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Some new inequalities for eigenvalues of the Hadamard product and the Fan product of matrices

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Abstract—Let A and B be nonnegative matrices. A new upper bound on the spectral radius $\rho(A \circ B)$ is obtained. Meanwhile, a new lower bound on the smallest eigenvalue $q(A \star B)$ for the Fan product, and a new lower bound on the minimum eigenvalue $q(B \circ A^{-1})$ for the Hadamard product of B and A^{-1} of two nonsingular M-matrices A and B are given. Some results of comparison are also given in theory. To illustrate our results, numerical examples are considered.

Keywords—Hadamard product, Fan product; nonnegative matrix, M-matrix, Spectral radius, Minimum eigenvalue, 1-path cover.

I. Introduction

$R^{N\times M}$ and N denote the set of all $n\times m$ real matrices and the $\{1, 2, \cdots, n\}$, respectively. If $A = (a_{ij}) \in R^{n\times m}$, $B = (b_{ij}) \in R^{n\times m}$ and $a_{ij} - b_{ij} \geq 0$, we say that $A \geq B$, and if $a_{ij} \geq 0$, we say that A is nonnegative. If $A \in R^{n\times n}$ is a nonnegative matrix, the Perron-Frobenius theorem guarantees that $\rho(A) \in \sigma(A)$, where the set $\sigma(A)$ denotes the spectrum of A, $\rho(A)$ denotes the spectral radius of A. $\emptyset$ denotes the empty set.

A matrix A is irreducible if there does not exist a permutation matrix P such that

$$PAP^T = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix},$$

where A_{11} and A_{22} are square matrices, then A is called irreducible. The set $Z_n \subset R^{n\times n}$ is defined by

$$Z_n = \{A = (a_{ij}) \in R^{n\times n} : a_{ij} \leq 0, \text{if } i \neq j,\ i, j = 1, \cdots, n\}.$$

Let $A = (a_{ij}) \in Z_n$ and suppose $A = sI - B$ with $s \in R$ and $B \geq 0$. Then $s - \rho(B)$ is an eigenvalue of A, every eigenvalue of A lies in the disc $\{z \in C :| z - s |\leq \rho(B)\}$, and hence every eigenvalue λ of A satisfies $Re\lambda \geq s - \rho(B)$. In particular, a matrix $A \in Z_n$ is called an M-matrix if $s \geq \rho(B)$. If $s > \rho(B)$ we call A is nonsingular M-matrix, and denote the class of nonsingular M-matrices by M_n.

Let $A = (a_{ij}) \in Z_n$, we denote $\min\{Re(\lambda) : \lambda \in \sigma(A)\}$ by $q(A)$, $q(A)$ is called the minimum eigenvalue of A.

The Hadamard product of $A = (a_{ij}) \in R^{n\times n}$ and $B = (b_{ij}) \in R^{n\times n}$ is defined by $A \circ B = (a_{ij}b_{ij}) \in R^{n\times n}$. Let $A = (a_{ij}), B = (b_{ij}) \in R^{n\times n}$, the Fan product of A and B is denoted by $A \star B = C = (c_{ij}) \in R^{n\times n}$, and is defined by

$$c_{ij} = \begin{cases} -a_{ij}b_{ij}, & \text{if } i \neq j, \\ a_{ii}b_{ii}, & \text{if } i = j. \end{cases}$$

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Let $A = (a_{ij})$ be an $n \times n$ matrix with all diagonal entries being nonzero throughout. For any $i, j, k \in N$, denote

$$\begin{aligned} R_i &= \sum_{k\neq i}^{n} |a_{ik}| \\ d_i &= \frac{R_i}{a_{ii}} \\ r_{ji} &= \frac{|a_{ji}|}{|a_{jj}| - \sum\limits_{k\neq j,i} |a_{jk}|}, \quad j \neq i \\ r_i &= \max_{j\neq i}\{r_{ji}\} \\ s_{ji} &= \frac{|a_{ji}| + \sum\limits_{k\neq j,i} |a_{jk}| r_i}{|a_{jj}|}, \quad j \neq i \\ s_j &= \max_{i\neq j}\{s_{ji}\} \end{aligned}$$

Denote the set of all simple circuits in the digraph Γ_A of A by $\Psi(A)$. A circuit of length k in Γ_A is an ordered sequence $\gamma = (i_1, \cdots, i_k, i_{k+1})$, where $i_1, \cdots, i_k \in N$ are all distinct, and $i_{k+1} = i_1$. The set $\{i_1, \cdots, i_k\}$ is called the support of γ and is denoted by $\bar{\gamma}$. The length of the circuit γ is denoted by $|\gamma|$, η is the greatest common divisor of 2 and s, $\tau = \frac{s}{\eta}$. $E(A) = \{e_{i,j} | a_{i,j} \neq 0, i, j \in N\}$ is the set of directed edge of $\Gamma(A)$. We say $\{e_{i_1,i_2}, e_{i_1+\eta,i_2+\eta}, \cdots, e_{i_2+(\tau-1)\eta,i_3+(\tau-1)\eta}\}$ is the odd 1-path cover; $\{e_{i_2,i_3}, e_{i_2+\eta,i_3+\eta}, \cdots, e_{i_2+(\tau-1)\eta,i_3+(\tau-1)\eta}\}$ is the even 1-path cover; The certain 1-path cover of γ recorded as $p^1(\gamma)$. When s is an positive odd number, the odd and even 1-path cover is the same, namely, only one 1-path cover contains all the directed edge of γ. We denote $p^1(A) = \bigcup\limits_{\gamma\in\Psi(A)} p^1(\gamma)$ is a 1-path cover of $\Gamma(A)$. For any $i, j \in N$, denote, $\alpha = \{i \in N | i \in \gamma \in \Psi(A)\}$, $\Theta_A = \{a_{ii} | i \in N \setminus \alpha\}$,

$$A^\circ = \begin{pmatrix} A_{i_1i_1} & A_{i_1i_2} & \cdots & A_{i_1i_m} \\ A_{i_2i_1} & A_{i_2i_2} & \cdots & A_{i_2i_m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{i_mi_1} & A_{i_mi_2} & \cdots & A_{i_mi_m} \end{pmatrix}, \{i_1, i_2, \cdots, i_m\} = \alpha$$

$$m_c^r(A) = \max\{\max_{\gamma\in\Psi(A)} r_A(\gamma), \max \Theta_A\},$$

$$M_c^r(A) = \max\{\min_{\gamma\in\Psi(A)} r_A(\gamma), \max \Theta_A\},$$

$r_A(\gamma)$ denotes the real roots of the equation

$$\prod_{i\in\bar{\gamma}} (x - a_{ii}) = \prod_{i\in\bar{\gamma}} R_i(A^\circ),$$

which greater than $\max_{i\in\bar{\gamma}}\{a_{ii}\}$.

II. Main results

For convenience, we give some lemmas which are useful for obtaining the main results.

Lemma 2.1 [1]. Let $A \in R^{n\times n}$ be an irreducible nonnegative matrix. Then

1) A has a positive real eigenvalue equals to its spectral radius;

2) To $\rho(A)$ there corresponds an eigenvector $x > 0$.

Lemma 2.2 [2]. Let $A, B \in R^{n\times n}$. If E, F are diagonal matrices of order n, then

$$\begin{aligned} E(A\circ B)F &= (EAF)\circ B = (EA)\circ(BF) \\ &= (AF)\circ(EB) = A\circ(EBF) \end{aligned}$$

and

$$\begin{aligned} E(A\star B)F &= (EAF)\star B = (EA)\star(BF) \\ &= (AF)\star(EB) = A\star(EBF). \end{aligned}$$

Lemma 2.3 [1]. Let $A \in R^{n\times n}$, with $n \geq 2$. Then, if λ is an eigenvalue of A, there is a pair (i, j) of positive integers with $i \neq j, (1 \leq i, j \leq n)$ such that

$$|\lambda - a_{ii}||\lambda - a_{jj}| \leq R_i R_j.$$

Lemma 2.4 [2]. Let $A = (a_{ij}) \in R^{n\times n}$ be diagonally dominant M-matrix. Then, for $A^{-1} = (\beta_{ij})$, we have

$$\beta_{ji} \leq \frac{|a_{ji}| + \sum\limits_{k\neq j,i}|a_{jk}|r_k}{a_{jj}}\beta_{ii}, \quad \text{for all } j \neq i.$$

Lemma 2.5 [2]. Let $A = (a_{ij}) \in R^{n\times n}$ be a strictly row diagonally dominant M-matrix. Then, for $A^{-1} = (\beta_{ij})$, we have

$$\beta_{ji} \leq s_{ji}\beta_{ii}, \quad \text{for all } j \neq i.$$

Lemma 2.6 [3]. Let $A = (a_{ij}) \in M_n$ be a strictly row diagonally dominant M-matrix. Then, for $A^{-1} = (\beta_{ij})$, we have

$$\beta_{ii} \geq \frac{1}{a_{ii}}.$$

Lemma 2.7 [4]. Let $A = (a_{ij}) \in R^{n\times n}$ be nonnegative matrix, then

$$m_c^r(A) \leq \rho(A) \leq M_c^r(A).$$

Theorem 2.1 [5]. Let $A = (a_{ij}) \in M_n$ be a strictly row diagonally dominant M-matrix. Then, for $A^{-1} = (\beta_{ij})$, $B = (b_{ij}) \in M_n$, we have

$$q(B\circ A^{-1}) \geq q(B)\min_i \beta_{ii}. \tag{1}$$

Theorem 2.2 [6]. Let $A = (a_{ij}) \in M_n$ be a strictly row diagonally dominant M-matrix. Then, for $A^{-1} = (\beta_{ij})$, $B = (b_{ij}) \in M_n$, we have

$$q(B\circ A^{-1}) \geq \frac{1-\rho(J_A)\rho(J_B)}{1+\rho^2(J_B)}\min_i \frac{b_{ii}}{a_{ii}}. \tag{2}$$

Theorem 2.3 [7]. Let $A = (a_{ij}) \in M_n$ be a strictly row diagonally dominant M-matrix. Then, for $A^{-1} = (\beta_{ij})$, $B = (b_{ij}) \in M_n$, we have

$$q(B\circ A^{-1}) \geq \min_{1\leq i\leq n}\left\{\frac{b_{ii} - s_i\sum\limits_{j\neq i}|b_{ji}|}{a_{ii}}\right\}. \tag{3}$$

Theorem 2.4 Let $A = (a_{ij}) \in M_n$ be a strictly row diagonally dominant M-matrix. Then, for $A^{-1} = (\beta_{ij})$, $B = (b_{ij}) \in M_n$, we have

$$\begin{aligned} q(B\circ A^{-1}) \geq \min_{i\neq j}\tfrac{1}{2}\Bigg\{ & b_{ii}\beta_{ii} + b_{jj}\beta_{jj} - \Bigg[(b_{ii}\beta_{ii} \\ & -b_{jj}\beta_{jj})^2 + 4s_i s_j\beta_{ii}\beta_{jj}\sum_{j\neq i}|b_{ji}|\sum_{l\neq j}|b_{lj}|\Bigg]^{\frac{1}{2}}\Bigg\}. \end{aligned} \tag{4}$$

Proof: If A is irreducible, then $0 < s_i < 1$, for any $i \in N$. Since $q(B\circ A^{-1})$ is an eigenvalue of $B\circ A^{-1}$. From Lemma 2.2 and Lemma 2.5, $q(B\circ A^{-1}) = q(D^{-1}(B\circ A^{-1})D) = q(D(B^T\circ(A^{-1})^T)D^{-1})$. Let $D = (s_1, s_2, \cdots, s_n) > 0$

$$\begin{aligned} R_i(B\circ A^{-1}) &= R_i(D^{-1}(B\circ A^{-1})D) \\ &= R_i(D(B^T\circ(A^{-1})^T)D^{-1}) \\ &= \sum_{j\neq i}|b_{ji}\beta_{ji}|\tfrac{s_i}{s_j} \\ &\leq s_i\sum_{j\neq i}\tfrac{1}{s_j}|b_{ji}|s_{ji}|\beta_{ii}| \\ &\leq s_i\sum_{j\neq i}\tfrac{1}{s_j}|b_{ji}|s_j|\beta_{ii}| \\ &= s_i|\beta_{ii}|\sum_{j\neq i}|b_{ji}|. \end{aligned}$$

Thus, by Lemma 2.3, there exists a pair (i, j) of positive integers with $i \neq j$ $(1 \leq i, j \leq n)$ such that

$$\begin{aligned} &|q(B\circ A^{-1}) - b_{ii}\beta_{ii}||q(B\circ A^{-1}) - b_{jj}\beta_{jj}| \\ &\leq s_i\beta_{ii}\sum_{j\neq i}|b_{ji}|s_j\beta_{jj}\sum_{l\neq j}|b_{lj}|. \end{aligned}$$

From the above inequality and $0 \leq q(B\circ A^{-1}) \leq a_{ii}b_{ii}$, $\forall i \in N$, we have

$$\begin{aligned} &(q(B\circ A^{-1}) - b_{ii}\beta_{ii})(q(B\circ A^{-1}) - b_{jj}\beta_{jj}) \\ &\leq s_i\beta_{ii}\sum_{j\neq i}|b_{ji}|s_j\beta_{jj}\sum_{l\neq j}|b_{lj}|. \end{aligned} \tag{5}$$

Thus, from (5), we have

$$\begin{aligned} q(B\circ A^{-1}) &\geq \tfrac{1}{2}\Bigg\{b_{ii}\beta_{ii} + b_{jj}\beta_{jj} - \Bigg[(b_{ii}\beta_{ii} - b_{jj}\beta_{jj})^2 \\ &\quad +4s_i s_j\beta_{ii}\beta_{jj}\sum_{j\neq i}|b_{ji}|\sum_{l\neq j}|b_{lj}|\Bigg]^{\frac{1}{2}}\Bigg\} \\ &\geq \min_{i\neq j}\tfrac{1}{2}\Bigg\{b_{ii}\beta_{ii} + b_{jj}\beta_{jj} - \Bigg[(b_{ii}\beta_{ii} - b_{jj}\beta_{jj})^2 \\ &\quad +4s_i s_j\beta_{ii}\beta_{jj}\sum_{j\neq i}|b_{ji}|\sum_{l\neq j}|b_{lj}|\Bigg]^{\frac{1}{2}}\Bigg\}. \end{aligned}$$

If A is reducible, it is well known that a matrix in Z_n is a nonsingular M-matrix if and only if all its leading principle minors are positive. If we denote by $D = (d_{ij})$ the $n\times n$ permutation matrix with $d_{12} = d_{23} = \cdots = d_{n-1n} = d_{n1} = 1$, the remaining d_{ij} zero, then $A-tD$ is irreducible nonsingular M-matrices for any chosen positive real number t, sufficiently small such that all the leading principle minors of $A-tD$ is positive. Now we substitute $A-tD$ for A in the previous case, and then letting $t \to 0$, the result follows by continuity. ∎

Remark 2.1 We next give a simple comparison between the lower bound in (4) and the lower bound in (3). Without loss of generality, for $i \neq j$, assume that

$$b_{ii}\beta_{ii} - s_i\beta_{ii}\sum_{j\neq i}|b_{ji}| \leq b_{jj}\beta_{jj} - s_j\beta_{jj}\sum_{l\neq j}|b_{lj}|. \quad (6)$$

Thus, we can write (6) equivalently as

$$s_j\beta_{jj}\sum_{l\neq j}|b_{lj}| \leq b_{jj}\beta_{jj} - b_{ii}\beta_{ii} + s_i\beta_{ii}\sum_{j\neq i}|b_{ji}|.$$

From (4) and the above inequality, we get

$$\begin{aligned} & b_{ii}\beta_{ii} + b_{jj}\beta_{jj} - \Bigg[(b_{ii}\beta_{ii} - b_{jj}\beta_{jj})^2 \\ & \quad +4s_is_j\beta_{ii}\beta_{jj}\sum_{j\neq i}|b_{ji}|\sum_{l\neq j}|b_{lj}|\Bigg]^{\frac{1}{2}} \\ & \geq b_{ii}\beta_{ii} + b_{jj}\beta_{jj} - \Bigg[(b_{ii}\beta_{ii} - b_{jj}\beta_{jj})^2 \\ & \quad +4(b_{jj}\beta_{jj} - b_{ii}\beta_{ii})s_i\beta_{ii}\sum_{j\neq i}|b_{ji}| \\ & \quad +(2s_i\beta_{ii}\sum_{j\neq i}|b_{ji}|)^2\Bigg]^{\frac{1}{2}} \\ & = 2b_{ii}\beta_{ii} - 2s_i\beta_{ii}\sum_{j\neq i}|b_{ji}|. \end{aligned}$$

From Lemma 2.6, we have

$$\begin{aligned} & q(B\circ A^{-1}) \geq \min_{i\neq j}\tfrac{1}{2}\Bigg\{b_{ii}\beta_{ii} + b_{jj}\beta_{jj} - \\ & \Bigg[(b_{ii}\beta_{ii} - b_{jj}\beta_{jj})^2 + 4s_is_j\beta_{ii}\beta_{jj}\sum_{j\neq i}|b_{ji}|\sum_{l\neq j}|b_{lj}|\Bigg]^{\frac{1}{2}}\Bigg\} \\ & = \min_{i\neq j}\{b_{ii}\beta_{ii} - s_i\beta_{ii}\sum_{j\neq i}|b_{ji}|\} \\ & \geq \min_{i\neq j}\Bigg\{\frac{b_{ii}-s_i\sum_{j\neq i}|b_{ji}|}{a_{ii}}\Bigg\} \end{aligned}$$

Hence, the bound (4) is sharper than the bound (3).

Theorem 2.5 If $A=(a_{ij})\in R^{n\times n}$, $B=(b_{ij})\in R^{n\times n}$, are two nonnegative matrices, then

$$\rho(A\circ B) \leq \max\{\min_{\gamma\in\Psi(A\circ B)} r_{A\circ B}(\gamma), \max\Theta_{A\circ B}\}.$$

$r_{A\circ B}(\gamma)$ denotes the real roots of the equation $\prod_{i\in\bar{\gamma}}(x - a_{ii}b_{ii}) = \prod_{i\in\bar{\gamma}}R_i(A\circ B)^{\circ}$ which greater than $\max_{i\in\bar{\gamma}}\{a_{ii}b_{ii}\}$.

Proof: From Lemma 2.7 it is easy to obtained the desired result. ∎

Theorem 2.6 Let $A=(a_{ij})\in M_n$ and $B=(b_{ij})\in M_n$. Then

$$\begin{aligned} q(A\star B) \geq \min_{i\neq j}\tfrac{1}{2}\Bigg\{ & a_{ii}b_{ii} + a_{jj}b_{jj} - \Bigg[(a_{ii}b_{ii} - a_{jj}b_{jj})^2 \\ & +4\alpha_i\alpha_j(b_{ii}-q(B))(b_{jj}-q(B))\Bigg]^{\frac{1}{2}}\Bigg\} \end{aligned} \quad (7)$$

where $\alpha_i = \max_{k\neq i}\{|a_{ki}|\}, \forall i\in N$.

Proof: If $A\star B$ is irreducible, then A and B are irreducible. Since, $A-q(A)I$ and $B-q(B)I$ are singular irreducible M-matrices. Then

$$a_{ii} - q(A) > 0, \forall i\in N.$$

and

$$b_{ii} - q(B) > 0, \forall i\in N.$$

Since $A=(a_{ij})$, $B=(b_{ij})$ are irreducible nonsingular M-matrices, then there exists two positive vectors u, v Such that $Au=q(A)u$, $Bv=q(B)v$. Thus, we have

$$a_{ii} - \sum_{j\neq i}\frac{|\, a_{ij}\,|\, u_j}{u_i} = q(A),$$

or equivalently,

$$\sum_{j\neq i}|\, a_{ij}\,|\, u_j = [a_{ii}-q(A)]u_i$$

and

$$b_{ii} - \sum_{j\neq i}\frac{|\, b_{ij}\,|\, v_j}{v_i} = q(B),$$

or equivalently,

$$\sum_{j\neq i}|\, b_{ij}\,|\, v_j = [b_{ii}-q(B)]v_i$$

For convenience, let denote $\alpha_i = \max_{k\neq i}\{|\, a_{ki}\,|\}, \forall i\in N$. Since A is an irreducible matrix, $\alpha_i>0, \forall i\in N$. Define a positive diagonal matrix $Z=diag(z_1,\cdots,z_n)$, where

$$z_i = \frac{v_i}{\alpha_i} > 0, \forall i\in N.$$

By Lemma 2.2, we have $q(A\star B) = q(Z^{-1}(A\star B)Z) = q(A\star(Z^{-1}BZ))$. For convenience, let $\hat{B}=(\hat{b}_{ij})=Z^{-1}BZ$. So we have

$$\begin{aligned} R_i(Z^{-1}(A\star B)Z) &= R_i(A\star\hat{B}) \\ &= \sum_{j\neq i}|\, a_{ij}\,||\, b_{ij}\,|\,\tfrac{z_j}{z_i} \\ &\leq \sum_{j\neq i}|\, b_{ij}\,|\, v_j\tfrac{\alpha_i}{v_i} \\ &= (b_{ii}-q(B))\alpha_i. \end{aligned}$$

According to Lemma 2.3, there exists a pair (i,j) of positive integers with $i\neq j (1\leq i,j\leq n)$, such that

$$|\, q(A\star B) - a_{ii}b_{ii}\,||\, q(A\star B) - a_{jj}b_{jj}\,| \leq (b_{ii}-q(B))\alpha_i(b_{jj}-q(B))\alpha_j$$

From the above inequality and $0\leq q(A\star B)\leq a_{ii}b_{ii}, \forall i\in N$, we have

$$\begin{aligned} &(q(A\star B)-a_{ii}b_{ii})(q(A\star B)-a_{jj}b_{jj}) \\ &\quad \leq \alpha_i\alpha_j(b_{ii}-q(B))(b_{jj}-q(B)) \end{aligned}$$

$$\begin{aligned} q(A\star B) &\geq \tfrac{1}{2}\Big\{a_{ii}b_{ii}+a_{jj}b_{jj}-\Big[(a_{ii}b_{ii}-a_{jj}b_{jj})^2 \\ &\quad +4\alpha_i\alpha_j(b_{ii}-q(B))(b_{jj}-q(B))\Big]^{\frac{1}{2}}\Big\} \\ &\geq \min_{i\neq j}\tfrac{1}{2}\Big\{a_{ii}b_{ii}+a_{jj}b_{jj}-\Big[(a_{ii}b_{ii}-a_{jj}b_{jj})^2 \\ &\quad +4\alpha_i\alpha_j(b_{ii}-q(B))(b_{jj}-q(B))\Big]^{\frac{1}{2}}\Big\}. \end{aligned}$$

If $A\star B$ is reducible. It is well known that a matrix in Z_n is a nonsingular M-matrix if and only if all its leading principal minors are positive. If we denote by $D=(d_{ij})$ the $n\times n$ permutation matrix with $d_{12}=d_{23}=\cdots=d_{n-1n}=d_{n1}=1$, the remaining d_{ij} zero, then both $A-tD$ and $B-tD$ are irreducible nonsingular M-matrices for any chosen positive real number t, sufficiently small such that all the leading principal minors of both $A-tD$ and $B-tD$ are positive. Now we substitute $A-tD$ and $B-tD$ for A and B, respectively in the previous case, and then letting $t\to 0$, the result follows by continuity. ■

Theorem 2.7 Let $A=(a_{ij})\in M_n$ and $B=(b_{ij})\in M_n$. Then

$$\begin{aligned} q(A\star B) &\geq \min_{i\neq j}\tfrac{1}{2}\Big\{a_{ii}b_{ii}+a_{jj}b_{jj}-\Big[(a_{ii}b_{ii}-a_{jj}b_{jj})^2 \\ &\quad +4\beta_i\beta_j(a_{ii}-q(A))(a_{jj}-q(A))\Big]^{\frac{1}{2}}\Big\} \end{aligned}$$

where $\beta_i=\max_{k\neq i}\{|b_{ki}|\}, \forall i\in N$.

According to Theorem 2.6 and Theorem 2.7, it is easy to obtain the following corollary.

Corollary 2.1 If $A=(a_{ij})$ and $B=(b_{ij})$ are two $n\times n$ nonsingular M-matrices, then

$$\begin{aligned} q(A\star B) &\geq \max\Big\{\min_{i\neq j}\tfrac{1}{2}\Big\{a_{ii}b_{ii}+a_{jj}b_{jj} \\ &-\Big[(a_{ii}b_{ii}-a_{jj}b_{jj})^2+4\alpha_i\alpha_j(b_{ii}-q(B))(b_{jj}-q(B))\Big]^{\frac{1}{2}}\Big\}, \\ &\min_{i\neq j}\tfrac{1}{2}\Big\{a_{ii}b_{ii}+a_{jj}b_{jj}-\Big[(a_{ii}b_{ii}-a_{jj}b_{jj})^2 \\ &+4\beta_i\beta_j(a_{ii}-q(A))(a_{jj}-q(A))\Big]^{\frac{1}{2}}\Big\}\Big\} \end{aligned}$$

where $\alpha_i=\max_{k\neq i}\{|a_{ki}|\}$ and $\beta_i=\max_{k\neq i}\{|b_{ki}|\} \quad \forall i\in N$.

Corollary 2.2 If $A=(a_{ij})$ and $B=(b_{ij})$ are two $n\times n$ nonsingular M-matrices, then

$$\begin{aligned} &|det(A\star B)|\geq[q(A\star B)]^n \\ &\geq \min_{i\neq j}\tfrac{1}{2^n}\Big\{a_{ii}b_{ii}+a_{jj}b_{jj}-\Big[(a_{ii}b_{ii}-a_{jj}b_{jj})^2 \\ &\quad +4\alpha_i\alpha_j(b_{ii}-q(B))(b_{jj}-q(B))\Big]^{\frac{1}{2}}\Big\}^n, \end{aligned}$$

and

$$\begin{aligned} &|det(A\star B)|\geq[q(A\star B)]^n \\ &\geq \min_{i\neq j}\tfrac{1}{2^n}\Big\{a_{ii}b_{ii}+a_{jj}b_{jj}-\Big[(a_{ii}b_{ii}-a_{jj}b_{jj})^2 \\ &\quad +4\beta_i\beta_j(a_{ii}-q(A))(a_{jj}-q(A))\Big]^{\frac{1}{2}}\Big\}^n. \end{aligned}$$

III. Numerical examples

Example 3.1

$$A=\begin{bmatrix} 4 & -1 & -1 & -1 \\ -2 & 5 & -1 & -1 \\ 0 & -2 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix},$$

$$B=\begin{bmatrix} 1 & -1/2 & 0 & 0 \\ -1/2 & 1 & -1/2 & 0 \\ 0 & -1/2 & 1 & -1/2 \\ 0 & 0 & -1/2 & 1 \end{bmatrix}.$$

By calculation, we have $q(B\circ A^{-1})=0.2148$. By the inequality (1), we get

$$q(B\circ A^{-1})\geq 0.07$$

By the inequality (2), we get

$$q(B\circ A^{-1})\geq 0.052$$

By the inequality (3), we get

$$q(B\circ A^{-1})\geq 0.075$$

By Theorem 2.4, we have

$$q(B\circ A^{-1})\geq 0.1729.$$

Example 3.2

$$A=\begin{bmatrix} 8 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 5 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 8 \end{bmatrix} B=\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

It is easy to calculate that $\rho(A\circ B)=\rho(A)=8.1801$. If we use Gersgorin theorem and Brauer theorem, we have

$$\rho(A\circ B)\leq 9.$$

and

$$\rho(A\circ B)\leq 9.$$

If we take $p^1(A)=\{e_{1,2},e_{2,3},e_{3,4},e_{4,5}\}$, $r_{A\circ B}(1,2)=r_{A\circ B}(1,4)=r_{A\circ B}(2,5)=8.3166$, $r_{A\circ B}(1,5)=9$, $r_{A\circ B}(2,4)=4$, $r_{A\circ B}(2,3)=r_{A\circ B}(3,4)=6$, $r_{A\circ B}(1,3)=r_{A\circ B}(3,5)=8.5616$.

From Theorem 2.5 we get

$$\begin{aligned} \rho(A\circ B) &\leq M_c^r(A\circ B) \\ &= \max\Big\{\min_{\gamma\in\Psi(A\circ B)} r_{A\circ B}(\gamma), \max\Theta_{A\circ B}\Big\} \\ &= 8.3166. \end{aligned}$$

Example 3.3

$$A=\begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix} B=\begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$$

By calculation, we have

$$q(A\star B)=6$$

By Theorem 2.6, we get

$$q(A\star B)=6.$$

IV. Conclusions

In this paper, we give some inequalities for the spectral radius of the Hadamard product of two nonnegative matrices. These bounds improve some existing results and numerical examples illustrate that our results are superior.

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