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Tensorial Transformations of Double Gai sequence spaces

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Abstract—The precise form of tensorial transformations acting on a given collection of infinite matrices into another ; for such classical ideas connected with the summability field of double gai sequence spaces. In this paper the results are impose conditions on the tensor g so that it becomes a tensorial transformations from the metric space χ^2 to the metric space $\mathbb{C}$

Keywords—tensorial transformations,double gai sequences , double analytic,dual.

I. Introduction

Let (x_{mn}) be a double sequence of real or complex numbers. Then the series $\sum_{m,n=1}^{\infty} x_{mn}$ is called a double series. The double series $\sum_{m,n=1}^{\infty} x_{mn}$is said to be convergent if and only if the double sequence (s_{mn}) is convergent, where

$$s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n = 1, 2, 3, ...)$$

see[1]). We denote w^2 as the class of all complex double sequences (x_{mn}) . Let Ω be the family of infinite matrices endowed with usual operations of pointwise addition and scalar multiplication.
A sequence $x = (x_{mn}) \in \Omega$ is said to be double analytic if

$$sup_{mn} |x_{mn}|^{1/m+n} < \infty.$$

The vector space of all prime sense double analytic sequences are usually denoted by Λ^2. A sequence $x = (x_{mn}) \in \Omega$ is called a double gai sequence if

$$((m + n)! |x_{mn}|)^{1/m+n} \to 0 \text{ as } m + n \to \infty.$$

We denote χ^2 as the class of prime sense double gai sequences. The spaces Λ^2 and χ^2 are metric spaces with metrics

$$d(x, y) = sup_{mn} \left\{ \left(|x_{mn} - y_{mn}|^{1/m+n} \right) : mn = 1, 2, \cdots \right\}$$

for all $x = (x_{mn})$ and $y = (y_{mn})$ in Λ^2 and

$$\tilde{d}(x, y) = sup_{mn} \left\{ ((m + n)! |x_{mn} - y_{mn}|)^{1/m+n} : m, n = 1, 2, \cdots \right\}$$

for all $x = (x_{mn})$ and $y = (y_{mn})$ in χ^2, respectively.

$$\ell^2 = \{x = (x_{mn}) \in \Omega : \sum\sum |x_{mn}| < \infty\}.$$

The space χ^2 can be then regarded as the space of gai functions of two variables equipped with the topology of uniform convergence on compact sets $C \times C$, where C is the complex plane. These spaces are known to be Frechet spaces.

For any double sequence $x = (x_{mn})$ the $(m, n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by

$$x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij} \zeta_{ij} \text{ for all } m, n \in \aleph,$$

where

$$\zeta_{mn} = \begin{pmatrix} 0, & 0, & ...0, & 0, & ... \\ 0, & 0, & ...0, & 0, & ... \\ . \\ . \\ . \\ 0, & 0, & ...1, & -1, & 0, & ... \\ 0, & 0, & ...0, & 0, & ... \end{pmatrix}$$

with 1 in the $(m, n)^{th}$ and -1 $(m + 1, n + 1)^{th}$ position and zero other wise.

An infinite matrix shall be denoted by $x = (x_{mn})$

$$x = \begin{pmatrix} x_{00}, & x_{01}, & ...x_{0n}, & ... \\ x_{10}, & x_{11}, & ...x_{1n}, & ... \\ . \\ . \\ . \\ x_{m0}, & x_{m1}, & ...x_{mn}, & ... \\ . \\ . \\ . \end{pmatrix}$$

where x_{mn}'s belong to the field K of scalars. Denote by N the set of all non-negative integers. Thus Ω is a vector space over K. By a matrix space X we mean any subspace Ω. The matrix space generated by $\{\zeta_{mn} : m, n \in N\}$ shall be denoted by φ. If $N \in N$ and $x \in \Omega$, we define

$$x^N = \sum\sum_{0<m+n<N} x_{mn} \zeta_{mn}$$

and call it as the N^{th}place section of the matrix x. For a matrix space X, we define X' by

$$X' = \{y = (y_{mn}) : y \in \Omega\, with \sum\sum |x_{mn} y_{mn}| < \infty\, forall\, x \in X\}$$

where $\sum\sum x_{mn} y_{mn} = lim_{N\to\infty} \sum\sum_{0<m+n<N} x_{mn} y_{mn}$ and term it as the $K-$ dual of X. Clearly X' is a vector space over K and contains φ.

We assume that each matrix space X contains φ under

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this assumption, X and $X^{'}$ form a dual system which express as $\left(X, X^{'}\right)$. Hence, the weak topology $\sigma\left(X, X^{'}\right)$, the Mackey topology $\tau\left(X, X^{'}\right)$, the strong toplogy $\beta\left(X, X^{'}\right)$ and so on.

$K-$ normal and $K-$ perfect matrix spaces:

A matrix space is called $K-$ normal provided $x = (x_{mn}) \in X$ whenever $|x_{mn}| \leq |y_{mn}|$ for $m + n \geq 0$, for some $y = (y_{mn}) \in X$. Clearly $X^{'}$ is $K-$ normal for any matrix space X. A matrix X is said to be $K-$ perfect, if $X = \left(X^{''}\right) = \left(X^{'}\right)^{'}$; observe that $X \subset X^{''}$ is always true.

II. Preliminaries

Some initial works on double sequence spaces is found in Bromwich[3]. Later on, it was investigated by Hardy[5], Moricz[7], Moricz and Rhoades[8], Basarir and Solankan[2], Tripathy[10], Colak and Turkmenoglu[4], Turkmenoglu[11], Patterson [9] and many others. In this paper we study some of the properties of transformations resulting from a tensor of order four, which relate various matrix spaces. Indeed, if $g = \left(\chi^2\right)^{pq}_{mn}$ is a tensor of order four having values in the field of scalars for fixed pair of integers p, q and m, n, we assume that its multiplication with any preassigned matrix $y = (y_{pq})$ is defined for all indices $m, n \geq 0$, namely

$$g.y = \sum\sum{}_{p+q\geq 0} \left(\chi^2\right)^{pq}_{mn} .y_{pq} = x_{mn} \qquad (1)$$

is well defined for all $m, n \geq 0$. In the following result we impose conditions on the tensor g so that it becomes a tensorial transformation from the metric space χ^2 to the metric space C.

III. Main Results

A. Theorem

We have $\left(\chi^2\right)^{'} = \Lambda^2$ and $\left(\Lambda^2\right)^{'} = \chi^2$.Thus χ^2 and Λ^2 are $K-$ perfect

Proof: We prove only $\left(\chi^2\right)^{'} = \Lambda^2$; the proof of $\left(\Lambda^2\right)^{'} = \chi^2$ is similar. Now observe that $\Lambda^2 \subset \left(\chi^2\right)^{'}$ is obvious.

For $\left(\chi^2\right)^{'} \subset \Lambda^2$, let $x \in \left(\chi^2\right)^{'}$ and $x \notin \Lambda^2$. For each integer $i \geq 1$ there exist sequences (m_i) and (n_i) (atleast one of which tends to infinity with i) such that

$$|x_{m_i n_i}| > \frac{i^{2(m_i+n_i)}}{(m_i+n_i)!}$$

Define the matrix y by

$$y_{mn} = \begin{cases} i^{-m_i-n_i}, & \text{if } m = m_i, n = n_i; \\ 0, & \text{otherwise} \end{cases}$$

Thus $y \in \chi^2$. However $\sum\sum |x_{mn}y_{mn}| = \infty$ and so $x \notin \left(\chi^2\right)^{'}$, a contradiction. This completes the proof.

B. Theorem

Suppose eqn. (1) is true for each $y \in \chi^2$. Then $x = (x_{mn}) \in C$ if and only if there exists a constant $M > 0$ such that

$$\left|\chi^2\right|^{00}_{mn}; \left((p+q)! \left|\chi^2\right|^{pq}_{mn}\right)^{1/p+q} \leq M,\ for\,all\, m, n, p, q \in N, \qquad (2)$$

and

$$lim_{m+n\to\infty} \left(\chi^2\right)^{pq}_{mn} = \Lambda^2_{pq}\, exists\, for\, every\, p, q \geq 0 \qquad (3)$$

Proof: The proof of the sufficiency part is straight forward and is therefore omitted.

For converse, let $x \in C$ where $x = (x_{mn})$ is given by eqn.(1). For $y \in \chi^2$, define the matrix $f = (f_{mx})$ of functionals by

$$f_{mx}(y) = x_{mn} = \sum\sum{}_{p+q\geq 0} \left(\chi^2\right)^{pq}_{mn} y_{pq}.$$

Since the set

$$\left\{\left|\chi^2\right|^{00}_{mn}, \left((p+q)! \left|\chi^2\right|^{pq}_{mn}\right)^{1/p+q}, p+q \geq 1\right\}$$

is analytic for fixed pair of integers m, n; it follows that the functionals $f^{'}_{mx}$ are continuous. Moreover, therese functionals are pointwise analytic. Therefore by uniform boundness principle there exists a ball $B_\epsilon(z)$ such that for all $y \in B_\epsilon(z)$.

$$|f_{mx}(y)| \leq M,\ for\,all\, m, n \geq 0$$

where M is a constant and all y with $|y| \leq \epsilon$. Choosing y to be the matrices y^{pq} for $p+q \geq 0$ respectively, where $y^{pq} = (\epsilon_{ij})$

$$\epsilon_{ij} = \begin{cases} \frac{\epsilon^{p+q}}{(p+q)!}, & \text{if } i = p, j = q; \\ 0, & \text{otherwise} \end{cases}$$

when $p + q > 0$ and $y^{00} = (\epsilon_{ij})$, $\chi^2_{00} = \epsilon, \epsilon_{ij} = 0, i + j \geq 1$. We obtain $\left|\chi^2\right|^{00}_{mn} \epsilon \leq M$ for all $m, n \geq 0$ and $\left|\chi^2\right|^{pq}_{mn} \frac{\epsilon^{p+q}}{(p+q)!} \leq M$, for all $m, n \geq 0$ and $p + q \geq 1$. Thus

$$\left|\chi^2\right|^{00}_{mn} \leq \frac{M}{\epsilon}, \left((p+q)! \left|\chi^2\right|^{pq}_{mn}\right)^{1/p+q} \leq M^{1/p+q} \times \frac{1}{\epsilon} \times \frac{1}{(p+q)!}$$

for $m + n \geq 0$ and $p + q > 0$.

Since $M^{1/p+q} \times \frac{1}{(p+q)!} \leq M$ for $p + q > 0$ it follows that

$$\left|\chi^2\right|^{00}_{mn}, \left(\left|\chi^2\right|^{pq}_{mn}\right)^{1/p+q} \leq \frac{1}{(p+q)!} \frac{1}{\epsilon} \times M^{1/p+q}\, for\, m+n \geq 0\, and\, p+q > 0.$$

This proves eqn. (2). The condition of eqn. (3) obviously follows.

This completes the proof.

C. Theorem

Let eqn.(1) be true for $y \in \ell^2$. Then $x = (x_{mn}) \in \chi^2$ if and only if

$$\left((m+n)! \left|\chi^2\right|^{pq}_{mn}\right)^{1/m+n} \to 0\, as\, m+n \to \infty \qquad (4)$$

uniformly in p and q.

Proof: Sufficiency follows by straightforward calculations. For necessity, assume that eqn. (4) is not true. Then for $\epsilon > 0$, and any $N \in N$, there exist integers m, n and p, q such that $m + n > N$ and

$$\left((m+n)! \left|\chi^2\right|^{pq}_{mn}\right)^{1/m+n} > \epsilon \qquad (5)$$

Since a maps ℓ^2 in χ^2, it follows a transforms ℓ^2 into itself and therefore

$$sup\left\{\sum\sum_{m+n\geq 0}\left|\chi^2\right|^{pq}_{mn} : p+q\geq 0\right\}\leq M.$$

Then we write

$$w_{mn} = sup_{p+q\geq 0}\left|\chi^2\right|^{pq}_{mn}, \, we\, can\, find\, a\, constant\, K > 0\, such\, that$$

$$|w_{mn}|\leq \frac{K}{2}\, for\, all\, m,n\geq 0 \tag{6}$$

We also have

$$\left((m+n)!\left|\chi^2\right|^{pq}_{mn}\right)^{1/m+n}\to 0\, as\, m+n\to\infty \tag{7}$$

for each fixed p and q. By eqn. (5) we can find $m_1 n_1$ and $p_1 q_1$ such that

$$\left((m_1+n_1)!\left|\chi^2\right|^{p_1q_1}_{m_1n_1}\right)^{1/m_1+n_1} > \epsilon/2 \tag{8}$$

Now from the relations eqn. (5) to eqn. (7), choose m_2, n_2 sufficiently large with $m_2+n_2 > m_1+n_1$ and p_2, q_2 with $p_2+q_2 > p_1+q_1$ such that

$$\left|\frac{K}{2^{m_2+n_2}}\right| < \left(\frac{\epsilon}{8}\right)^{m_1+n_1}\times\frac{1}{(m_1+n_1)!} \tag{9}$$

$$\left((m_2+n_2)!\left|\chi^2\right|^{p_2q_2}_{m_2n_2}\right)^{1/m_2+n_2} > \epsilon/2 \tag{10}$$

and

$$\left(\frac{1}{(m_2+n_2)!}\left|\chi^2\right|^{p_1q_1}_{m_2n_2}\right)^{m_2+n_2} < \frac{\epsilon}{16} \tag{11}$$

Proceeding in this way, we get sequences $\{m_k\}\,\{n_k\}\,,\{p_k\}\; and \;\{q_k\}$ with $m_k + n_k > m_{k-1}+n_{k-1}, p_k+q_k > p_{k-1}+q_{k-1}; k\geq 2$ such that

$$\left|\frac{K}{2^{m_k+n_k}}\right| < \left(\frac{\epsilon}{8(k-1)}\right)^{m_{k-1}+n_{k-1}}\times\frac{1}{(m_{k-1}+n_{k-1})!} \tag{12}$$

$$\left((m_k+n_k)!\left|\chi^2\right|^{p_kq_k}_{m_kn_k}\right)^{1/m_k+n_k} > \epsilon/2 \tag{13}$$

and

$$\left((m_k+n_k)!\left|\chi^2\right|^{p_jq_j}_{m_kn_k}\right)^{1/m_k+n_k} > \epsilon/8k\, where\, 1\leq j\leq k-1. \tag{14}$$

Let us now introduce the matrix $y=(y_{pq})\in\ell^2$ as follows

$$y_{pq}=\begin{cases}\frac{1}{2^{m_k+n_k}}, & \text{if } p=p_k, q=q_k, k=1,2,3,\cdots;\\ 0, & \text{otherwise}\end{cases}$$

It is easily verified that $x=(x_{mn})\notin\chi^2$ where

$$x_{mn}=\sum\sum_{p+q\geq 0}\left(\chi^2\right)^{pq}_{mn} y_{pq}\, for\, all\, m,n\geq 0$$

Indeed, $\left((m_k+n_k)!\left|\chi^2_{m_kn_k}\right|\right)^{1/m_k+n_k}$

$$\geq\frac{1}{2}\left((m_k+n_k)!\left|\chi^2\right|^{p_kq_k}_{m_kn_k}\right)^{1/m_k+n_k} -$$

$$\left((m_k+n_k)!\left|\sum_{j<k}\chi^2\right|^{p_jq_j}_{m_kn_k} y_{p_jq_j}\right)^{1/m_k+n_k} -$$

$$\left((m_k+n_k)!\left|\sum_{j>k}\chi^2\right|^{p_jq_j}_{m_kn_k} y_{p_jq_j}\right)^{1/m_k+n_k}$$

$$> \frac{\epsilon}{4}-\frac{(k-1)\epsilon}{8k}-\frac{\epsilon}{8k}=\frac{\epsilon}{8}$$

for all $k\geq 1$. Hence it is a contradiction and the result follows.

Similarly, we can prove the following result

D. Theorem

Let eqn.(1) be true for $y\in\ell^2$. Then $x=(x_{mn})\in\Lambda^2$ if and only if

$$\left((m+n)!\left|\chi^2\right|^{pq}_{mn}\right)^{1/m+n}\leq M,$$

uniformly in p,q and m,n; where M is a positive constant.

IV. Conclusion

Tensorial transformation of classical ideas connected with the field of double gai sequence spaces.

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References

[1] T.Apostol, Mathematical Analysis, *Addison-wesley , London*, **1**978.
[2] M.Basarir and O.Solancan, On some double sequence spaces, *J. Indian Acad. Math.*, **21(2)** (1999), 193-200.
[3] T.A.I_A. Browmich, An Introduction to the Theory of Infinite Series, *Macmillan Co. Ltd. New York*, **1965**.
[4] R.Colak and A.Turkmenoglu, The double sequence spaces $\ell^2_\infty(p), c^2_0(p)$ and $c^2(p)$, *(to appear)*.
[5] G.H.Hardy, On the convergence of certain multiple series, *Proc. Camb. Phil. Soc.*, **19** (1917), 86-95.
[6] P.K.Kamthan and M.Gupta,sequence spaces and series, *Marcel Dekker, New York, Basel*, **1981**.
[7] F.Moricz, Extention of the spaces c and c_0 from single to double sequences, *Acta. Math. Hungerica*, **57(1-2)**, (1991), 129-136.
[8] F.Moricz and B.E.Rhoades, Almost convergence of double sequences and strong regularity of summability matrices, *Math. Proc. Camb. Phil. Soc.*, **104**, (1988), 283-294.
[9] R.F.Patterson, Analogue of some fundamental theorems of summability theory, *Internat. J. Math. Math. Sci.*, **23(1)**, (2000), 1-9.
[10] B.C.Tripathy, On statistically convergent double sequences, *Tamkang J. Math.*, **34(3)**, (2003), 231-237.
[11] A.Turkmenoglu, Matrix transformation between some classes of double sequences, *Jour. Inst. of math. and Comp. Sci. (Math. Seri.)*, **12(1)**, (1999), 23-31.
[12] A.Wilansky, Summability Through Functional Analysis , *North-Holland, Amsterdam* , **1**984.