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The Sizes of Large Hierarchical Long-Range Percolation Clusters

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Abstract—We study a long-range percolation model in the hierarchical lattice Ω_N of order N where probability of connection between two nodes separated by distance k is of the form $\min\{\alpha\beta^{-k}, 1\}$, $\alpha \geq 0$ and $\beta > 0$. The parameter α is the percolation parameter, while β describes the long-range nature of the model. The Ω_N is an example of so called ultrametric space, which has remarkable qualitative difference between Euclidean-type lattices. In this paper, we characterize the sizes of large clusters for this model along the line of some prior work. The proof involves a stationary embedding of Ω_N into $\mathbb{Z}$. The phase diagram of this long-range percolation is well understood.

Keywords—percolation, component, hierarchical lattice, phase transition.

I. Introduction

PERCOLATION theory in the Euclidean lattice $\mathbb{Z}^d$ started with the work of Broadbent and Hammersley in 1957. The infinity of the space of vertices and its geometry are principal features of this model; see e.g. [11] and references therein. Some questions of percolation in other non-Euclidean infinite systems is formulated in [4]. The study of long-range percolation on $\mathbb{Z}^d$ traces back to [15] and leads to a range of interesting results in probability theory and statistical physics [1], [5], [6], [8], [18], [21]. On the other hand, hierarchical structures have been used in applications in the physics, genetics and social sciences thanks to the multi-scale organization of many natural objects [3], [13], [19], [20].

Recently, long-range percolation is studied on the hierarchical lattice Ω_N of order N (to be defined below), where classical methods for the usual lattice break down. The asymptotic long-range percolation on Ω_N is addressed in [10] for $N \to \infty$. The work [9], [12], [16] and [17] analyze the phase transition of long-range percolation on Ω_N for finite N using different connection probabilities and methodologies. The contact process on Ω_N for fixed N has been investigated in [2]. In this paper, we investigate the sizes of large connected components (or clusters) in the resulting percolation graph on Ω_N for fixed N. The form of the connection probabilities used here follow from a prior work [16].

For an integer $N \geq 2$, we define the set

$$\Omega_N := \Big\{\mathbf{x} = (x_1, x_2, \cdots) : x_i \in \{0, 1, \cdots, N-1\}, \quad i = 1, 2, \cdots, \sum_{i=1}^{\infty} x_i < \infty\Big\}, \quad (1)$$

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and define a metric d on it:

$$d(\mathbf{x}, \mathbf{y}) = \begin{cases} 0, & \mathbf{x} = \mathbf{y}, \\ \max\{i : x_i \neq y_i\}, & \mathbf{x} \neq \mathbf{y}. \end{cases} \quad (2)$$

The pair (Ω_N, d) is referred to as the hierarchical lattice of order N, which may be thought of as the set of leaves at the bottom of an infinite regular tree without a root, where the distance between two vertices is the number of levels (generations) from the bottom to their most recent common ancestor. Figure 1 shows the lattice Ω_2 along with its metric generating tree.

Such a distance d satisfies the strong triangle inequality

$$d(\mathbf{x}, \mathbf{y}) \leq \max\{d(\mathbf{x}, \mathbf{z}), d(\mathbf{z}, \mathbf{y})\}, \quad (3)$$

for any triple $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \Omega_N$. Hence, (Ω_N, d) is an ultrametric (or non-Archimedean) space [14]. From its ultrametricity, it is clear that for every $\mathbf{x} \in \Omega_N$ there are $(N-1)N^{k-1}$ vertices at distance k from it.

Now consider a long-range percolation on Ω_N. For each $k \geq 1$, the probability of connection between $\mathbf{x}$ and $\mathbf{y}$ such that $d(\mathbf{x}, \mathbf{y}) = k$ is given by

$$p_k = \min\left\{\frac{\alpha}{\beta^k}, 1\right\}, \quad (4)$$

where $0 \leq \alpha < \infty$ and $0 < \beta < \infty$, all connections being independent. Two vertices $\mathbf{x}, \mathbf{y} \in \Omega_N$ are in the same cluster if there exists a finite sequence $\mathbf{x} = \mathbf{x}_0, \mathbf{x}_1, \cdots, \mathbf{x}_n = \mathbf{y}$ of vertices such that each pair $(\mathbf{x}_{i-1}, \mathbf{x}_i)$, $i = 1, \cdots, n$, of vertices presents an edge.

The rest of the paper is organized as follows. In Section 2, we provide the main results and Section 3 is devoted to the proofs.

II. Main results

Let $\mathbb{N}$ be the non-negative integers including 0, and denote by $\ell := \min\{k \in \mathbb{N} : \alpha \leq \beta^{k+1}\}$. Let $|S|$ be the size of a set S. The connected component containing the node $\mathbf{x} \in \Omega_N$ is denoted by $C(\mathbf{x})$. Since, for every node $\mathbf{x}$, $|C(\mathbf{x})|$ has the same distribution, it suffices to consider only $|C(\mathbf{0})|$. The percolation probability is defined as

$$\theta(\alpha, \beta) := P(|C(\mathbf{0})| = \infty), \quad (5)$$

and the critical percolation value is defined as

$$\alpha_c(\beta) := \inf\{\alpha \geq 0 : \theta(\alpha, \beta) > 0\}. \quad (6)$$

The following theorem characterizes the phase transition for this model.

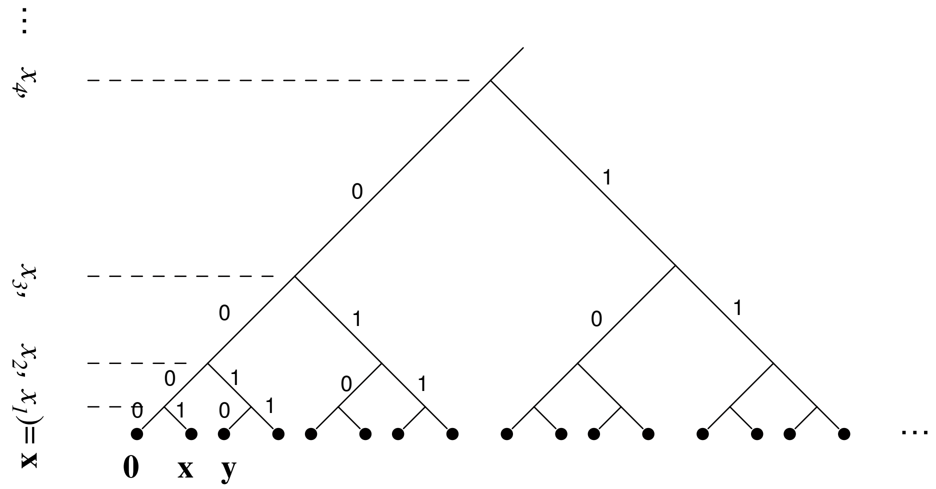

Fig. 1. An illustration of hierarchical lattice Ω_2 of order 2. The distances between three vertices $\mathbf{0} = (0, 0, 0, \cdots)$, $\mathbf{x} = (1, 0, 0, \cdots)$ and $\mathbf{y} = (0, 1, 0, \cdots)$ are $d(\mathbf{0}, \mathbf{x}) = 1$ and $d(\mathbf{0}, \mathbf{y}) = d(\mathbf{x}, \mathbf{y}) = 2$.

Theorem 1. ([16])

(i) *If $\beta \le N$, then $\alpha_c(\beta) = 0$;*
(ii) *If $N < \beta < N^2$, then $0 < \alpha_c(\beta) < \infty$;*
(iii) *If $\beta \ge N^2$, then $\alpha_c(\beta) = \infty$.*

The uniqueness of infinite component is established in the following result.

Theorem 2. ([17]) *For $0 \le \alpha < \infty$ and $0 < \beta < \infty$, there is at most one infinite component almost surely.*

Before presenting our main result, we give some notations. For any vertex $\mathbf{x} \in \Omega_N$, define $B_r(\mathbf{x})$ the ball of radius r around $\mathbf{x}$, that is, $B_r(\mathbf{x}) = \{\mathbf{y} : d(\mathbf{x}, \mathbf{y}) \le r\}$. From this definition we make the following observations. Firstly, for any $\mathbf{x} \in \Omega_N$, $B_r(\mathbf{x})$ contains N^r vertices. Secondly, $B_r(\mathbf{x}) = B_r(\mathbf{y})$ if $d(\mathbf{x}, \mathbf{y}) \le r$. Finally, for any $\mathbf{x}$, $\mathbf{y}$ and r, we either have $B_r(\mathbf{x}) = B_r(\mathbf{y})$ or $B_r(\mathbf{x}) \cap B_r(\mathbf{y}) = \emptyset$.

For a set S of vertices, denote by $\bar{S} = \Omega_N \backslash S$ its complement. Let $C_n(\mathbf{x})$ be the cluster of vertices that are connected to $\mathbf{x}$ by a path using only vertices within $B_n(\mathbf{x})$. For disjoint sets $S_1, S_2 \subseteq \Omega_N$, we denote by $S_1 \leftrightarrow S_2$ the event that at least one edge joins a vertex in S_1 to a vertex in S_2. $S_1 \not\leftrightarrow S_2$ means the event that such an edge does not exist. Let $C_n^m(\mathbf{x})$ be the largest clusters in $B_n(\mathbf{x})$. If there are more than one such clusters, just take any one of them as $C_n^m(\mathbf{x})$. It is clear that $|C_n^m(\mathbf{x})| = \max_{\mathbf{y} \in B_n(\mathbf{x})} |C_n(\mathbf{y})|$. Our main result is the following.

Theorem 3. *Suppose that α and β are such that $\theta := \theta(\alpha, \beta) > 0$, i.e., $0 < \beta < N^2$. Therefore, for every $\varepsilon > 0$,*

$$\lim_{k \to \infty} P(|C_k^m(\mathbf{0})| > (\theta - \varepsilon) N^k) = 1. \tag{7}$$

III. Proof of Theorem 3

In this section, we provide the complete proof of Theorem 3, which is similar to that of Theorem 5 in [12]. We will need the following lemmas.

Lemma 1. *For any constant $K > 0$,*

$$1_{\{\{|C(\mathbf{0})| = \infty\} \cap \{|C_n(\mathbf{0})| < K(\beta/N)^n\}\}} \to 0, \tag{8}$$

almost surely as $n \to \infty$.

Proof. By multiplication principle, we only need to show that the conditional probability

$$P\left(|C(\mathbf{0})| = \infty \,\middle|\, \left|\left\{n \in \mathbb{N} : |C_n(\mathbf{0})| \le K\left(\frac{\beta}{N}\right)^n\right\}\right| = \infty\right) = 0. \tag{9}$$

First, we assume that $\beta > N$. Let n_1 be the smallest n for which $C_n(\mathbf{0}) \le K(\beta/N)^n$. If $C_{n_i}(\mathbf{0}) \not\leftrightarrow \overline{B_{n_i}(\mathbf{0})}$, then $n_{i+1} = n_i$. If $C_{n_i}(\mathbf{0}) \leftrightarrow \overline{B_{n_i}(\mathbf{0})}$, then n_{i+1} is the smallest $n > n_i$ such that $C_{n_i}(\mathbf{0}) \not\leftrightarrow \overline{B_n(\mathbf{0})}$ and $|C_n(\mathbf{0})| \le K(\beta/N)^n$.

Note that $|C_{n_i}(\mathbf{0})| \le K(\beta/N)^{n_i}$, and then we have

$$\begin{aligned}
& P(C_{n_i}(\mathbf{0}) \leftrightarrow \overline{B_{n_i}(\mathbf{0})}) \\
\le\ & P\left(C_{n_i}(\mathbf{0}) \leftrightarrow \overline{B_{n_i}(\mathbf{0})} \,\middle|\, |C_{n_i}(\mathbf{0})| = \left\lfloor K\left(\frac{\beta}{N}\right)^{n_i} \right\rfloor\right) \\
=\ & 1 \\
& - \prod_{j=n_i+1}^{\infty} (1 - \min\{\alpha\beta^{-j}, 1\})^{K(\beta/N)^{n_i}(N-1)N^{j-1}}
\end{aligned} \tag{10}$$

If $n_i + 1 \le \ell$, then we have a trivial bound, i.e., the above probability less than 1. If $n_i + 1 > \ell$, then

$$\begin{aligned}
& P(C_{n_i}(\mathbf{0}) \leftrightarrow \overline{B_{n_i}(\mathbf{0})}) \\
\le\ & 1 - \prod_{j=n_i+1}^{\infty} (1 - \alpha\beta^{-j})^{K(\beta/N)^{n_i}(N-1)N^{j-1}} \\
<\ & 1 \\
& - \exp\left\{-\frac{1}{\beta^j \alpha^{-1} - 1}\left(K\left(\frac{\beta}{N}\right)^{n_i}(N-1)N^{j-1}\right)\right\} \\
<\ & 1 - \exp\left\{-\alpha K \frac{N-1}{\beta - N}\right\},
\end{aligned} \tag{11}$$

involving the inequality $\exp\left(-\frac{1}{x-1}\right) < 1 - \frac{1}{x}$ as in [16]. The right-hand side of (11) is strictly less than 1 and is independent of n_i. Recall that $\{C_{n_i}(\mathbf{0}) \leftrightarrow \overline{B_{n_i}(\mathbf{0})}\}_{i \ge 1}$ are independent events. If there are infinitely many different n_i, then there must be some n_i for which $\{C_{n_i}(\mathbf{0}) \not\leftrightarrow \overline{B_{n_i}(\mathbf{0})}\}$ holds. If there are only finitely many different n_i, then by definition the same thing holds. The above comments clearly yield (9) for any $\beta > N$. By monotonicity, we know that (9) holds for any $0 < \beta < N^2$. □

Lemma 2. *For any constant $K > 0$. The fraction of the vertices in $B_n(\mathbf{0})$ which are in a cluster of size at least $K(\beta/N)^n$, converges to θ almost surely as $n \to \infty$.*

Proof. First assume that $\beta > N$. We will use the random embedding of the hierarchical lattice in $\mathbb{Z}$ [17]. From the ergodic theorem we obtain for any $k > 0$,

$$\frac{1}{2N^n+1}\sum_{\mathbf{x}=-N^n}^{N^n} 1_{\{\cap_{j=k}^{\infty}\{|C_j(\mathbf{x})|>K(\beta/N)^j\}\}} \to P(\cap_{j=k}^{\infty}\{|C_j(\mathbf{x})| > K(\beta/N)^j\}), \quad (12)$$

almost surely as $n \to \infty$.

By virtue of Lemma 1, the right-hand side of (12) increases to θ as $k \to \infty$. Hence, we have

$$A(n) := \frac{1}{2N^n+1}\sum_{\mathbf{x}=-N^n}^{N^n} 1_{\{|C_n(\mathbf{x})|>K(\beta/N)^n\}} \to \theta, \quad (13)$$

almost surely as $n \to \infty$. By our construction in [17], the collection vertices $\{-N^n, -N^n+1, -N^n+2, \ldots, N^n\}$ contains the image under the embedding of the ball $B_n(\mathbf{0})$ and this image contains a fraction $N^n/(2N^n+1)$ of those vertices. The events $\{|C_n(\mathbf{x})| > K(\beta/N)^n\}$ are independent for vertices in different n-balls, and then

$$A_1(n) := \frac{1}{2N^n+1}\sum_{\mathbf{x}\in B_n(\mathbf{0})} 1_{\{|C_n(\mathbf{x})|>K(\beta/N)^n\}} \quad (14)$$

and $A_2(n) := A(n) - A_1(n)$ are independent.

It is easy to see that $A_1(n)$ and $A_2(n)$ are bounded above by 1 and have asymptotically the same mean. By (13) we obtain that

$$\frac{1}{N^n}\sum_{\mathbf{x}\in B_n(\mathbf{0})} 1_{\{|C_n(\mathbf{x})|>K(\beta/N)^n\}} \to \theta, \quad (15)$$

almost surely as $n \to \infty$ for $\beta > N$. When $\beta \le N$, we have $\theta = 1$ by Theorem 1. It is direct to check that the above derivations still hold. □

Proof of Theorem 3. From Lemma 2 we have for every $K > 0$ and $\varepsilon > 0$

$$P\left(\left|\left\{\mathbf{x}\in B_n(\mathbf{0}) : |C_n(\mathbf{x})| > K\left(\frac{\beta}{N}\right)^n\right\}\right| > (\theta-\varepsilon)N^n\right) > 1-\varepsilon, \quad (16)$$

for n large enough. A ball $B_n(\mathbf{y})$ is said to be good if and only if

$$\left|\left\{\mathbf{x}\in B_n(\mathbf{y}) : |C_n(\mathbf{x})| > K\left(\frac{\beta}{N}\right)^n\right\}\right| > (\theta-\varepsilon)N^n. \quad (17)$$

In what follows, we condition on the event that all n-balls in $B_{n+1}(\mathbf{0})$ are good. The probability of this event is bounded below by $(1-\varepsilon)^N \ge 1 - N\varepsilon$.

For each good ball $B_n(\mathbf{y})$, $\mathbf{y} \in \Omega_N$, we make a partition of the set

$$B_n'(\mathbf{y}) := \left\{\mathbf{x}\in B_n(\mathbf{y}) : |C_n(\mathbf{x})| > K\left(\frac{\beta}{N}\right)^n\right\} \quad (18)$$

into super vertices. For $\mathbf{x} \in B_n'(\mathbf{y})$ we make a partition of $C_n(\mathbf{x})$ into $\lfloor |C_n(\mathbf{x})|/\lceil K(\beta/N)^n\rceil \rfloor$ super vertices, all of which have size at least $\lceil K(\beta/N)^n \rceil$. Denote by V_n the collection of super vertices that contain vertices in $B_{n+1}(\mathbf{0})$. For K large enough, if $B_n(\mathbf{y})$ is good, then V_n contains at least $(\theta-\varepsilon)N^n/\lceil 2K(\beta/N)^n\rceil \ge (\theta-\varepsilon)N^n/(3K(\beta/N)^n)$ super vertices.

As in [12], we construct a new N-partite graph on V_n as follows. Let V_n be the vertex set and let E_n be the edge sets. Choose $\lceil K(\beta/N)^n\rceil$ original vertices from every super vertex in V_n. Choosing those vertices may be done in any way that is independent of the presence of edges of length $\ge n+1$. Denote these sets by A_n. The super vertices $x, y \in V_n$ are connected by an edge if there is at least one edge in the original graph which is present between vertices that make up the sets in A_n corresponding to x and y, respectively, and if the original vertices that make up x and y are at distance $n+1$ of each other. Otherwise, there is no edge between the super vertices. Since $\beta < N^2$, $(\theta-\varepsilon)N^n/(3K(\beta/N)^n)$ tends to infinity as $n \to \infty$. Hence, the expected degree of a vertex in V_n is larger than

$$\begin{aligned}&\frac{(N-1)(\theta-\varepsilon)N^n}{3K(\beta/N)^n}\left(1-\left(1-\frac{\alpha}{\beta^{n+1}}\right)^{K^2(\beta/N)^{2n}}\right)\\ > \;&\frac{(N-1)(\theta-\varepsilon)N^n}{3K(\beta/N)^n}\\ &\cdot\left(1-\exp\left\{-\frac{\alpha}{\beta^{n+1}}K^2\left(\frac{\beta}{N}\right)^{2n}\right\}\right),\end{aligned} \quad (19)$$

which exceeds $\lambda := (N-1)(\theta-\varepsilon)\alpha K/(6\beta)$ for large n. Clearly, the parameter λ can be mae large enough by choosing K large enough.

The N-partite graph (V_n, E_n) is an inhomogeneous random graphs; see [7] for backgrounds. The degree of every super vertex is asymptotically Poisson distributed, with mean bounded below by λ. The unique largest cluster of such an N-partite graph contains a fraction η of the super vertices almost surely as $n \to \infty$, where η is the largest solution of the equation

$$1 - \eta = e^{-\lambda\eta}. \quad (20)$$

We can choose λ sufficiently large and $\eta > 1-\varepsilon$. Hence, for each $\varepsilon > 0$ and large n, the graph (V_n, E_n) contains a unique giant cluster containing a fraction $(1-\varepsilon)N$ of the vertices in V_n with probability at least $1-\varepsilon$.

Since we have conditioned on the event that all n-balls in $B_{n+1}(\mathbf{0})$ are good, the fraction of vertices in $B_{n+1}(\mathbf{0})$ that are part of vertices in V_n is larger than $\theta - 2\varepsilon$. Accordingly, conditional on the same event, the largest cluster in $B_{n+1}(\mathbf{0})$ is at least of size $(\eta-\varepsilon)(\theta-2\varepsilon)N^n > (1-2\varepsilon)(\theta-2\varepsilon)N^n$ with probability at least $1-\varepsilon$. By the multiplication principle, we have the probability that the largest cluster in $B_{n+1}(\mathbf{0})$ is at least of size $(1-2\varepsilon)(\theta-2\varepsilon)N^n$ is bounded below by

$(1-\varepsilon)(1-N\varepsilon)$. Now, choosing $\varepsilon' < \varepsilon/\max\{4, N+1\}$, we finally obtain that

$$P(|C_n^m(\mathbf{0})| > (\theta - \varepsilon')N^n) \geq 1 - \varepsilon'. \qquad (21)$$

The proof then readily follows. □

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REFERENCES

[1] M. Aizenman and C. M. Newman, Discontinuity of the percolation density in one-dimensional $1/|x-y|^2$ percolation models. *Comm. Math. Phys.*, 107(1986) 611–647.
[2] S. R. Athreya and J. M. Swart, Survival of contact processes on the hierarchical group. *Probab. Theory Relat. Fields*, 147(2010) 529–563.
[3] A. L. Barabási and E. Ravaz, Hierarchical organization in complex networks. *Phys. Rev. E*, 67(2003) 026112.
[4] I. Benjamini and O. Schramm, Percolation beyond $\mathbb{Z}^d$, many questions and a few answers. *Elect. Commun. Probab.*, 1(1996) 71–82.
[5] N. Berger, Transience, recurrence and critical behavior for long-range percolation. *Comm. Math. Phys.*, 226(2002) 531–558.
[6] M. Biskup, On the scaling of chemical distance in long-range percoaltion models. *Ann. Probab.*, 32(2004) 2938–2977.
[7] B. Bollobás, S. Janson and O. Riordan, The phase transition in inhomogeneous random graphs. *Random Struct. Algorithms*, 31(2007) 3–122.
[8] D. Coppersmith, D. Gamarnik and M. Sviridenko, The diameter of a long-range percolation graph. *Random Struct. Algorithms*, 21(2002) 1–13.
[9] D. A. Dawson and L. G. Gorostiza, Percolation in an ultrametric space. *arXiv:1006.4400v2*, 2011.
[10] D. A. Dawson and L. G. Gorostiza, Percolation in a hierarchical random graph. *Comm. Stochastic Analysis*, 1(2007) 29–47.
[11] G. Grimmett, *Percolation*. New York: Springer, 1999.
[12] V. Koval, R. Meester and P. Trapman, Long-range percolation on the hierarchical lattice. *arXiv:1004.1251v1*, 2010.
[13] R. Rammal, G. Toulouse and M. A. Virasoro, Ultrametricity for physicists. *Rev. Mod. Phys.*, 58(1986) 765–788.
[14] P. Schneider, *Nonarchimedean Functional Analysis*. New York: Springer, 2002.
[15] L. S. Schulman, Long range percolation in one dimension. *J. Phys. A: Math. Gen.*, 16(1983) L639–L641.
[16] Y. Shang, Percolation in a hierarchical lattice. Submitted to *Ann. Probab.*
[17] Y. Shang, Uniqueness of the infinite component for percolation on a hierarchical lattice. Submitted to *Random Struct. Algorithms*.
[18] Y. Shang, The giant component in a random subgraph of a weak expander. *Int. J. Math. Comput. Sci.*, 7(2011) 95–99.
[19] Y. Shang, Leader-following consensus problems with a time-varying leader under measurement noises. *arXiv:0909.4349*, to appear in *Adv. Dyn. Syst. Appl.*
[20] J. Shen, Cucker-Smale flocking under hierarchical leadership. *SIAM J. Appl. Math.*, 68(2007) 694–719.
[21] P. Trapman, The growth of the infinite long-range percolation cluster. *Ann. Probab.*, 38(2010) 1583–1608.

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