I-Vague Groups

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Abstract—The notions of I-vague groups with membership and non-membership functions taking values in an involutary dually residuated lattice ordered semigroup are introduced which generalize the notions with truth values in a Boolean algebra as well as those usual vague sets whose membership and non-membership functions taking values in the unit interval [0, 1]. Moreover, various operations and properties are established.

Keywords—Involutary dually residuated lattice ordered semigroup, I-vague set and I-vague group.

I. INTRODUCTION

The notion of fuzzy groups defined by A. Rosenfeld[13] is the first application of fuzzy set theory in Algebra. Since then a number of works have been done in the area of fuzzy algebra.


The vague sets of W. L. Gau and D. J. Buehrer[6] and G. L. Gau[9] are I-vague sets which are categorically equivalent to the class of MV-algebras of C. C. Chang[4] and well studied offer a natural generalization of the notion of vague groups with membership and non-membership functions taking values in an involutary DRL-semigroup promises a unified study of vague sets. In this paper using the definition of I-vague sets, we define a dually residuated lattice ordered semigroup in a DRL-semigroup if and only if

Definition 2.1: [14] A system $A = (A, +, \leq , - )$ is called a dually residuated lattice ordered semigroup (in short DRL-semigroup) if and only if

i) $A = (A, +)$ is a commutative semigroup with zero "0";

ii) $A = (A, \leq)$ is a lattice such that $a + (b \cup c) = (a + b) \cup (a + c)$ and $a + (b \cap c) = (a + b) \cap (a + c)$ for all $a, b, c \in A$;

iii) Given $a, b \in A$, there exists a least $x$ in $A$ such that $b + x \geq a$, and we denote this $x$ by $a - b$ (for a given $a, b$ this $x$ is uniquely determined);

iv) $(a - b) \cup 0 + b \leq a \cup b$ for all $a, b \in A$;

v) $a - a \geq 0$ for all $a \in A$.

Theorem 2.2: [14] Any DRL-semigroup is a distributive lattice.

Definition 2.3: [19] A DRL-semigroup A is said to be involutory if there is an element $i(\neq 0)$ (0 is the identity w.r.t. +) such that

i) $a + (1 - a) = 1 + 1$;

ii) $1 - (1 - a) = a$ for all $a \in A$.

Theorem 2.4: [15] In a DRL-semigroup with 1, 1 is unique.

Theorem 2.5: [15] If a DRL-semigroup contains a least element $x$, then $x = 0$. Dually, if a DRL-semigroup with 1 contains a largest element $a$, then $a = 1$.

In his thesis Z. Teshome[19] studied the concept of I-vague sets. In this paper using the definition of I-vague sets, we defined and studied I-vague groups where I is an involutor DRL-semigroup. In this paper we shall recall some basic results in [14], [15], [19] without proof. Moreover, notation, terminology and results of [19] are used in this paper. Throughout this paper, we shall denote the identity element of a group G by e and the order of an element x of G by O(x). Moreover, for $x \in G$, $x < x$ denotes the cyclic group generated by x.

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II. PRELIMINARIES

Definition 2.8: [19] An I-vague set $A = (t_A, f_A)$ of G is an I-vague set with membership and non-membership functions taking values in an involutary DRL-semigroup if and only if

i) $A = (t_A, f_A)$ is a commutative semigroup with zero "0";

ii) $A = (t_A, f_A)$ is a lattice such that $a + (b \cup c) = (a + b) \cup (a + c)$ and $a + (b \cap c) = (a + b) \cap (a + c)$ for all $a, b, c \in A$.

Definition 2.9: [19] The interval $[t_A(x), 1 - f_A(x)]$ is called the I-vague value of $x \in G$ and is denoted by $V(x)$.

Definition 2.10: [19] Let $B = \{a_1, b_1\}$ and $B_2 = \{a_2, b_2\}$ be two I-vague values. We say $B_1 \geq B_2$ if and only if $a_1 \geq a_2$ and $b_1 \geq b_2$.

Definition 2.11: [19] An I-vague set $A = (t_A, f_A)$ of G is said to be contained in an I-vague set $B = (t_B, f_B)$ of G written as $A \subseteq B$ if and only if $t_A(x) \leq t_B(x)$ and $f_A(x) \geq f_B(x)$ for all $x \in G$. A is said to be equal to B written as $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.
Definition 2.12: [19] An I-vague set $A$ of $G$ with $V_A(x) = V_A(y)$ for all $x, y \in G$ is called a constant I-vague set of $G$.

Definition 2.13: [19] Let $A$ be an I-vague set of a nonempty set $G$. Let $A_{(\alpha, \beta)} = \{x \in G : V_A(x) \geq \alpha, \beta\}$ where $\alpha, \beta \in I$ and $\alpha \leq \beta$. Then $A_{(\alpha, \beta)}$ is called the $(\alpha, \beta)$ cut of the I-vague set $A$.

Definition 2.14: [19] Let $S \subseteq G$. The characteristic function of $S$ denoted as $\chi_S = (t_{\chi_S}, f_{\chi_S})$, which takes values in $I$ is defined as follows:

$$t_{\chi_S}(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_{\chi_S}(x) = \begin{cases} 0 & \text{if } x \in S \\ 1 & \text{otherwise}. \end{cases}$$

$\chi_S$ is called the I-vague characteristic set of $S$ in $I$. Thus

$$V_{\chi_S} = \{1, 1\} \text{ if } x \in S; \{0, 0\} \text{ otherwise.}$$

Definition 2.15: [19] Let $A = (t_A, f_A)$ and $B = (t_B, f_B)$ be I-vague sets of a set $G$.

(i) Their union $A \cup B$ is defined as $A \cup B = (t_{A \cup B}, f_{A \cup B})$ where $t_{A \cup B}(x) = t_A(x) \lor t_B(x)$ and $f_{A \cup B}(x) = f_A(x) \lor f_B(x)$ for each $x \in G$.

(ii) Their intersection $A \cap B$ is defined as $A \cap B = (t_{A \cap B}, f_{A \cap B})$ where $t_{A \cap B}(x) = t_A(x) \land t_B(x)$ and $f_{A \cap B}(x) = f_A(x) \land f_B(x)$ for each $x \in G$.

Definition 2.16: [19] Let $B_1 = [a_1, b_1]$ and $B_2 = [a_2, b_2]$ be I-vague values. Then

(i) $\sup(B_1, B_2) = \sup\{a_1, a_2, \sup\{b_1, b_2\}\}$.

(ii) $\inf(B_1, B_2) = \inf\{a_1, a_2, \inf\{b_1, b_2\}\}$.

Lemma 2.17: [19] Let $A$ and $B$ be I-vague sets of a set $G$. Then $A \cup B$ and $A \cap B$ are also I-vague sets of $G$.

Let $x \in G$. From the definition of $A \cup B$ and $A \cap B$ we have

(i) $V_{A \cup B}(x) = \inf\{V_A(x), V_B(x)\}$.

(ii) $V_{A \cap B}(x) = \inf\{V_A(x), V_B(x)\}$.

Definition 2.18: [19] Let I be complete and $\{A_i : i \in \Delta\}$ be a non empty family of I-vague sets of $G$ where $A_i = (t_{A_i}, f_{A_i})$. Then

(i) $\bigwedge_{i \in \Delta} A_i = (\bigwedge_{i \in \Delta} t_{A_i}, \bigvee_{i \in \Delta} f_{A_i})$.

(ii) $\bigvee_{i \in \Delta} A_i = (\bigvee_{i \in \Delta} t_{A_i}, \bigwedge_{i \in \Delta} f_{A_i})$.

Lemma 2.19: [19] Let I be complete. If $\{A_i : i \in \Delta\}$ is a non empty family of I-vague sets of $G$, then $\bigwedge_{i \in \Delta} A_i$ and $\bigvee_{i \in \Delta} A_i$ are I-vague sets of $G$.

Definition 2.20: [19] Let I be complete and $\{A_i : \{A_{1i}, A_{2i} : i \in \Delta\}$ be a non empty family of I-vague sets of $G$. Then for each $x \in G$,

(i) $\sup\{V_{A_i}(x) : i \in \Delta\} = \bigwedge_{i \in \Delta} t_{A_i}(x), \bigvee_{i \in \Delta} (1 - f_{A_i}(x))$.

(ii) $\inf\{V_{A_i}(x) : i \in \Delta\} = \bigvee_{i \in \Delta} t_{A_i}(x), \bigwedge_{i \in \Delta} (1 - f_{A_i}(x))$.

III. I-VAGUE GROUPS

Definition 3.1: Let G be a group. An I-vague set A of a group G is called an I-vague group of G if

(i) $V_A(xy) \geq \inf\{V_A(x), V_A(y)\}$ for all $x, y \in G$ and

(ii) $V_A(x^{-1}) \geq V_A(x)$ for all $x \in G$.

Lemma 3.2: If A is an I-vague group of a group G, then $V_A(x) = V_A(x^{-1})$ for all $x \in G$.

Proof: Since A is an I-vague group of G, $V_A(x^{-1}) \geq V_A(x)$ for all $x \in G$. Therefore $V_A(e) \geq V_A(x)$ for all $x \in G$.

Lemma 3.3: If A is an I-vague group of a group G, then $V_A(e) \geq V_A(x)$ for all $x \in G$.

Proof: Let $x \in G$.

$V_A(e) = V_A(xx^{-1}) \geq \inf\{V_A(x), V_A(x^{-1})\} = V_A(x)$ for all $x \in G$. Therefore $V_A(e) \geq V_A(x)$ for all $x \in G$.

Lemma 3.4: Let $m \in Z$. If A is an I-vague group of a group G, then $V_A(x^m) \geq V_A(x)$ for all $x \in G$.

Proof: Let $m \in Z$. We prove that $V_A(x^m) \geq V_A(x)$ for all $x \in G$. Since $V_A(e) \geq V_A(x)$ for all $x \in G$ by lemma 3.3, the statement is true for $m = 0$.

First we prove that the lemma is true for positive integers by induction.

Since $V_A(x) \geq V_A(e)$, it is true for $m = 1$.

Assume it is true for $m$.

Thus $V_A(x^{m+1}) \geq \inf\{V_A(x^m), V_A(x)\} = V_A(x)$. Therefore $V_A(x^m) \geq V_A(x)$.

Consequently, $V_A(x^m) \geq V_A(x)$ for all $x \in G$ and for every integer m. Hence the lemma follows.

Lemma 3.5: A necessary and sufficient condition for an I-vague set A of a group G to be an I-vague group of G is that $V_A(xy^{-1}) \geq \inf\{V_A(x), V_A(y)\}$ for all $x, y \in G$.

Proof: Let A be an I-vague set of G. Suppose that $V_A(xy^{-1}) \geq \inf\{V_A(x), V_A(y)\}$ for all $x, y \in G$. Let $x \in G$.

Then $V_A(e) = V_A(xx^{-1}) \geq \inf\{V_A(x), V_A(x^{-1})\} = V_A(x)$. Thus $V_A(e) \geq V_A(x)$ for all $x \in G$.

Therefore $V_A(x^m) \geq V_A(e)$ for each $x \in G$.

Let $x, y \in G$. Then

$V_A(xy) = V_A((x^2y^{-1})^{-1}) \geq \inf\{V_A(x), V_A(y^{-1})\}$

$\geq \inf\{V_A(x), V_A(y)\}$. Hence

$V_A(xy) \geq \inf\{V_A(x), V_A(y)\}$ for each $x, y \in G$, so A is an I-vague group of G.

Conversely, suppose that A is an I-vague group of G. Let $x, y \in G$. Then

$V_A(xy) \geq \inf\{V_A(x), V_A(y^{-1})\} = \inf\{V_A(x), V_A(y)\}$. Therefore $V_A(xy^{-1}) \geq \inf\{V_A(x), V_A(y)\}$ for all $x, y \in G$.

Hence the theorem follows.

Lemma 3.6: Let H be a subgroup of G and $[\gamma, \delta] \subseteq [\alpha, \beta] \subseteq I$ where $\alpha \leq \beta$ and $\gamma \leq \delta$. Then the I-vague set A of G defined by

$$V_A(x) = \begin{cases} [\alpha, \beta] & \text{if } x \in H \\ [\gamma, \delta] & \text{otherwise} \end{cases}$$

is an I-vague group of G.

Proof: Let H be a subgroup of G. We prove that the I-vague set A defined above is an I-vague group of G.
Let $x, y \in G$. If $xy^{-1} \in H$, then $V_A(xy^{-1}) = [\alpha, \beta]$.

Hence $V_A(xy^{-1}) \geq \inf \{V_A(x), V_A(y)\}$.

If $xy^{-1} \notin H$, then either $x \notin H$ or $y \notin H$.

Thus, $\inf \{V_A(x), V_A(y)\} = \{\gamma, \delta\}$. It follows that $V_A(xy^{-1}) \geq \inf \{V_A(x), V_A(y)\}$. Hence $V_A(xy^{-1}) \geq \inf \{V_A(x), V_A(y)\}$ for every $x, y \in G$.

Therefore $A$ is an I-vague group of $G$.

**Example:** Consider the group $(Z, +)$. Let $I$ be the unit interval $[0, 1]$ of real numbers. Let $a \oplus b = \min \{1, a + b\}$.

With the usual ordering $(I, \oplus, \leq)$, $-I$ is an involutary DRL-semigroup.

Define the I-vague set $A$ of $G$ as follows:

$$V_A(x) = \begin{cases} [a_1, b_1] & \text{if } x \in 4\mathbb{Z}; \\ [a_2, b_2] & \text{if } x \in 2\mathbb{Z} - 4\mathbb{Z}; \\ [a_3, b_3] & \text{otherwise} \end{cases}$$

where $[a_3, b_3] = [a_2, b_2] \leq [a_1, b_1]$ and $a_i, b_i \in [0, 1]$ for $i = 1, 2, 3$. Then $A$ is an I-vague group of $G$.

We prove that $V_A(xy^{-1}) \geq \inf \{V_A(x), V_A(y)\}$ for all $x, y \in G$.

(i) If $xy^{-1} \in 4\mathbb{Z}$, then $V_A(xy^{-1}) = [a_1, b_1] \geq \inf \{V_A(x), V_A(y)\}$.

(ii) If $xy^{-1} \in 2\mathbb{Z} - 4\mathbb{Z}$, then there exist $x, y \in Z$ such that $x \notin 4\mathbb{Z}$ or $y \notin 4\mathbb{Z}$. This implies $\inf \{V_A(x), V_A(y)\} \leq [a_2, b_2] = V_A(xy^{-1})$.

(iii) If $xy^{-1} = x - y$ is odd, then one of them must be odd. Hence $\inf \{V_A(x), V_A(y)\} = [a_3, b_3] \leq a(x^{-1})$.

Therefore $A$ is an I-vague group of $G$.

**Lemma 3.7:** Let $H \neq \emptyset$ and $H \subseteq G$. The I-vague characteristic set of $H$, $\chi_H$, is an I-vague group of $G$ iff $H$ is a subgroup of $G$.

**Proof:** Suppose that $H$ is a subgroup of $G$. By Lemma 3.6, $\chi_H$ is an I-vague group of $G$.

Conversely, suppose that $\chi_H$ is an I-vague group of $G$.

We show that $H$ is a subgroup of $G$. Let $x, y \in H$. Then $V_H(xy^{-1}) \geq \inf \{V_H(x), V_H(y)\} = [1, 1]$. Hence $V_H(x^{-1}) = [1, 1]$. So $x^{-1} \in H$. Therefore $H$ is a subgroup of $G$. Hence the lemma follows.

**Lemma 3.8:** If $A$ and $B$ are I-vague groups of a group $G$, then $A \cap B$ is also an I-vague group of $G$.

**Proof:** Let $A$ and $B$ be I-vague groups of $G$.

Then $A \cap B$ is an I-vague set of $G$ by lemma 2.17. Now we show that $V_{A \cap B}(xy^{-1}) \geq \inf \{V_{A \cap B}(x), V_{A \cap B}(y)\}$ for each $x, y \in G$.

Let $x, y \in G$. Then $V_{A \cap B}(xy^{-1}) = \inf \{V_A(xy^{-1}), V_B(xy^{-1})\}$.

Thus $V_{A \cap B}(xy^{-1}) \geq \inf \{V_{A \cap B}(x), V_{A \cap B}(y)\}$ for every $x, y \in G$. Therefore $A \cap B$ is an I-vague group of $G$.

**Lemma 3.9:** Let $I$ be complete. If $\{A_i : i \in \Delta\}$ is a non empty family of I-vague groups of $G$, then $\bigcap_{i \in \Delta} A_i$ is an I-vague group of $G$.

**Proof:** Let $A = \bigcap_{i \in \Delta} A_i$. Then $A$ is an I-vague set of $G$ by lemma 2.19.

Now we prove that $V_A(xy^{-1}) \geq \inf \{V_A(x), V_A(y)\}$ for every $x, y \in G$. Let $x, y \in G$. Then $V_A(xy^{-1}) = V \bigcap_{i \in \Delta} A_i(xy^{-1})$.

Thus $V_A(xy^{-1}) \geq \inf \{V_{A \cap B}(x), V_{A \cap B}(y)\}$ for all $x, y \in G$. Hence $A \cap B$ is an I-vague group of $G$.

**Theorem 3.11:** An I-vague set $A$ of a group $G$ is an I-vague group of $G$ if and only if for all $\alpha, \beta \in I$ with $\alpha \leq \beta$, the I-vague cut $A_{(\alpha, \beta)}$ is a subgroup of $G$ whenever it is non empty.

**Proof:** Let $A$ be an I-vague set of $G$.

Suppose that $A$ is an I-vague group of $G$. We prove that $A_{(\alpha, \beta)}$ is a subgroup of $G$ whenever it is non empty.

Let $x, y \in A_{(\alpha, \beta)}$. Then $V_A(x) \geq [\alpha, \beta]$ and
**Theorem 3.12:** Let \( A \) be an I-vague group of a group \( G \).

If \( V_A(xy^{-1}) = V_A(x) \) for \( x, y \in G \), then \( V_A(x) = V_A(y) \).

**Proof:** Suppose that \( V_A(xy^{-1}) = V_A(x) \) for \( x, y \in G \).

\[
V_A(x) = V_A(ce) = V_A(xy^{-1}y) \supseteq \text{inf}(V_A(xy^{-1})), \ V_A(x) = \text{inf}(V_A(x), V_A(y)) = V_A(y).
\]

Thus \( V_A(x) \geq V_A(y) \).

Therefore \( V_A(x) = V_A(y) \).

Hence the theorem follows.

The following example shows that the converse of the preceding theorem is not true.

**Example:** Let \( I \) be the unit interval \([0, 1]\) of real numbers. Define \( \oplus = \min \{1, a + b\} \). With the usual ordering \((I, \oplus, \leq, -)\) is an involutory DRL-semigroup.

Consider \( G = (Z, +) \) and \( H = (3Z, +) \). Let \( A \) be the I-vague group of \( G \) defined by

\[
A(x) = \left\{ \begin{array}{ll}
\left[ \frac{1}{2}, 1 \right] & \text{if } x \in H ; \\
0 & \text{otherwise.}
\end{array} \right.
\]

Let \( x = 2 \) and \( y = 1 \). Then \( A(x) = V_A(y) = [0, \frac{3}{2}] \) but \( A(xy^{-1}) = V_A(2 - 1) = V_A(1) = [0, \frac{3}{2}] \neq A(0) \).

**Theorem 3.13:** Let \( A \) be an I-vague group of a group \( G \) and \( x \in G \). Then \( V_A(xy) = V_A(x) \) for all \( y \in G \) iff \( V_A(x) = V_A(e) \).

**Proof:** Let \( A \) be an I-vague group of a group \( G \) and \( x \in G \).

Suppose that \( V_A(xy) = V_A(x) \) for all \( y \in G \) and \( V_A(x) = V_A(e) \).

We prove that \( V_A(xy) = V_A(x) \) for all \( y \in G \).

For any \( y \in G \), \( V_A(xy) \subseteq V_A(x) \).

\[
V_A(xy) \supseteq \text{inf}(V_A(x), V_A(y)) = V_A(y).
\]

Hence \( V_A(xy) \supseteq V_A(y) \).

Similarly, \( V_A(xy) \subseteq V_A(x) \).

Thus \( V_A(xy) = V_A(x) \) and \( V_A(xy) = V_A(y) \).

The converse follows.

Theorem 3.14: Let \( A \) be an I-vague group of a group \( G \).

Then \( GVA = \{ x \in G : V_A(x) = V_A(e) \} \) is a subgroup of \( G \).

**Proof:** Let \( A \) be an I-vague group of a group \( G \). Since \( e \in GVA \), \( GVA \neq \emptyset \) and \( GVA \subseteq G \). Let \( x, y \in GVA \). We prove that \( x^{-1} \in GVA \).

\[
V_A(xy^{-1}) \supseteq \text{inf}(V_A(x), V_A(y)) = V_A(e).
\]

Thus \( V_A(xy^{-1}) \supseteq V_A(x) \) and \( V_A(xy^{-1}) \supseteq V_A(y) \).

Therefore \( GVA \) is a subgroup of \( G \).

**Lemma 3.15:** Let \( A \) be an I-vague group of a group \( G \).

If \( x < y \), then \( V_A(y) \leq V_A(x) \).

**Proof:** Suppose that \( x < y \). Then \( x \in L \). It follows that \( x = y^{-m} \) for some \( m \in Z \).

\[
V_A(x) = V_A(y^{-m}) \geq V_A(y) \Rightarrow V_A(x) \supseteq V_A(y).
\]

The following example shows that the converse of lemma 3.15 is not true.

**Example:** Let \( I \) be the unit interval \([0, 1]\) of real numbers. Let \( a + b = \min \{1, a + b\} \). With the usual ordering \((I, \oplus, \leq, -)\) is an involutory DRL-semigroup. Let \( G = \{ e, a, b, c \} \).

Define the I-vague set \( A \) of \( G \) by

\[
V_A(x) = \left\{ \begin{array}{ll}
\left[ \frac{1}{2}, 1 \right] & \text{if } x \in A > \\
\{0, \frac{3}{4}\} & \text{otherwise.}
\end{array} \right.
\]

Then \( V_A(c) = [0, \frac{3}{4}] \) but \( < a > \) is not a subset of \( < c > \).

**Definition 3.16:** Let \( A \) be an I-vague group of a group \( G \). Image of \( A \) is defined as \( ImA = \{ V_A(x) : x \in G \} \).

Since \( V_A(x) \supseteq V_A(x) \) for all \( x \in G \), \( V_A(e) \) is the greatest element of \( ImA \).

**Theorem 3.17:** Let \( A \) be an I-vague group of a group \( G \).

(i) If \( G \) is cyclic then \( ImA \) has a least element.

(ii) If \( V_A(x) \leq V_A(y) \) then \( x \supseteq y \) and \( ImA \) has a least element then \( G \) is cyclic.

**Proof:** Let \( A \) be an I-vague group of \( G \).

(i) Suppose that \( G \) is cyclic. Then \( V_A(x) \) for some \( x \in G \). We prove that \( V_A(x) \) is the least element of \( ImA \).

Let \( y \in G \). Then \( y = x^m \) for some \( m \in Z \).

\[
V_A(y) = V_A(x^m) \supseteq V_A(x).
\]

We have \( V_A(x) \leq V_A(y) \) for every \( y \in G \).

Thus \( V_A(x) \) is the least element of \( ImA \) of \( A \).

(ii) Suppose that \( ImA \) has a least element say \( V_A(x) \) for some \( x \in G \). Let \( y \in G \). Then \( V_A(y) \geq V_A(x) \) for all \( y \in G \).

By our condition we have \( < y > \subseteq < x > \).

Since \( y \in < y > \), \( y \in < x > \). Hence \( G < x < y > \).

Consequently, we have \( G = < x > \).

Therefore \( G \) is cyclic.

**Lemma 3.18:** Let \( A \) be an I-vague group of \( G \). Let \( x, y \in G \).

The two conditions

(i) \( V_A(x) = V_A(y) \Rightarrow < x > = < y > \)

(ii) \( V_A(x) > V_A(y) \Rightarrow < x > < y > < x > \) are equivalent to the condition \( V_A(x) \geq V_A(y) \Rightarrow < x > < y > \).

**Proof:** Assume that the two conditions are given.

We prove that \( V_A(x) \geq V_A(y) \Rightarrow < x > < y > \).

If \( V_A(x) > V_A(y) \), then \( < x > < y > < x > \).

If \( V_A(x) = V_A(y) \), then \( < x > < x > < y > < x > \).

We have \( < x > < x > \).

Conversely, assume that \( V_A(x) \geq V_A(y) \Rightarrow < x > < y > \).

(i) Suppose that \( V_A(x) = V_A(y) \).

\[
V_A(x) = V_A(y) \Rightarrow V_A(x) \geq V_A(y) \text{ and } V_A(x) \geq V_A(y).
\]

(ii) \( V_A(x) > V_A(y) \Rightarrow V_A(x) \geq V_A(y) \Rightarrow < x > < y > < x > \).

Thus \( V_A(x) < V_A(y) \Rightarrow < x > < y > \).

Therefore

(i) and (ii) are equivalent to \( V_A(x) \geq V_A(y) \iff < x > < y > \).
Moreover, since $G$ is a cyclic group of prime power order, $O = a_i$ satisfies $V_a(y) \Rightarrow x \Rightarrow y >$. Then $V_a(x) < V_a(y) \Rightarrow x > y >$. Let $A$ be a cyclic group of prime power order.

**Proof:** Let $A$ be an I-vague group of a group $G$ such that the image set of $A$ is given by $\text{Im}A = \{I_0 > I_1 > \ldots > I_n\}$ and such that

(i) $V_a(x) = V_a(y) \Rightarrow x \Rightarrow y >$;
(ii) $V_a(x) < V_a(y) \Rightarrow x > y >$.

By the definition of $A$ we have $a_i$ satisfies $V_a(y) \Rightarrow x \Rightarrow y >$. Thus $V_a(x) < V_a(y) \Rightarrow x > y >$.

Since $G$ is a cyclic group of prime power order, $O = a_i$ satisfies $V_a(y) \Rightarrow x \Rightarrow y >$. Hence $V_a(x) < V_a(y) \Rightarrow x > y >$.

**Step(2)** We show that $A$ is an I-vague group of $G$.

By the definition of $A$ we have $O(x) = O(y)$ implies $x \Rightarrow y >$. Hence $V_a(x) < V_a(y) \Rightarrow x > y >$.

Moreover, since $G$ is a cyclic group of prime power order, $< x > \wedge < y >$ or $< y > \wedge < x >$.

If $x < y$ then $x, y \in < y >$. Hence $< x > \wedge < y >$.

If $y \leq x$ then $x, y \in < x >$. Hence $< x > \wedge < y >$.

**Theorem 3.19:** Let $A$ be an I-vague group of a group $G$ such that the image set of $A$ is given by $\text{Im}A = \{I_0 > I_1 > \ldots > I_n\}$ and such that

(i) $V_a(x) = V_a(y) \Rightarrow x \Rightarrow y >$;
(ii) $V_a(x) < V_a(y) \Rightarrow x > y >$.

Then $A$ is a cyclic group of prime power order.

Suppose that $G = \mathbb{Z}/p\mathbb{Z}$, where $p$ is not a prime power. Then there exist prime numbers $p$ and $q$ such that $p \neq q$ which are factors of $m$.

Consider $V_a(p)$ and $V_a(q)$.

Since $\text{Im}A = \{I_0 > I_1 > \ldots > I_n\}$, either $V_a(p) \geq V_a(q)$ or $V_a(p) < V_a(q)$. It follows that $< p > \leq q >$ or $< q > \leq p >$, a contradiction.

Hence $G$ is not isomorphic to $\mathbb{Z}/p\mathbb{Z}$.

Thus $G \cong \mathbb{Z}/p\mathbb{Z}$ for some $m \in N$.

Since $G$ is not a prime power, then there exist prime numbers $p$ and $q$ such that $p \neq q$ which are factors of $m$.

Consider $V_a(p)$ and $V_a(q)$.

Suppose that $G$ is a cyclic group of order $p^m$ where $p$ is prime and $m \in N \cup \{0\}$. We find an $I$ and an I-vague group $A$ of $G$ satisfying (i) and (ii).

**Step(3)** We show that $A$ satisfies the conditions (i) and (ii) of the theorem.

(a) Suppose that $V_a(x) = V_a(y)$ for $x, y \in G$.

Since $G$ is a cyclic group of prime power order, $O(x) = O(y)$ implies $x \Rightarrow y >$. Hence $V_a(x) = V_a(y) \Rightarrow x > y >$.

(b) Suppose that $V_a(x) = V_a(y)$ for $x, y \in G$. Then $I_j \geq I_k$.

It follows that $< j > \leq < k >$.

Since $G$ is a cyclic group of order $p^m$ and $O(x) \leq O(y)$, $< x > \leq < y >$. Hence $V_a(x) = V_a(y) \Rightarrow x \leq y$.

Therefore $A$ satisfies (i) and (ii).

Hence the theorem follows.

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**References**


