \dfrac{\lambda}{\dfrac{\mu\pi_n^- + 2\mu\pi_n^+}{\pi_n}} \pi_{n-1} = \dfrac{\lambda}{\mu\left(1 + \dfrac{\pi_n^+}{\pi_n}\right)} \pi_{n-1}$ so, $\pi_n \leq \rho^n$ which is more intersting if $0 < \rho < 1$, with a special look for ρ close to 0. Thus: $\sum_{n=N}^{\infty} \pi_n \leq \dfrac{\rho^N}{1-\rho}$.

So, for a given precision ϵ, the computation of the integer N satisfying $\sum_{n=N+1}^{\infty} \pi_n \leq \varepsilon$ can be done easily. As an example for $\rho = 1$ and $\epsilon = 10^{-10}$ we find $N = 34$. We then need $p(i,i)$ $1 \leq i \leq 17$. The determination of N can also be regarded as the stopping rule for the computation algorithm described below. We note that the upper bound ϵ of the error is made on the total sum of probabilities. We see further in the numerical results that ϵ is around the upper bound of the error made between the computed value of $p(i,i)$ and $p(i,i)$ itself.

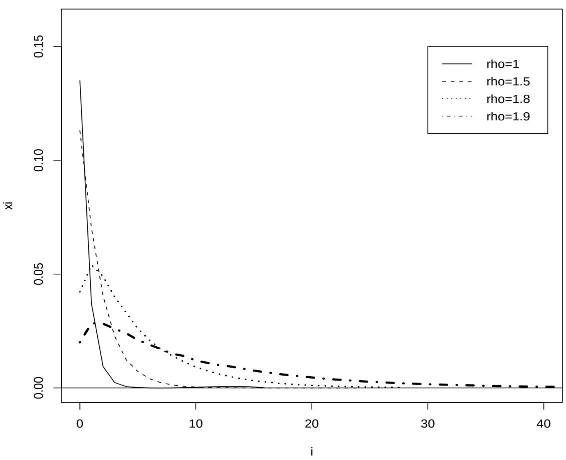

Fig. 3. Evolution of the diagonal probabilities

IV. Computation of the steady state probabilities and numerical results

A. Computation of $y_i^{(N)}$ $1 \leq i \leq \left[\frac{N}{2}\right]$

We recall that the integer N is so that all the probabilities $\{p(i,j)/i+j \geq N+1\}$, having a total sum smaller than a given ϵ, are neglected.

While the $\left[\frac{N}{2}\right] + 1$ components of the vector $Y_{2\left[\frac{N}{2}\right]}^{(N)}$ are of the forme:

$$\left(\sum_{i=1}^{i=\left[\frac{N}{2}\right]} \alpha_{i,j}^{(N)} y_i^{(N)} + \alpha_{0,j}^{(N)}\right) y_0^{(N)};\ 1 \leq j \leq \left[\frac{N}{2}\right] + 1$$

then the unknowns $\left\{y_i^{(N)}\ ; 1 \leq i \leq \left[\frac{N}{2}\right]\right\}$ are in the set solution of the system of linear inequalities:

$$\sum_{i=1}^{i=\left[\frac{N}{2}\right]} \alpha_{i,j}^{(N)} y_i^{(N)} + \alpha_{0,j}^{(N)} > 0 \text{ and } y_i^{(N)} > 0 \tag{7}$$

for $1 \leq j \leq \left[\frac{N}{2}\right] + 1$ and $1 \leq i \leq \left[\frac{N}{2}\right]$.

We use the simplex algorithm to obtain a lower and an upper bound for each $y_i^{(N)}$ noted respectively $y_{i,\min}^{(N)}$ and $y_{i,\max}^{(N)}$. Because of the unicity due to the ergodicity of the related Markov process, those bounds are almost equal for a large value of N, as to be seen in further computation (see table 1). While $y_i^{(N)} > 0$, we can take the objective function $\sum_1^{\left[\frac{N}{2}\right]} y_i^{(N)}$ (or any linear function of $\left\{y_i^{(N)}\ ; 1 \leq i \leq \left[\frac{N}{2}\right]\right\}$) and the constraints (7). While $\sum p(i,i) = \dfrac{1}{1+\rho}$ (see [6]), we then compute $y_0^{(N)}$ as a constant of normalisation by: $y_0^{(N)}\left(1 + \sum_1^{\left[\frac{N}{2}\right]} y_i^{(N)}\right) = \dfrac{1}{1+\rho}$ or $y_0^{(N)} = \dfrac{1}{(1+\rho)\left(1 + \sum_1^{\left[\frac{N}{2}\right]} y_i^{(N)}\right)}$ where $\left(y_i^{(N)}\right)_{1 \leq i \leq \left[\frac{N}{2}\right]}$ is any solution for (7). We can take the arithmetic mean: $y_i^{(N)} = \frac{1}{2}\left(y_{i,\min}^{(N)} + y_{i,\max}^{(N)}\right)$

B. Algorithm description

Step 1: set up a value of ρ.

Step 2: set up a precision ϵ and compute the corresponding value of N.

Step 3: put $y_i^{(N)}.y_0^{(N)} = r_N(i,i)$ $1 \leq i \leq \left[\frac{N}{2}\right]$ and get (by using a formal calculus) the $\left[\frac{N}{2}\right] + 1$ components of the vector $Y_{2\left[\frac{N}{2}\right]}^{(N)}$ denoted by $\sum_{i=1}^{i=\left[\frac{N}{2}\right]} \alpha_{i,j}^{\left(2\left[\frac{N}{2}\right]\right)} y_i^{(N)} + \alpha_0^{\left(2\left[\frac{N}{2}\right]\right)}$ $1 \leq$

$j \leq \left[\dfrac{N}{2}\right]+1$. This can be done whithout solving the formal systems of linear equations $A_n Y_n^{(N)} = D_n^{(N)}$ for $n = 2k$ and $n = 2k+1$. It's due to the fact that the inverse of the matrix:

$$\begin{pmatrix} 1 & 1 & 0 & . & . & 0 \\ 0 & 1 & 1 & 0 & . & . \\ . & 0 & 1 & 1 & 0 & . \\ . & . & 0 & 1 & 1 & 0 \\ . & . & . & 0 & 1 & 1 \\ 0 & . & . & . & 0 & 1 \end{pmatrix}$$

has one of the two forms:

$$\begin{pmatrix} 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & -1 \\ . & 0 & 1 & -1 & 1 \\ . & . & 0 & 1 & -1 \\ 0 & . & . & 0 & 1 \end{pmatrix}$$

$$or \quad \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 & -1 & 1 \\ . & 0 & 1 & -1 & 1 & -1 \\ . & . & 0 & 1 & -1 & 1 \\ . & . & . & 0 & 1 & -1 \\ 0 & . & . & . & 0 & 1 \end{pmatrix}$$

and the inverse of the matrix:

$$\begin{pmatrix} 1 & 1 & 0 & . & . & 0 \\ 0 & 1 & 1 & 0 & . & . \\ . & 0 & 1 & 1 & 0 & . \\ . & . & 0 & 1 & 1 & 0 \\ . & . & . & 0 & 1 & 1 \\ 0 & . & . & . & 0 & 2 \end{pmatrix}$$

has one of the two forms:

$$\begin{pmatrix} 1 & -1 & 1 & -1 & \frac{1}{2} \\ 0 & 1 & -1 & 1 & -\frac{1}{2} \\ . & 0 & 1 & -1 & \frac{1}{2} \\ . & . & 0 & 1 & -\frac{1}{2} \\ 0 & . & . & 0 & \frac{1}{2} \end{pmatrix}$$

$$or \quad \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -\frac{1}{2} \\ 0 & 1 & -1 & 1 & -1 & \frac{1}{2} \\ . & 0 & 1 & -1 & 1 & -\frac{1}{2} \\ . & . & 0 & 1 & -1 & \frac{1}{2} \\ . & . & . & 0 & 1 & -\frac{1}{2} \\ 0 & . & . & . & 0 & \frac{1}{2} \end{pmatrix}$$

So in the computation program we put: $Y_n^{(N)} = A_n^{-1} D_n^{(N)}$ for $n = 2k$ or $2k+1$.

Step 4: use the simplex method with objective function $\sum_{i=1}^{i=\left[\frac{N}{2}\right]} y_i^{(N)}$ and constraints:

$$\sum_{i=1}^{i=\left[\frac{N}{2}\right]} \alpha_{i,j}^{(m)} y_i^{(N)} + \alpha_{0,j}^{(m)} \geq 0 \quad 1 \leq m \leq \left[\frac{N}{2}\right]+1$$

$$y_i^{(N)} \geq 0 \quad 1 \leq i \leq \left[\frac{N}{2}\right].$$

Step 5: get a lower and an upper bound for $y_i^{(N)}$ denoted respectively $y_{i,\min}^{(N)}$ and $y_{i,\max}^{(N)}$ and put $y_i^{(N)} = \frac{1}{2}\left(y_{i,\max}^{(N)} + y_{i,\min}^{(N)}\right)$. So the set solution is a convex set denoted $C_N^{'}$.

Step 6: return to the matrix formulation for the computation of $p(i,j)$ $i+j \leq N$.

C. Numerical results and error analysis

Proposition 11: If we note e_0 and e_i the error computation made on $p(0,0)$ and $\dfrac{p(i,i)}{p(0,0)}$ respectively then:

$$e_0 \leq \frac{\sum_{i=1}^{\left[\frac{N}{2}\right]} \left(y_{i,\max}^{(N)} - y_{i,\min}^{(N)}\right)}{(1+\rho)\left(1+\sum_{i=1}^{\left[\frac{N}{2}\right]} y_{i,\min}^{(N)}\right)^2} + \frac{\epsilon\left(1+\sum_{i=1}^{\left[\frac{N}{2}\right]} y_{i,\max}^{(N)}\right)}{\left(1+\sum_{i=1}^{\left[\frac{N}{2}\right]} y_{i,\min}^{(N)}\right)^2} =$$

ϵ_0 and $e_i \leq \left(y_{i,\max}^{(N)} - y_{i,\min}^{(N)}\right) = \epsilon_{1,i}$

Proof: We know that $p(0,0) = \lim_{N\to+\infty} \dfrac{1}{(1+\rho)\left(1+\sum_{i=1}^{\infty} y_{i,0}^{(N)}\right)}$ where $\left(y_{i,0}^{(N)}\right)_{i\geq 1}$ is a sequence of real positive numbers having the property: $y_{i,\min}^{(N)} \leq y_{i,0}^{(N)} \leq y_{i,\max}^{(N)}$ $1 \leq i \leq N$ for $\left(y_1^{(N)}, y_2^{(N)}, ..., y_{\left[\frac{n}{2}\right]}^{(N)}\right) \in C_n$. Then $p(0,0)\left(1+\sum_{i=1}^{\left[\frac{N}{2}\right]} y_{i,0}^{(N)}\right) = \dfrac{1}{1+\rho} - \alpha_1$ which leads to:

$$p(0,0) = \frac{1-(1+\rho)\,\alpha_1}{(1+\rho)\left(1+\sum_{i=1}^{\left[\frac{N}{2}\right]} y_{i,0}^{(N)}\right)}$$ with $0 < \alpha_1 < \epsilon$ and the integer N satisfying: $\sum_{n=N+1}^{\infty} \pi_n \leq \epsilon$. So, the computed value of $p(0,0)$ is $y_0^{(N)} = \dfrac{1}{(1+\rho)\left(1+\sum_{1}^{\left[\frac{N}{2}\right]} y_i^{(N)}\right)}$ where

the sequence $\left(y_i^{(N)}\right)_{1\leq i\leq\left[\frac{N}{2}\right]}$ is chosen in the set C_n.

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We then have:

$$|p(0,0) - y_0| =$$

$$\left| \frac{1-(1+\rho)\,\alpha_1}{(1+\rho)\left(1+\sum_{i=1}^{\left[\frac{N}{2}\right]} y_{i,0}^{(N)}\right)} - \frac{1}{(1+\rho)\left(1+\sum_{1}^{\left[\frac{N}{2}\right]} y_{i}^{(N)}\right)} \right|$$

$$= \left| \frac{(1-(1+\rho)\,\alpha_1)\left(1+\sum_{1}^{\left[\frac{N}{2}\right]} y_{i}^{(N)}\right) - \left(1+\sum_{i=1}^{\left[\frac{N}{2}\right]} y_{i,0}^{(N)}\right)}{(1+\rho)\left(1+\sum_{i=1}^{\left[\frac{N}{2}\right]} y_{i,0}^{(N)}\right)\left(1+\sum_{1}^{\left[\frac{N}{2}\right]} y_{i}^{(N)}\right)} \right|$$

$$\leq \frac{\sum_{i=1}^{\left[\frac{N}{2}\right]}\left(y_{i,\max}^{(N)} - y_{i,\min}^{(N)}\right)}{(1+\rho)\left(1+\sum_{i=1}^{\left[\frac{N}{2}\right]} y_{i,\min}^{(N)}\right)^2} + \frac{\epsilon\left(1+\sum_{i=1}^{\left[\frac{N}{2}\right]} y_{i,\max}^{(N)}\right)}{\left(1+\sum_{i=1}^{\left[\frac{N}{2}\right]} y_{i,\min}^{(N)}\right)^2}$$

While the true value of $p(i,i)$ is given by: $p(i,i) = p(0,0)(\lim_{N\to+\infty} y_i^{(N)})$ where: $y_{i,\min}^{(N)} \leq y_i^{(N)} \leq y_{i,\max}^{(N)}$ for any i and N such that $1 \leq i \leq \left[\frac{N}{2}\right]$, then $y_{i,\min}^{(N)} \leq \dfrac{p\,(i,i)}{p\,(0,0)} \leq y_{i,\max}^{(N)}$ ■

We denote further $\epsilon_1 = \max \epsilon_{1,i}$ the upper bound of the error commited on the probabilities $p(i,i)$.

It is well known that the convergence to the solution is fast for small values of ρ $(\rho < 1)$. More ρ is close to 2, more the convergence is slow. This can be seen in the plots made in figure 3. We give then a sample of results obtained for different values of ρ. After each computation of a system of probabilities $p(i,j)$ we add a table in which we indicate the values of:

- ϵ of the error made on the total sum of probabilities
- ϵ_0 the upper bound of the error made on the computation of $p(0,0)$
- ϵ_1 the upper bound of the error made on the computation of the $\dfrac{p\,(i,i)}{p\,(0,0)}$ $i \geq 1$
- The integer N having the property $\sum_{i+j\geq N+1} p(i,j) \leq \epsilon$
- The computed sum $\sum_{i+j\leq N} p(i,j)$

While the diagonal probabilities are of a special interest we also give them for $\rho = 0.2$ and $\rho = 1.9$. Some other cases of computation are made in further tables with values of ρ close to 2.

Results comments: The obtained results match those given in [13]. The powerfulness of the method described in the present paper allows us to make computations for all values of ρ and this is done for $\rho = 0.2$ and $\rho = 1.9$. In addition to the precision with which are obtained the probabilities $p(i,j)$, this method allows us (by using the bounds given in [6]) to make an easy choice of the finite state space E_ϵ in which the computation is made. If E is the whole state space of the process $(X_t, Y_t)_t$ then $E - E_\epsilon$ represents the missing states with $P(E - E_\epsilon) \leq \epsilon$.

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TABLE I: SOME SIDES OF $\prod_{i=1}^{\left[\frac{N}{2}\right]}\left[y_{i,\min}^{(N)}, y_{i,\max}^{(N)}\right]$ FOR $\rho = 1.5$

$\left[\frac{N}{2}\right] = 60$		
i	$y_{i,\max}^{(N)}$	$y_{i,\min}^{(N)}$
1	0.91576961624328916380	0.91576961624328916379
10	0.00581353037003270174	0.00581353037003270100
20	0.00001843593794715056	0.00001843593794713455
30	$0.58464266523450423613\ 10^{-7}$	$0.58464266389157826824\ 10^{-7}$
40	$0.18540258000034771549\ 10^{-9}$	$0.18540205570576079581\ 10^{-9}$
50	$0.58795091077701311566\ 10^{-12}$	$0.58695403207873375878\ 10^{-12}$
60	$0.19227829860516430820\ 10^{-14}$	$0.64269894149493612264\ 10^{-15}$

TABLE II: THE $p(i,j)'s$ FOR $\rho = 0.2$

$j\backslash i$	0	1	2	3	4	5	6	7	8
0	$8.17\ 10^{-1}$								
1	$8.17\ 10^{-2}$	$1.5\ 10^{-2}$							
2	$7.12\ 10^{-4}$	$8.5\ 10^{-4}$	$1.6\ 10^{-4}$						
3	$3.52\ 10^{-6}$	$4.2\ 10^{-6}$	$8.5\ 10^{-6}$	$1.6\ 10^{-6}$					
4	$1.68\ 10^{-8}$	$2.0\ 10^{-8}$	$4.0\ 10^{-8}$	$8.5\ 10^{-8}$	$1.6\ 10^{-8}$				
5	$7.99\ 10^{-11}$	$9.5\ 10^{-11}$	$1.9\ 10^{-10}$	$4.0\ 10^{-10}$	$8.5\ 10^{-10}$	$1.6\ 10^{-10}$			
6	$3.81\ 10^{-13}$	$4.5\ 10^{-13}$	$9.2\ 10^{-13}$	$1.9\ 10^{-12}$	$4.0\ 10^{-12}$	$8.5\ 10^{-12}$	10^{-12}		
7	$1.82\ 10^{-15}$	$2.1\ 10^{-15}$	$4.3\ 10^{-15}$	$9.2\ 10^{-15}$	$1.9\ 10^{-14}$	$4.0\ 10^{-14}$	$8\ 10^{-14}$	10^{-14}	
8	$2.08\ 10^{-17}$	$1.2\ 10^{-17}$	$2.1\ 10^{-16}$	$4.3\ 10^{-17}$	$9.0\ 10^{-17}$	$1.9\ 10^{-16}$	$4\ 10^{-16}$	$8\ 10^{-16}$	10^{-16}
9	$1.24\ 10^{-17}$	$2.4\ 10^{-18}$	$4.5\ 10^{-19}$	$8.0\ 10^{-20}$	$1.3\ 10^{-20}$	$1.9\ 10^{-21}$	$2\ 10^{-22}$	$2\ 10^{-23}$	
10	$1.24\ 10^{-17}$	$2.4\ 10^{-18}$	$4.3\ 10^{-19}$	$7.2\ 10^{-20}$	$1.0\ 10^{-20}$	$1.3\ 10^{-21}$	$1\ 10^{-22}$		
11	$1.25\ 10^{-17}$	$2.3\ 10^{-18}$	$4.0\ 10^{-19}$	$6.1\ 10^{-20}$	$7.8\ 10^{-21}$	$7.1\ 10^{-22}$			
12	$1.27\ 10^{-17}$	$2.3\ 10^{-18}$	$3.4\ 10^{-19}$	$4.6\ 10^{-20}$	$4.2\ 10^{-21}$				
13	$1.29\ 10^{-17}$	$2.1\ 10^{-18}$	$2.8\ 10^{-19}$	$2.6\ 10^{-20}$					
14	$1.33\ 10^{-17}$	$1.8\ 10^{-18}$	$1.7\ 10^{-19}$						

ϵ	ϵ_0	ϵ_1	N	*the computed Sum*
10^{-12}	$0.981\ 10^{-12}$	$0.109\ 10^{-20}$	20	1.000000000000

TABLE III: THE $p(i,j)'s$ FOR $\rho = 1.9$

$i\backslash j$	0	1	2	3	4	5	6	7
0	$1.99\ 10^{-2}$							
1	$1.89\ 10^{-2}$	$2.85\ 10^{-2}$						
2	$7.56\ 10^{-3}$	$1.95\ 10^{-2}$	$2.82\ 10^{-2}$					
3	$2.44\ 10^{-3}$	$6.32\ 10^{-3}$	$1.81\ 10^{-2}$	$2.60\ 10^{-2}$				
4	$7.55\ 10^{-4}$	$1.96\ 10^{-3}$	$5.60\ 10^{-3}$	$1.64\ 10^{-2}$	$2.36\ 10^{-2}$			
5	$2.31\ 10^{-4}$	$6.00\ 10^{-4}$	$1.72\ 10^{-3}$	$5.03\ 10^{-3}$	$1.48\ 10^{-2}$	$2.13\ 10^{-2}$		
6	$7.08\ 10^{-5}$	$1.84\ 10^{-4}$	$5.26\ 10^{-4}$	$1.54\ 10^{-3}$	$4.54\ 10^{-3}$	$1.34\ 10^{-2}$	$1.92\ 10^{-2}$	
7	$2.17\ 10^{-5}$	$5.62\ 10^{-5}$	$1.61\ 10^{-4}$	$4.71\ 10^{-4}$	$1.39\ 10^{-3}$	$4.09\ 10^{-3}$	$1.21\ 10^{-5}$	$1.74\ 10^{-2}$
8	$6.63\ 10^{-6}$	$1.72\ 10^{-5}$	$4.92\ 10^{-5}$	$1.44\ 10^{-4}$	$4.24\ 10^{-4}$	$1.25\ 10^{-3}$	$3.69\ 10^{-3}$	$1.09\ 10^{-2}$
9	$2.03\ 10^{-6}$	$5.26\ 10^{-6}$	$1.51\ 10^{-5}$	$4.41\ 10^{-5}$	$1.30\ 10^{-4}$	$3.83\ 10^{-4}$	$1.13\ 10^{-3}$	$3.33\ 10^{-3}$
10	$6.21\ 10^{-7}$	$1.61\ 10^{-6}$	$4.61\ 10^{-6}$	$1.35\ 10^{-5}$	$3.98\ 10^{-5}$	$1.17\ 10^{-4}$	$3.46\ 10^{-4}$	
11	$1.90\ 10^{-7}$	$4.92\ 10^{-7}$	$1.41\ 10^{-6}$	$4.13\ 10^{-6}$	$1.22\ 10^{-5}$	$3.59\ 10^{-5}$		
12	$5.81\ 10^{-8}$	$1.51\ 10^{-7}$	$4.31\ 10^{-7}$	$1.26\ 10^{-6}$	$3.72\ 10^{-6}$			
13	$1.78\ 10^{-8}$	$4.61\ 10^{-8}$	$1.32\ 10^{-7}$	$3.86\ 10^{-7}$				
14	$5.44\ 10^{-9}$	$1.41\ 10^{-8}$	$4.03\ 10^{-8}$					

ϵ	ϵ_0	ϵ_1	N	*the computed Sum*
10^{-5}	$0.2\ 10^{-6}$	$1.5\ 10^{-5}$	226	1.00000

V. Conclusion

This new numerical method is very simple for use and easily implemented for computation. The tools used are classical linear algebra and simplex method for which the software is available for solving big linear systems of inequalities and a great number of variables. Just a formal calculus (in terms of $\left\{y_i^{(N)}\ ;\ 1 \leq i \leq \left[\frac{N}{2}\right]\right\}$) of $\{r_N\left(i,j\right); i+j=N\}$ and the use of the simplex algorithm (known as one of the most efficient algorithm regarding it's complexity) for solving the system of inequalities $\{r_N\left(i,j\right) \geq 0; i+j=N\}$. This will permit us to get the unknown steady state probabilities with a desired precision. For small values of ρ this method seems faster compared with the other ones. More ρ is close 0 more the number of operations is greater (see for example [13]). An other point of interest is the adaptation of the method for other cases (due to the same structure of the steady state balance equations) of the shortest queue problem**.**

References

[1] Adan, I.J.B.F. and Wessels, J. and Zijm W.H.M. , Analysis of the symmetric shortest queueing problem, Stoch. Models, 1990 **6**, 691-713 .
[2] Cohen, J.W. and Boxma, O.J. , Boundary Problems in Queuing Systems Analysis, North- Holland, Amsterdam, 1983
[3] Flatto, L. and McKean, H.P. , Two queues in parallel, Comm. Pure. Appl. Math, 1977 **30**, 255-263 .
[4] Gertsbakh, I. , The shorter queue problem: A numerical study using the matrix geometric solution, European J. Operation Research, 1984 **15**, 374-381.
[5] Haight, F.A. , Two queues in parallel, Biometrika, 1958 **48**, 401-410 .
[6] Halfin, S. , The shortest queue problem, J. Appl. Probab., 1985 **22**, 865-878 .
[7] Kingman, J.F.C. . Two similar queues in parallel, Ann. Math. Stat., 1961 **32**, 1314-1323.
[8] Lardjane, T. and Messaci, R. , On a new numerical computation of the steady state solution of two infinite server parallel queues, Applied Mathematical Sciences, 2011 Vol 5, **78**, 3875-3891.
[9] Neuts, M. F. , Matrix geometric Solutions in Stochastic Models. John Hopkins University Press, Baltimore, 1980
[10] Tarabia, A.M.K., Analysis of two queues in parallel with jockeying and restricted capacities, Appl. Math.Modell., 2008 **32**(5), 802-810.
[11] Wang, P. and Locker, V.F. , Steady state distributions of parallel queues, INFOR, 2001 **39**(1).
[12] Yao, H. and Knessl, C. , On the infinite server shortest queue problem: symmetric case, Stochastic Models, 2005 **21**(1), 101-132 .
[13] Zhao, Y. and Grassman, W.K. , A numerically stable algorithm for two server queue model, Queueing Systems, 1991, **8**, 59-79 .