

Some new inequalities for eigenvalues of the Hadamard product and the Fan product of matrices

Jing Li and Guang Zhou

Abstract—Let A and B be nonnegative matrices. A new upper bound on the spectral radius $\rho(A \circ B)$ is obtained. Meanwhile, a new lower bound on the smallest eigenvalue $q(A \star B)$ for the Fan product, and a new lower bound on the minimum eigenvalue $q(B \circ A^{-1})$ for the Hadamard product of B and A^{-1} of two nonsingular M -matrices A and B are given. Some results of comparison are also given in theory. To illustrate our results, numerical examples are considered.

Keywords—Hadamard product, Fan product; nonnegative matrix, M -matrix, Spectral radius, Minimum eigenvalue, 1-path cover.

I. INTRODUCTION

$R^{N \times M}$ and N denote the set of all $n \times m$ real matrices and the $\{1, 2, \dots, n\}$, respectively. If $A = (a_{ij}) \in R^{n \times m}$, $B = (b_{ij}) \in R^{n \times m}$ and $a_{ij} - b_{ij} \geq 0$, we say that $A \geq B$, and if $a_{ij} \geq 0$, we say that A is nonnegative. If $A \in R^{n \times n}$ is a nonnegative matrix, the Perron-Frobenius theorem guarantees that $\rho(A) \in \sigma(A)$, where the set $\sigma(A)$ denotes the spectrum of A , $\rho(A)$ denotes the spectral radius of A . \emptyset denotes the empty set.

A matrix A is irreducible if there does not exist a permutation matrix P such that

$$PAP^T = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix},$$

where A_{11} and A_{22} are square matrices, then A is called irreducible. The set $Z_n \subset R^{n \times n}$ is defined by

$$Z_n = \{A = (a_{ij}) \in R^{n \times n} : a_{ij} \leq 0, \text{ if } i \neq j, i, j = 1, \dots, n\}.$$

Let $A = (a_{ij}) \in Z_n$ and suppose $A = sI - B$ with $s \in R$ and $B \geq 0$. Then $s - \rho(B)$ is an eigenvalue of A , every eigenvalue of A lies in the disc $\{z \in C : |z - s| \leq \rho(B)\}$, and hence every eigenvalue λ of A satisfies $Re\lambda \geq s - \rho(B)$. In particular, a matrix $A \in Z_n$ is called an M -matrix if $s \geq \rho(B)$. If $s > \rho(B)$ we call A is nonsingular M -matrix, and denote the class of nonsingular M -matrices by M_n .

Let $A = (a_{ij}) \in Z_n$, we denote $\min\{Re(\lambda) : \lambda \in \sigma(A)\}$ by $q(A)$, $q(A)$ is called the minimum eigenvalue of A .

The Hadamard product of $A = (a_{ij}) \in R^{n \times n}$ and $B = (b_{ij}) \in R^{n \times n}$ is defined by $A \circ B = (a_{ij}b_{ij}) \in R^{n \times n}$. Let $A = (a_{ij}), B = (b_{ij}) \in R^{n \times n}$, the Fan product of A and B is denoted by $A \star B = C = (c_{ij}) \in R^{n \times n}$, and is defined by

$$c_{ij} = \begin{cases} -a_{ij}b_{ij}, & \text{if } i \neq j, \\ a_{ii}b_{ii}, & \text{if } i = j. \end{cases}$$

Jing Li and Guang Zhou are with the School of Mathematics Science, University Electronic Science and Technology of China, Chengdu 611731, PR China.

Email address: zhoguang@163.com.

Let $A = (a_{ij})$ be an $n \times n$ matrix with all diagonal entries being nonzero throughout. For any $i, j, k \in N$, denote

$$\begin{aligned} R_i &= \sum_{k \neq i}^n |a_{ik}| \\ d_i &= \frac{R_i}{a_{ii}} \\ r_{ji} &= \frac{|a_{ji}|}{|a_{jj}| - \sum_{k \neq j, i} |a_{jk}|}, \quad j \neq i \\ r_i &= \max_{j \neq i} \{r_{ji}\} \\ s_{ji} &= \frac{|a_{ji}| + \sum_{k \neq j, i} |a_{jk}|r_i}{|a_{jj}|}, \quad j \neq i \\ s_j &= \max_{i \neq j} \{s_{ji}\} \end{aligned}$$

Denote the set of all simple circuits in the digraph Γ_A of A by $\Psi(A)$. A circuit of length k in Γ_A is an ordered sequence $\gamma = (i_1, \dots, i_k, i_{k+1})$, where $i_1, \dots, i_k \in N$ are all distinct, and $i_{k+1} = i_1$. The set $\{i_1, \dots, i_k\}$ is called the support of γ and is denoted by $\bar{\gamma}$. The length of the circuit γ is denoted by $|\gamma|$, η is the greatest common divisor of 2 and s , $\tau = \frac{s}{\eta}$. $E(A) = \{e_{i,j} | a_{i,j} \neq 0, i, j \in N\}$ is the set of directed edge of $\Gamma(A)$. We say $\{e_{i_1, i_2}, e_{i_1+\eta, i_2+\eta}, \dots, e_{i_2+(\tau-1)\eta, i_3+(\tau-1)\eta}\}$ is the odd 1-path cover; $\{e_{i_2, i_3}, e_{i_2+\eta, i_3+\eta}, \dots, e_{i_2+(\tau-1)\eta, i_3+(\tau-1)\eta}\}$ is the even 1-path cover; The certain 1-path cover of γ recorded as $p^1(\gamma)$. When s is an positive odd number, the odd and even 1-path cover is the same, namely, only one 1-path cover contains all the directed edge of γ . We denote $p^1(A) = \bigcup_{\gamma \in \Psi(A)} p^1(\gamma)$ is a 1-path cover of $\Gamma(A)$. For any $i, j \in N$,

denote, $\alpha = \{i \in N | i \in \gamma \in \Psi(A)\}$, $\Theta_A = \{a_{ii} | i \in N \setminus \alpha\}$,

$$A^\circ = \begin{pmatrix} A_{i_1 i_1} & A_{i_1 i_2} & \cdots & A_{i_1 i_m} \\ A_{i_2 i_1} & A_{i_2 i_2} & \cdots & A_{i_2 i_m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{i_m i_1} & A_{i_m i_2} & \cdots & A_{i_m i_m} \end{pmatrix}, \{i_1, i_2, \dots, i_m\} = \alpha$$

$$m_c^r(A) = \max_{\gamma \in \Psi(A)} \{ \max r_A(\gamma), \max \Theta_A \},$$

$$M_c^r(A) = \max_{\gamma \in \Psi(A)} \{ \min r_A(\gamma), \max \Theta_A \},$$

$r_A(\gamma)$ denotes the real roots of the equation

$$\prod_{i \in \bar{\gamma}} (x - a_{ii}) = \prod_{i \in \bar{\gamma}} R_i(A^\circ),$$

which greater than $\max_{i \in \bar{N}} \{a_{ii}\}$.

II. MAIN RESULTS

For convenience, we give some lemmas which are useful for obtaining the main results.

Lemma 2.1 [1]. Let $A \in R^{n \times n}$ be an irreducible nonnegative matrix. Then

1) A has a positive real eigenvalue equals to its spectral radius;

2) To $\rho(A)$ there corresponds an eigenvector $x > 0$.

Lemma 2.2 [2]. Let $A, B \in R^{n \times n}$. If E, F are diagonal matrices of order n , then

$$\begin{aligned} E(A \circ B)F &= (EAF) \circ B = (EA) \circ (BF) \\ &= (AF) \circ (EB) = A \circ (EBF) \end{aligned}$$

and

$$\begin{aligned} E(A \star B)F &= (EAF) \star B = (EA) \star (BF) \\ &= (AF) \star (EB) = A \star (EBF). \end{aligned}$$

Lemma 2.3 [1]. Let $A \in R^{n \times n}$, with $n \geq 2$. Then, if λ is an eigenvalue of A , there is a pair (i, j) of positive integers with $i \neq j$, $(1 \leq i, j \leq n)$ such that

$$|\lambda - a_{ii}| |\lambda - a_{jj}| \leq R_i R_j.$$

Lemma 2.4 [2]. Let $A = (a_{ij}) \in R^{n \times n}$ be diagonally dominant M -matrix. Then, for $A^{-1} = (\beta_{ij})$, we have

$$\beta_{ji} \leq \frac{|a_{ji}| + \sum_{k \neq j, i} |a_{jk}| r_k}{a_{jj}} \beta_{ii}, \quad \text{for all } j \neq i.$$

Lemma 2.5 [2]. Let $A = (a_{ij}) \in R^{n \times n}$ be a strictly row diagonally dominant M -matrix. Then, for $A^{-1} = (\beta_{ij})$, we have

$$\beta_{ji} \leq s_j \beta_{ii}, \quad \text{for all } j \neq i.$$

Lemma 2.6 [3]. Let $A = (a_{ij}) \in M_n$ be a strictly row diagonally dominant M -matrix. Then, for $A^{-1} = (\beta_{ij})$, we have

$$\beta_{ii} \geq \frac{1}{a_{ii}}.$$

Lemma 2.7 [4]. Let $A = (a_{ij}) \in R^{n \times n}$ be nonnegative matrix, then

$$m_c^r(A) \leq \rho(A) \leq M_c^r(A).$$

Theorem 2.1 [5]. Let $A = (a_{ij}) \in M_n$ be a strictly row diagonally dominant M -matrix. Then, for $A^{-1} = (\beta_{ij})$, $B = (b_{ij}) \in M_n$, we have

$$q(B \circ A^{-1}) \geq q(B) \min_i \beta_{ii}. \quad (1)$$

Theorem 2.2 [6]. Let $A = (a_{ij}) \in M_n$ be a strictly row diagonally dominant M -matrix. Then, for $A^{-1} = (\beta_{ij})$, $B = (b_{ij}) \in M_n$, we have

$$q(B \circ A^{-1}) \geq \frac{1 - \rho(J_A)\rho(J_B)}{1 + \rho^2(J_B)} \min_i \frac{b_{ii}}{a_{ii}}. \quad (2)$$

Theorem 2.3 [7]. Let $A = (a_{ij}) \in M_n$ be a strictly row diagonally dominant M -matrix. Then, for $A^{-1} = (\beta_{ij})$, $B = (b_{ij}) \in M_n$, we have

$$q(B \circ A^{-1}) \geq \min_{1 \leq i \leq n} \left\{ \frac{b_{ii} - s_i \sum_{j \neq i} |b_{ji}|}{a_{ii}} \right\}. \quad (3)$$

Theorem 2.4 Let $A = (a_{ij}) \in M_n$ be a strictly row diagonally dominant M -matrix. Then, for $A^{-1} = (\beta_{ij})$, $B = (b_{ij}) \in M_n$, we have

$$\begin{aligned} q(B \circ A^{-1}) &\geq \min_{i \neq j} \frac{1}{2} \left\{ b_{ii} \beta_{ii} + b_{jj} \beta_{jj} - \left[(b_{ii} \beta_{ii} \right. \right. \\ &\quad \left. \left. - b_{jj} \beta_{jj})^2 + 4s_i s_j \beta_{ii} \beta_{jj} \sum_{j \neq i} |b_{ji}| \sum_{l \neq j} |b_{lj}| \right]^{\frac{1}{2}} \right\}. \end{aligned} \quad (4)$$

Proof: If A is irreducible, then $0 < s_i < 1$, for any $i \in N$. Since $q(B \circ A^{-1})$ is an eigenvalue of $B \circ A^{-1}$. From Lemma 2.2 and Lemma 2.5, $q(B \circ A^{-1}) = q(D^{-1}(B \circ A^{-1})D) = q(D(B^T \circ (A^{-1})^T)D^{-1})$. Let $D = (s_1, s_2, \dots, s_n) > 0$

$$\begin{aligned} R_i(B \circ A^{-1}) &= R_i(D^{-1}(B \circ A^{-1})D) \\ &= R_i(D(B^T \circ (A^{-1})^T)D^{-1}) \\ &= \sum_{j \neq i} |b_{ji} \beta_{ji}| \frac{s_i}{s_j} \\ &\leq s_i \sum_{j \neq i} \frac{1}{s_j} |b_{ji}| s_j |\beta_{ii}| \\ &\leq s_i \sum_{j \neq i} \frac{1}{s_j} |b_{ji}| s_j |\beta_{ii}| \\ &= s_i |\beta_{ii}| \sum_{j \neq i} |b_{ji}|. \end{aligned}$$

Thus, by Lemma 2.3, there exists a pair (i, j) of positive integers with $i \neq j$ ($1 \leq i, j \leq n$) such that

$$\begin{aligned} |q(B \circ A^{-1}) - b_{ii} \beta_{ii}| |q(B \circ A^{-1}) - b_{jj} \beta_{jj}| \\ \leq s_i \beta_{ii} \sum_{j \neq i} |b_{ji}| s_j \beta_{jj} \sum_{l \neq j} |b_{lj}|. \end{aligned}$$

From the above inequality and $0 \leq q(B \circ A^{-1}) \leq a_{ii} b_{ii}$, $\forall i \in N$, we have

$$\begin{aligned} (q(B \circ A^{-1}) - b_{ii} \beta_{ii})(q(B \circ A^{-1}) - b_{jj} \beta_{jj}) \\ \leq s_i \beta_{ii} \sum_{j \neq i} |b_{ji}| s_j \beta_{jj} \sum_{l \neq j} |b_{lj}|. \end{aligned} \quad (5)$$

Thus, from (5), we have

$$\begin{aligned} q(B \circ A^{-1}) &\geq \frac{1}{2} \left\{ b_{ii} \beta_{ii} + b_{jj} \beta_{jj} - \left[(b_{ii} \beta_{ii} - b_{jj} \beta_{jj})^2 \right. \right. \\ &\quad \left. \left. + 4s_i s_j \beta_{ii} \beta_{jj} \sum_{j \neq i} |b_{ji}| \sum_{l \neq j} |b_{lj}| \right]^{\frac{1}{2}} \right\} \\ &\geq \min_{i \neq j} \frac{1}{2} \left\{ b_{ii} \beta_{ii} + b_{jj} \beta_{jj} - \left[(b_{ii} \beta_{ii} - b_{jj} \beta_{jj})^2 \right. \right. \\ &\quad \left. \left. + 4s_i s_j \beta_{ii} \beta_{jj} \sum_{j \neq i} |b_{ji}| \sum_{l \neq j} |b_{lj}| \right]^{\frac{1}{2}} \right\}. \end{aligned}$$

If A is reducible, it is well known that a matrix in Z_n is a nonsingular M -matrix if and only if all its leading principle minors are positive. If we denote by $D = (d_{ij})$ the $n \times n$ permutation matrix with $d_{12} = d_{23} = \dots = d_{n-1n} = d_{n1} = 1$,

the remaining d_{ij} zero, then $A - tD$ is irreducible nonsingular M -matrices for any chosen positive real number t , sufficiently small such that all the leading principle minors of $A - tD$ is positive. Now we substitute $A - tD$ for A in the previous case, and then letting $t \rightarrow 0$, the result follows by continuity. ■

Remark 2.1 We next give a simple comparison between the lower bound in (4) and the lower bound in (3). Without loss of generality, for $i \neq j$, assume that

$$b_{ii}\beta_{ii} - s_i\beta_{ii} \sum_{j \neq i} |b_{ji}| \leq b_{jj}\beta_{jj} - s_j\beta_{jj} \sum_{l \neq j} |b_{lj}|. \quad (6)$$

Thus, we can write (6) equivalently as

$$s_j\beta_{jj} \sum_{l \neq j} |b_{lj}| \leq b_{jj}\beta_{jj} - b_{ii}\beta_{ii} + s_i\beta_{ii} \sum_{j \neq i} |b_{ji}|.$$

From (4) and the above inequality, we get

$$\begin{aligned} & b_{ii}\beta_{ii} + b_{jj}\beta_{jj} - \left[(b_{ii}\beta_{ii} - b_{jj}\beta_{jj})^2 \right. \\ & \quad \left. + 4s_i s_j \beta_{ii} \beta_{jj} \sum_{j \neq i} |b_{ji}| \sum_{l \neq j} |b_{lj}| \right]^{\frac{1}{2}} \\ & \geq b_{ii}\beta_{ii} + b_{jj}\beta_{jj} - \left[(b_{ii}\beta_{ii} - b_{jj}\beta_{jj})^2 \right. \\ & \quad \left. + 4(b_{jj}\beta_{jj} - b_{ii}\beta_{ii}) s_i \beta_{ii} \sum_{j \neq i} |b_{ji}| \right. \\ & \quad \left. + (2s_i \beta_{ii} \sum_{j \neq i} |b_{ji}|)^2 \right]^{\frac{1}{2}} \\ & = 2b_{ii}\beta_{ii} - 2s_i \beta_{ii} \sum_{j \neq i} |b_{ji}|. \end{aligned}$$

From Lemma 2.6, we have

$$\begin{aligned} q(B \circ A^{-1}) & \geq \min_{i \neq j} \frac{1}{2} \left\{ b_{ii}\beta_{ii} + b_{jj}\beta_{jj} - \right. \\ & \quad \left[(b_{ii}\beta_{ii} - b_{jj}\beta_{jj})^2 + 4s_i s_j \beta_{ii} \beta_{jj} \sum_{j \neq i} |b_{ji}| \sum_{l \neq j} |b_{lj}| \right]^{\frac{1}{2}} \Big\} \\ & = \min_{i \neq j} \left\{ b_{ii}\beta_{ii} - s_i \beta_{ii} \sum_{j \neq i} |b_{ji}| \right\} \\ & \geq \min_{i \neq j} \left\{ \frac{b_{ii} - s_i \sum_{j \neq i} |b_{ji}|}{a_{ii}} \right\} \end{aligned}$$

Hence, the bound (4) is sharper than the bound (3).

Theorem 2.5 If $A = (a_{ij}) \in R^{n \times n}$, $B = (b_{ij}) \in R^{n \times n}$, are two nonnegative matrices, then

$$\rho(A \circ B) \leq \max \left\{ \min_{\gamma \in \Psi(A \circ B)} r_{A \circ B}(\gamma), \max \Theta_{A \circ B} \right\}.$$

$r_{A \circ B}(\gamma)$ denotes the real roots of the equation $\prod_{i \in \bar{\gamma}} (x - a_{ii}b_{ii}) = \prod_{i \in \bar{\gamma}} R_i(A \circ B)^\circ$ which greater than $\max_{i \in \bar{\gamma}} \{a_{ii}b_{ii}\}$.

Proof: From Lemma 2.7 it is easy to obtained the desired result. ■

Theorem 2.6 Let $A = (a_{ij}) \in M_n$ and $B = (b_{ij}) \in M_n$. Then

$$\begin{aligned} q(A \star B) & \geq \min_{i \neq j} \frac{1}{2} \left\{ a_{ii}b_{ii} + a_{jj}b_{jj} - \left[(a_{ii}b_{ii} - a_{jj}b_{jj})^2 \right. \right. \\ & \quad \left. \left. + 4\alpha_i \alpha_j (b_{ii} - q(B))(b_{jj} - q(B)) \right]^{\frac{1}{2}} \right\} \end{aligned} \quad (7)$$

where $\alpha_i = \max_{k \neq i} \{ |a_{ki}| \}$, $\forall i \in N$.

Proof: If $A \star B$ is irreducible, then A and B are irreducible. Since, $A - q(A)I$ and $B - q(B)I$ are singular irreducible M -matrices. Then

$$a_{ii} - q(A) > 0, \forall i \in N.$$

and

$$b_{ii} - q(B) > 0, \forall i \in N.$$

Since $A = (a_{ij})$, $B = (b_{ij})$ are irreducible nonsingular M -matrices, then there exists two positive vectors u, v Such that $Au = q(A)u$, $Bv = q(B)v$. Thus, we have

$$a_{ii} - \sum_{j \neq i} \frac{|a_{ij}| u_j}{u_i} = q(A),$$

or equivalently,

$$\sum_{j \neq i} |a_{ij}| u_j = [a_{ii} - q(A)] u_i$$

and

$$b_{ii} - \sum_{j \neq i} \frac{|b_{ij}| v_j}{v_i} = q(B),$$

or equivalently,

$$\sum_{j \neq i} |b_{ij}| v_j = [b_{ii} - q(B)] v_i$$

For convenience, let denote $\alpha_i = \max_{k \neq i} \{ |a_{ki}| \}$, $\forall i \in N$. Since A is an irreducible matrix, $\alpha_i > 0$, $\forall i \in N$. Define a positive diagonal matrix $Z = \text{diag}(z_1, \dots, z_n)$, where

$$z_i = \frac{v_i}{\alpha_i} > 0, \forall i \in N.$$

By Lemma 2.2, we have $q(A \star B) = q(Z^{-1}(A \star B)Z) = q(A \star (Z^{-1}BZ))$. For convenience, let $\hat{B} = (\hat{b}_{ij}) = Z^{-1}BZ$. So we have

$$\begin{aligned} R_i(Z^{-1}(A \star B)Z) & = R_i(A \star \hat{B}) \\ & = \sum_{j \neq i} |a_{ij}| |\hat{b}_{ij}| \frac{z_j}{z_i} \\ & \leq \sum_{j \neq i} |b_{ij}| v_j \frac{\alpha_i}{v_i} \\ & = (b_{ii} - q(B)) \alpha_i. \end{aligned}$$

According to Lemma 2.3, there exists a pair (i, j) of positive integers with $i \neq j (1 \leq i, j \leq n)$, such that

$$|q(A \star B) - a_{ii}b_{ii}| |q(A \star B) - a_{jj}b_{jj}| \leq (b_{ii} - q(B)) \alpha_i (b_{jj} - q(B)) \alpha_j$$

From the above inequality and $0 \leq q(A \star B) \leq a_{ii}b_{ii}$, $\forall i \in N$, we have

$$\begin{aligned} (q(A \star B) - a_{ii}b_{ii})(q(A \star B) - a_{jj}b_{jj}) \\ \leq \alpha_i \alpha_j (b_{ii} - q(B))(b_{jj} - q(B)) \end{aligned}$$

$$q(A \star B) \geq \frac{1}{2} \left\{ a_{ii}b_{ii} + a_{jj}b_{jj} - \left[(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4\alpha_i\alpha_j(b_{ii} - q(B))(b_{jj} - q(B)) \right]^{\frac{1}{2}} \right\}$$

$$\geq \min_{i \neq j} \frac{1}{2} \left\{ a_{ii}b_{ii} + a_{jj}b_{jj} - \left[(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4\alpha_i\alpha_j(b_{ii} - q(B))(b_{jj} - q(B)) \right]^{\frac{1}{2}} \right\}.$$

If $A \star B$ is reducible. It is well known that a matrix in Z_n is a nonsingular M-matrix if and only if all its leading principal minors are positive. If we denote by $D = (d_{ij})$ the $n \times n$ permutation matrix with $d_{12} = d_{23} = \dots = d_{n-1n} = d_{n1} = 1$, the remaining d_{ij} zero, then both $A - tD$ and $B - tD$ are irreducible nonsingular M-matrices for any chosen positive real number t , sufficiently small such that all the leading principal minors of both $A - tD$ and $B - tD$ are positive. Now we substitute $A - tD$ and $B - tD$ for A and B , respectively in the previous case, and then letting $t \rightarrow 0$, the result follows by continuity. ■

Theorem 2.7 Let $A = (a_{ij}) \in M_n$ and $B = (b_{ij}) \in M_n$. Then

$$q(A \star B) \geq \min_{i \neq j} \frac{1}{2} \left\{ a_{ii}b_{ii} + a_{jj}b_{jj} - \left[(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4\beta_i\beta_j(a_{ii} - q(A))(a_{jj} - q(A)) \right]^{\frac{1}{2}} \right\}$$

where $\beta_i = \max_{k \neq i} \{|b_{ki}|\}$, $\forall i \in N$.

According to Theorem 2.6 and Theorem 2.7, it is easy to obtain the following corollary.

Corollary 2.1 If $A = (a_{ij})$ and $B = (b_{ij})$ are two $n \times n$ nonsingular M-matrices, then

$$q(A \star B) \geq \max \left\{ \min_{i \neq j} \frac{1}{2} \left\{ a_{ii}b_{ii} + a_{jj}b_{jj} - \left[(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4\alpha_i\alpha_j(b_{ii} - q(B))(b_{jj} - q(B)) \right]^{\frac{1}{2}} \right\}, \right.$$

$$\left. \min_{i \neq j} \frac{1}{2} \left\{ a_{ii}b_{ii} + a_{jj}b_{jj} - \left[(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4\beta_i\beta_j(a_{ii} - q(A))(a_{jj} - q(A)) \right]^{\frac{1}{2}} \right\} \right\}$$

where $\alpha_i = \max_{k \neq i} \{|a_{ki}|\}$ and $\beta_i = \max_{k \neq i} \{|b_{ki}|\}$ $\forall i \in N$.

Corollary 2.2 If $A = (a_{ij})$ and $B = (b_{ij})$ are two $n \times n$ nonsingular M-matrices, then

$$|\det(A \star B)| \geq [q(A \star B)]^n$$

$$\geq \min_{i \neq j} \frac{1}{2^n} \left\{ a_{ii}b_{ii} + a_{jj}b_{jj} - \left[(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4\alpha_i\alpha_j(b_{ii} - q(B))(b_{jj} - q(B)) \right]^{\frac{1}{2}} \right\}^n,$$

and

$$|\det(A \star B)| \geq [q(A \star B)]^n$$

$$\geq \min_{i \neq j} \frac{1}{2^n} \left\{ a_{ii}b_{ii} + a_{jj}b_{jj} - \left[(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4\beta_i\beta_j(a_{ii} - q(A))(a_{jj} - q(A)) \right]^{\frac{1}{2}} \right\}^n.$$

III. NUMERICAL EXAMPLES

Example 3.1

$$A = \begin{bmatrix} 4 & -1 & -1 & -1 \\ -2 & 5 & -1 & -1 \\ 0 & -2 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & -1/2 & 0 & 0 \\ -1/2 & 1 & -1/2 & 0 \\ 0 & -1/2 & 1 & -1/2 \\ 0 & 0 & -1/2 & 1 \end{bmatrix}.$$

By calculation, we have $q(B \circ A^{-1}) = 0.2148$. By the inequality (1), we get

$$q(B \circ A^{-1}) \geq 0.07$$

By the inequality (2), we get

$$q(B \circ A^{-1}) \geq 0.052$$

By the inequality (3), we get

$$q(B \circ A^{-1}) \geq 0.075$$

By Theorem 2.4, we have

$$q(B \circ A^{-1}) \geq 0.1729.$$

Example 3.2

$$A = \begin{bmatrix} 8 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 5 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 8 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

It is easy to calculate that $\rho(A \circ B) = \rho(A) = 8.1801$. If we use Gersgorin theorem and Brauer theorem, we have

$$\rho(A \circ B) \leq 9.$$

and

$$\rho(A \circ B) \leq 9.$$

If we take $p^1(A) = \{e_{1,2}, e_{2,3}, e_{3,4}, e_{4,5}\}$, $r_{A \circ B}(1, 2) = r_{A \circ B}(1, 4) = r_{A \circ B}(2, 5) = 8.3166$, $r_{A \circ B}(1, 5) = 9$, $r_{A \circ B}(2, 4) = 4$, $r_{A \circ B}(2, 3) = r_{A \circ B}(3, 4) = 6$, $r_{A \circ B}(1, 3) = r_{A \circ B}(3, 5) = 8.5616$.

From Theorem 2.5 we get

$$\rho(A \circ B) \leq M_c^r(A \circ B)$$

$$= \max \left\{ \min_{\gamma \in \Psi(A \circ B)} r_{A \circ B}(\gamma), \max \Theta_{A \circ B} \right\}$$

$$= 8.3166.$$

Example 3.3

$$A = \begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$$

By calculation, we have

$$q(A \star B) = 6$$

By Theorem 2.6, we get

$$q(A \star B) = 6.$$

IV. CONCLUSIONS

In this paper, we give some inequalities for the spectral radius of the Hadamard product of two nonnegative matrices. These bounds improve some existing results and numerical examples illustrate that our results are superior.

REFERENCES

- [1] R.A. Horn and C.R. Johnson, *Matrix Analysis*, Cambridge University Press, 1985.
- [2] Y.T.Li, F.B.Chen, D.F.Wang, New lower bounds on eigenvalue of the Hadamard product of an M -matrix and its inverse, *Linear Algebra Appl.* 430(2009) 1423-1431.
- [3] X.R.Yong, Z.Wang, On a conjecture of Fiedler and Markham, *Linear Algebra Appl.* 288(1999) 259-267.
- [4] H.B.Lu, Estimate for the Perron Root of a Nonnegative matrix, *Chinese Journal of Engineering. Mathematics.* 25(2008) 68-73.
- [5] R.A.Horn, C.R.Johnson, *Topics in Matrix Analysis*, Cambridge University Press, 1991.
- [6] R.Huang, Some inequalities of the Hadamard product and the Fan product of matrices. *Linear Algebr.Appl.* 2008, 428, 1551-1559.
- [7] D.M.Zhou, G.L.Chen, G.X.Wu, X.Y.Zhang. On some new bounds for eigenvalues of the Hadamard product and the Fan product of matrices. *Linear Algebra Appl.* 2012, In Press.

Jing Li was born in Henan Province, China, in 1988. She received the B.S. degree from Huaibei Normal University in 2011. She is currently pursuing the M.S. degree from University of Electronic Science and Technology of China. Her research interests are numerical algebra and matrix analysis.

Guang Zhou was born in Anhui Province, China, in 1987. He received the B.S. degree from Fuyang University in 2011. He is currently pursuing the M.S. degree from University of Electronic Science and Technology of China. His research interests the stability of neural networks. His research interests include chaos synchronization, switch and delay dynamic systems.