Tensorial Transformations of Double Gai sequence spaces

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Abstract — The precise form of tensorial transformations acting on a given collection of infinite matrices into another ; for such classical ideas connected with the summability field of double gai sequence spaces. In this paper the results are impose conditions on the tensor $g$ so that it becomes a tensorial transformations from the metric space $\chi^2$ to the metric space $\mathbb{C}$

Keywords — tensorial transformations, double gai sequences, double analytic, dual.

I. Introduction

Let $(x_{mn})$ be a double sequence of real or complex numbers. Then the series $\sum_{m,n=1}^{\infty} x_{mn}$ is called a double series. The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is said to be convergent if and only if the double sequence $(s_{mn})$ is convergent, where

$$s_{mn} = \sum_{i,j=1}^{m,n} x_{ij}(m, n = 1, 2, 3, \ldots)$$

see[1]. We denote $\omega^2$ as the class of all complex double sequences $(x_{mn})$. Let $\Omega$ be the family of infinite matrices endowed with usual operations of pointwise addition and scalar multiplication.

A sequence $x = (x_{mn}) \in \Omega$ is said to be double analytic if

$$\sup_{mn} |x_{mn}|^{1/m+n} < \infty.$$ 

The vector space of all prime sense double analytic sequences are usually denoted by $\Lambda^2$. A sequence $x = (x_{mn}) \in \Omega$ is called a double gai sequence if

$$(m+n)! |x_{mn}|^{1/m+n} \to 0 	ext{ as } m+n \to \infty.$$ 

We denote $\chi^2$ as the class of prime sense double gai sequences. The spaces $\Lambda^2$ and $\chi^2$ are metric spaces with metrics

$$d(x, y) = \sup_{mn} \left\{ \left( |x_{mn} - y_{mn}|^{1/m+n} : mn = 1, 2, \ldots \right) \right\}$$

for all $x = (x_{mn})$ and $y = (y_{mn})$ in $\Lambda^2$ and

$$\tilde{d}(x, y) = \sup_{mn} \left\{ ((m+n)! |x_{mn} - y_{mn}|^{1/m+n} : m, n = 1, 2, \ldots \right\}$$

for all $x = (x_{mn})$ and $y = (y_{mn})$ in $\chi^2$, respectively.

$$\ell^2 = \{ x = (x_{mn}) \in \Omega : \sum |x_{mn}| < \infty \} .$$

The space $\chi^2$ can be then regarded as the space of gai functions of two variables equipped with the topology of uniform convergence on compact sets $C \times C$, where $C$ is the complex plane. These spaces are known to be Frechet spaces.

For any double sequence $x = (x_{mn})$ the $(m, n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by

$$x^{[m,n]} = \sum_{i,j=1}^{m,n} x_{ij} \zeta_{ij} \text{ for all } m, n \in \mathbb{N},$$

where

$$\zeta_{mn} = \begin{cases} 0, & 0, \ldots, 0, 0, \ldots \cr 0, & 0, \ldots, 0, 0, \ldots \cr \ldots \cr 0, & 0, \ldots, 1, -1, 0, \ldots \cr 0, & 0, \ldots, 0, 0, \ldots \cr \end{cases}$$

with 1 in the $(m, n)^{th}$ and -1 $(m+1, n+1)^{th}$ position and zero other wise.

An infinite matrix shall be denoted by $x = (x_{mn})$

$$\begin{pmatrix} x_{00} & \ldots & x_{0n} & \ldots \ x_{10} & \ldots & x_{1n} & \ldots \ \vdots & \ddots & \vdots & \ddots \ x_{m0} & \ldots & x_{mn} & \ldots \ \vdots & \ddots & \vdots & \ddots \ \end{pmatrix}$$

where $x_{mn}$’s belong to the field $K$ of scalars. Denote by $N$ the set of all non-negative integers. Thus $\Omega$ is a vector space over $K$. By a matrix space $X$ we mean any subspace $\Omega$. The matrix space generated by $\{ \zeta_{mn} : m, n \in \mathbb{N} \}$ shall be denoted by $\varphi$. If $N \in \mathbb{N}$ and $x \in \Omega$, we define

$$x^N = \sum_{0 < m+n < N} x_{mn} \zeta_{mn}$$

and call it as the $N^{th}$ place section of the matrix $x$. For a matrix space $X$, we define $X'$ by

$$X' = \{ y = (y_{mn}) : y \in \Omega \text{ with } \sum |x_{mn}y_{mn}| < \infty \text{ for all } x \in X \}$$

where

$$\lim_{N \to \infty} \sum_{0 < m+n < N} x_{mn}y_{mn} = \sum_{0 < m+n < N} x_{mn}y_{mn}$$

and term it as the $K$– dual of $X$. Clearly $X'$ is a vector space over $K$ and contains $\varphi$.

We assume that each matrix space $X$ contains $\varphi$ under
this assumption, $X$ and $X'$ form a dual system which express as $(X, X')$. Hence, the weak topology $\sigma (X, X')$, the Mackey topology $\tau (X, X')$, the strong topology $\beta (X, X')$ and so on.

**K− normal and K− perfect matrix spaces:**

A matrix space is called $K$− normal provided $x = (x_{mn}) \in X$ whenever $|x_{mn}| \leq |y_{mn}|$ for $m + n \geq 0$, for some $y = (y_{mn}) \in X$. Clearly $X$ is $K$− normal for any matrix space $X$. A matrix $X$ is said to be $K$− perfect, if $X = (X')'$; observe that $X \subset X'$ is always true.

**II. PRELIMINARIES**

Some initial works on double sequence spaces is found in Bromwich[3]. Later on, it was investigated by Hardy[5], Moricz[7], Moricz and Rheades[8], Basarir and Solanki[2], Tripathy[10], Colak and Turkmenoglu[4], Turkmenoglu[11], Patterson [9] and many others. In this paper we study some of the properties of transformations resulting from a tensor of order four, which relate various matrix spaces. Indeed, if $g = (\chi^{pq}_{mn})$ is a tensor of order four having values in the field of scalars for fixed pair of integers $p, q$ and $m, n$, we assume that its multiplication with any preassigned matrix $y = (y_{pq})$ is defined for all indices $m, n \geq 0$, namely $g.y = \sum_{p+q \geq 0} (\chi^{pq}_{mn} y_{pq} = x_{mn}$

(1)

is well defined for all $m, n \geq 0$. In the following result we impose conditions on the tensor $g$ so that it becomes a tensorial transformation from the metric space $\chi^2$ to the metric space $C$.

**III. MAIN RESULTS**

**A. Theorem**

We have $(\chi^2)' = \Lambda^2$ and $(\Lambda^2)' = \chi^2$. Thus $\chi^2$ and $\Lambda^2$ are $K$− perfect.

**Proof:** We prove only $(\chi^2)' = \Lambda^2$; the proof of $(\Lambda^2)' = \chi^2$ is similar. Now observe that $\Lambda^2 \subset (\chi^2)'$ is obvious. For $(\chi^2) \subset \Lambda^2$, let $x \in (\chi^2)$ and $x \notin \Lambda^2$. For each integer $i \geq 1$ there exist sequences $(m_i)$ and $(n_i)$ (atleast one of which tends to infinity with $i$) such that $|x_{m_in_i}| > \frac{|\chi^{m_in_i}|}{(m_i+n_i)}$

Define the matrix $y$ by

$y_{mn} = \begin{cases} \chi_{m_in_i} & \text{if } m = m_i, n = n_i; \\ 0 & \text{otherwise} \end{cases}$

Thus $y \in \chi^2$. However $\sum |x_{m_n}y_{mn}| = \infty$ and so $x \notin (\chi^2)$, a contradiction. This completes the proof.

**B. Theorem**

Suppose eqn. (1) is true for each $y \in \chi^2$. Then $x = (x_{mn}) \in \chi^2$ if and only if there exists a constant $M > 0$ such that

$|\chi^2_{m_n}| \cdot (p + q)! |\chi^2_{pq}|^{1/(p+q)} < M, \text{ for all } m, n, p, q \in N$. 

(2)

and

$\lim_{m+n \to \infty} (\chi^2_{pq})^{1/(m+n)} = \Lambda^2_{pq} \exists m, n, p, q \geq 0$ (3)

**Proof:** The proof of the sufficiency part is straight forward and is therefore omitted.

For converse, let $x \in C$ where $x = (x_{mn})$ is given by eqn.(1). For $y \in \chi^2$, define the matrix $f = (f_{mn})$ of functionals by $f_{mn} (y) = x_{mn} = \sum_{p+q \geq 0} (\chi^2_{pq})^{1/(p+q)} y_{pq}$.

Since the set $\{ |\chi^2_{m_n}| \cdot (p + q)! |\chi^2_{pq}|^{1/(p+q)} , p + q \geq 1 \}$ is analytic for fixed pair of integers $m, n$; it follows that the functionals $f_{mn}$ are continuous. Moreover, therese functionals are pointwise analytic. Therefore by uniform boundness principle there exists a ball $B(.)$ such that for all $y \in B(.)$

$|f_{mn} (y)| \leq M, \text{ for all } m, n \geq 0$ where $M$ is a constant and all $y$ with $|y| \leq \epsilon$. Choosing $y$ to be the matrices $y^{pq}$ for $p + q \geq 0$ respectively, where $y^{pq} = (\epsilon_{ij})$

$\epsilon_{ij} = \begin{cases} \frac{\epsilon^{p+q}}{(p+q)!} & \text{if } i = p, j = q; \\ 0, & \text{otherwise} \end{cases}$

when $p + q > 0$ and $y^{00} = (\epsilon_{ij}) = (\epsilon_{ij}) \cdot \chi^2 = \epsilon, \epsilon_{ij} = 0, i + j \geq 1$. We obtain $|\chi^2_{m_n}| \epsilon \leq M$ for all $m, n \geq 0$ and $|\epsilon^{p+q} |\leq M, \text{ for all } m, n \geq 0$ and $|p + q| \geq 1$. Thus $|\chi^2_{m_n}| \leq M, (p + q)! |\chi^2_{pq}|^{1/(p+q)} \leq M^{1/p+q} \times \frac{1}{(p+q)!}$ for $m + n \geq 0$ and $p + q > 0$.

Since $M^{1/p+q} \times \frac{1}{(p+q)!} \leq M$ for $p + q > 0$ it follows that

$|\chi^2_{m_n}| \cdot (\chi^2_{pq})^{1/(m+n)} \leq \frac{1}{(p+q)!} \frac{1}{\epsilon} \times M^{1/p+q}$ for $m + n \geq 0$ and $p + q > 0$.

This proves eqn. (2). The condition of eqn. (3) obviously follows.

This completes the proof.

**C. Theorem**

Let eqn.(1) be true for $y \in \ell^2$. Then $x = (x_{mn}) \in \chi^2$ if and only if

$\left( (m + n)! |\chi^2_{pq}|^{1/m+n} \rightarrow 0 \right. \text{ as } m + n \rightarrow \infty$ (4)

uniformly in $p$ and $q$.

**Proof:** Sufficiency follows by straightforward calculations. For necessity, assume that eqn. (4) is not true. Then for $\epsilon > 0$, and any $N \in N$, there exist integers $m, n$ and $p, q$ such that $m + n > N$ and

$\left( (m + n)! |\chi^2_{pq}|^{1/m+n} \geq \epsilon \right.$ (5)

Since a maps $\ell^2$ in $\chi^2$, it follows a transforms $\ell^2$ into itself and therefore
Then we write
\[ w_{mn} = \sup_{p+q \geq 0} |x^{pq}_{mn}|, \text{ we can find a constant } K > 0 \text{ such that} \]
\[ |w_{mn}| \leq \frac{K}{2} \text{ for all } m, n \geq 0 \quad (6) \]
We also have
\[ \left( (m + n)! \right)^{1/m+n} \to 0 \text{ as } m + n \to \infty \quad (7) \]
for each fixed \( p \) and \( q \). By eqn. (5) we can find \( m_1n_1 \) and \( p_1q_1 \) such that
\[ (m_1 + n_1)! |x^{p_1q_1}_{m_1n_1}|^{1/m_1+n_1} > \epsilon/2 \quad (8) \]
Now from the relations eqn. (5) to eqn. (7), choose \( m_2, n_2 \) sufficiently large with \( m_2 + n_2 > m_1 + n_1 \) and \( p_2, q_2 \) with \( p_2 + q_2 > p_1 + q_1 \) such that
\[ \left( \frac{K}{2m_2+n_2} \right) < \left( \frac{\epsilon}{8} \right) \times \frac{1}{(m_1 + n_1)!} \quad (9) \]
and
\[ (m_2 + n_2)! |x^{p_2q_2}_{m_2n_2}|^{1/m_2+n_2} > \epsilon/2 \quad (10) \]
Proceeding in this way, we get sequences \( \{m_k\}, \{n_k\}, \{p_k\} \) and \( \{q_k\} \) with \( m_k + n_k > m_{k-1} + n_{k-1}, p_k + q_k > p_{k-1} + q_{k-1} ; k \geq 2 \) such that
\[ \left| \frac{K}{2m_k+n_k} \right| < \left( \frac{\epsilon}{8(k-1)} \right) \times \frac{1}{(m_{k-1} + n_{k-1})!} \quad (12) \]
and
\[ (m_k + n_k)! |x^{p_kq_k}_{m_kn_k}|^{1/m_k+n_k} > \epsilon/8 \text{ where } 1 \leq j \leq k-1. \quad (13) \]
Let us now introduce the matrix \( y = (y_{pq}) \in \ell^2 \) as follows
\[ y_{pq} = \begin{cases} \frac{1}{2^{m_k+n_k}}, & \text{if } p = p_k, q = q_k, k = 1, 2, 3, \ldots; \\ 0, & \text{otherwise} \end{cases} \]
It is easily verified that \( x = (x_{mn}) \notin x^2 \) where
\[ x_{mn} = \sum_{p+q \geq 0} (x^{pq}_{mn})_{pq} \] for all \( m, n \geq 0 \)
Indeed,
\[ \left( (m_k + n_k)! \right)^{1/m_k+n_k} \geq \frac{1}{2} \left( (m_k + n_k)! \right)^{1/m_k+n_k} - \]
\[ - \left( (m_k + n_k)! \right)^{1/m_k+n_k} \sum_{j<k} \chi^{p_jq_j}_{m_kn_k} y_{p_jq_j} \]
\[ - \left( (m_k + n_k)! \right)^{1/m_k+n_k} \sum_{j>k} \chi^{p_jq_j}_{m_kn_k} y_{p_jq_j} > \frac{\epsilon}{4} - \frac{\epsilon}{4} = \frac{\epsilon}{8} \]
for all \( k \geq 1 \). Hence it is a contradiction and the result follows.

Similarly, we can prove the following result

**D. Theorem**

Let eqn.(1) be true for \( y \in \ell^2 \). Then \( x = (x_{mn}) \in \Lambda^2 \) if and only if
\[ \left( (m + n)! \right)^{1/m+n} \leq M, \]
uniformly in \( p, q \) and \( m, n \); where \( M \) is a positive constant.

**IV. Conclusion**

Tensorial transformation of classical ideas connected with the field of double gai sequence spaces.

**Acknowledgment**

I wish to thank the referees for their several remarks and valuable suggestions that improved the presentation of the paper.

**References**


[4] R.Colak and A.Turkmenoglu, The double sequence spaces \( \ell^2_\infty (p), c_0^2 (p) \) and \( c^2 (p) \), (to appear).


