Prime(Semiprime) Fuzzy h-ideal in Γ-hemiring

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Abstract—The notions of prime(semiprime) fuzzy h-ideal(h-bi-ideal, h-quasi-ideal) in Γ-hemiring and some of their characterizations are obtained by using "belongingness(∈)" and "quasi – coincidence(γ)". Cartesian product of prime(semiprime) fuzzy h-ideals of Γ-hemirings are also investigated.

Keywords—Γ-hemiring, Fuzzy h-ideals, Prime fuzzy left h-ideal, Prime(semiprime) (∈, ∈ ∨γ)-fuzzy left h-bi-ideal(h-ideal, h-quasi-ideal), Cartesian product

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I. INTRODUCTION

The concept of fuzzy set was introduced by Zadeh[17]. Jun and Lee[7] applied the concept of fuzzy sets to the theory of Γ-rings. The notion of Γ-semiring was introduced by Rao[12] as a generalization of Γ-ring as well as of semiring. Ideals of semiring play a crucial role in the structure theory but they do not in general coincide with the usual ring ideals if S is a ring and for this reason, their usage is somewhat limited when we try to obtain some analogues of ring theorems for semirings. Indeed, many results in rings apparently have no analogues in semirings by using only ideals. In this aspect, Henriksen[5] defined a special type of ideals in semirings, which are called k-ideals. A still more restricted class of ideals in hemirings has been given by Izuka[6]. However, a definition of ideal in any commutative semiring S can be given which coincides with Izuka’s definition provided S is a hemiring, and it is called h-ideal. LaTorre[9] investigated h-ideals and k-ideals in hemirings in an effort to obtain analogues of familiar ring theorems. Jun et al[8], Zhan et al[18], considered the fuzzy setting of h-ideals in Γ-hemiring. As a continuation of this investigation Yin et al[16], Ma et al[10] introduced and studied fuzzy h-bi-ideals, fuzzy h-quasi-ideals, in hemirings. In[2], Bhakat and Das introduced a new type of fuzzy subgroup using the combined notions of "belongingness(∈)" and "quasi – coincidence(γ)". For more results on fuzzy h-bi-ideals, fuzzy h-quasi-ideals in Γ-hemiring we refer to [15].

In this paper, we consider some properties of prime(semiprime) fuzzy left h-ideals in Γ-hemirings. The concepts of prime(semiprime) (∈, ∈ ∨γ)-fuzzy h-bi-ideals(h-ideals, h-quasi-ideals) in Γ-hemiring are described and some of their characterizations are obtained.

II. PRELIMINARIES

We recall the following preliminaries for subsequent use.

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Definition 2.1: [4] A hemiring [respectively semiring] is a nonempty set S on which operations addition and multiplication have been defined such that the following conditions are satisfied:
(i) (S,+) is a commutative monoid with identity 0.
(ii) (S,.) is a semigroup [respectively monoid with identity 1S].
(iii) Multiplication distributes over addition from either side.
(iv) 0s=0=s0 for all s S.
(v) 1S ≠ 0

For more preliminaries of semirings(hemirings) we refer to [4].

Definition 2.2: [14] Let S and Γ be two additive commutative semigroups with zero. Then S is called a Γ-hemiring if there exists a mapping S × Γ × S → S ( (x,a,y) ↦ axy for a, b ∈ S and x ∈ Γ) satisfying the following conditions:
(i) (a + b)x = ax + bx,
(ii) a(x + y) = ax + ay,
(iii) (a + b)x = ax + bx,
(iv) ax(βa) = (ab)xβ.

Let S be the additive commutative semigroup consisting of all non positive even integers. Then S is a semigroup of all non positive integers and Γ be the additive commutative semigroup of all non positive even integers. Then S is a Γ-hemiring.

Definition 2.3: [17] A fuzzy subset μ of a non-empty set S is defined to be a function μ : S → [0, 1].

Definition 2.4: A fuzzy set μ in S is called prime if μ(χy) = μ(x) or μ(xγy) = μ(y) for all x, y ∈ S and γ ∈ Γ.

Example 2.5: Let S = Γ, be the set of non-negative integers. Then S forms a Γ-hemiring. Define a fuzzy subset ζ of S by ζ(s) = \begin{cases} 1 & \text{if } s \text{ is even} \\ 0.1 & \text{otherwise} \end{cases}. Then ζ is a prime fuzzy h-ideal of S.
where $\Gamma$ is a h-ideal of $S$, $\mu$ is a fuzzy subset of $S$, defined as follows $\mu(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0.7 & \text{if } x \text{ is even} \\ 0.1 & \text{if } x \text{ is odd} \end{cases}$.

The fuzzy subset $\mu$ of $S$ is a fuzzy ideal of $S$.

Definition 2.8: [14] A left ideal $A$ of a h-ideal of $S$ is called a left h-ideal if for any $x, z \in S$, $a, b \in A$, $x + a + z = b + z \Rightarrow x \in A$.

A right h-ideal is defined analogously.

Definition 2.9: [14] A fuzzy left ideal $\mu$ of a h-ideal $S$ is called a fuzzy left h-ideal if for all $a, b, x, z \in S$, $x + a + z = b$, $z \Rightarrow \mu(x) \geq \min \{\mu(a), \mu(b)\}$.

A fuzzy right h-ideal is defined similarly.

By a fuzzy h-ideal $\mu$, we mean that $\mu$ is both fuzzy left and right h-ideal.

Example 2.10: In Example 2.7, $\mu$ is a fuzzy h-ideal also.

Definition 2.11: [13] Let $f$ be a function from a set $X$ to a set $Y$, $\mu$ be a fuzzy subset of $X$ and $\sigma$ be a fuzzy subset of $Y$.

Then image of $\mu$ under $f$, denoted by $f(\mu)$, is a fuzzy subset of $Y$ defined by $f(\mu)(y) = \sup_{x \in f^{-1}(y)} \{\mu(x) \}$ if $f^{-1}(y) \neq \emptyset$.

The pre-image of $\sigma$ under $f$, denoted by $f^{-1}(\sigma)(x) = \sigma(f(x))$ for all $x \in X$.

Definition 2.12: [13] Let $f$ be a function from a set $X$ to a set $Y$ and $\mu$ be a fuzzy subset of $X$. Then $\mu$ is said to be f-invariant if $f(\mu) = \mu(y)$ where $x,y \in Y$.

Theorem 2.13: [14] A fuzzy set $\mu$ of $S$ is a fuzzy left h-ideal of $S$ if and only if $\mu_i$ (level subset of $\mu$), $i \in [0,1]$ is a left h-ideal of $S$ whenever it is non-empty.

Corollary 2.14: [14] Let $A$ be a non-empty subset of a h-ideal of $S$. Then $A$ is a left h-ideal of $S$ if and only if $\chi_A$ (characteristic function of $A$) is a fuzzy left h-ideal of $S$.

Theorem 2.15: [14] Let $A$ be a non-empty subset of a h-ideal of $S$. Let $\mu$ be a fuzzy set in $S$ defined by $\mu(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$.

Then $\mu$ is a fuzzy left h-ideal of $S$ if and only if $A$ is a left h-ideal of $S$.

Definition 2.16: [11] Let $\mu$ and $\theta$ be two fuzzy sets of a h-ideal $S$. Define h-product of $\mu$ and $\theta$ by $\mu \ast \theta = \sup \{\min\{\mu(a), \theta(b)\}, \theta(b)\}$.

If $x$ cannot be expressed as above for $x,a,b_1,b_2, \gamma, \delta \in \Gamma$.

III. PRIME FUZZY LEFT H-IDEAL

Definition 3.1: [11] A fuzzy left(right) h-ideal $\mu$ of a h-ideal $S$ is said to be prime if $\mu$ is a non-constant function and for any two fuzzy left(right) h-ideals $\mu$ and $\nu$ of $S$, $\mu \cap \nu \subseteq \mu$ implies $\mu \subseteq \nu$ or $\nu \subseteq \mu$.

Theorem 3.2: A fuzzy set $\mu$ of a h-ideal of $S$ is a prime fuzzy left(right) h-ideal of $S$ if and only if $P(\mu) = 0$. A prime left(right) h-ideal of $S$ is either a prime left(right) h-ideal of $S$.

Theorem 3.3: [11] A fuzzy subset $\zeta$ of a h-ideal of $S$ is a prime fuzzy left(right) h-ideal of $S$ if and only if

(i) $\zeta_0 = \{x \in S : \zeta(x) = \zeta(0)\}$ is a prime left(right) h-ideal of $S$.

(ii) For all $x \in S$ contains exactly two elements.

(iii) $\zeta(0) = 1$.  

Example 3.4: Let $N_0$ be the set of all non-negative integers and $E(\Gamma)$ be the set of all non-negative even integers. Then $N_0$ is a h-ideal.

Consider the following fuzzy subsets of $N_0$:

$$\zeta(n) = \begin{cases} 0.5 & \text{if } n \text{ is even} \\ 0.2 & \text{if } n \text{ is even} \\ 0.1 & \text{if } n \text{ is even} \end{cases}$$

Definition 3.5: [13] Let $\mu_i : i \in \mathbb{Z}$, is any collection of non-constant prime semifinite fuzzy h-ideals of $S$ such that $\mu_i \subseteq \mu_1 \subseteq \mu_2 \subseteq \cdots \subseteq \mu_{n_0}$... then the following statements are true:

(a) $\mu_i$ is a prime(semi) fuzzy h-ideal of $S$.

(b) $\cap \mu_i$ is a prime(semi) fuzzy h-ideal of $S$.

Proof: It follows from Definition 3.3 that

(i) $1 \in \text{Im} \mu_i$.

(ii) The ideal $I_{\mu_i} = \{x \in S | \mu_i(x) = 1\}$ is prime.

(iii) There exists $\alpha_i, \beta_i \in [0,1]$ such that $\mu_i(x) = \alpha_i$ for all $x \in S = I_{\mu_i}$ for all $i \in \mathbb{Z}$.

Now, $I_{\mu_1} \subseteq \cdots \subseteq I_{\mu_i} \subseteq \cdots (since \mu_1 \subseteq \mu_2 \subseteq \cdots \subseteq \mu_{n_0})$... then the following statements are true.

(a) $\mu_i$ is a prime(semi) fuzzy h-ideal of $S$.

(b) $\cap \mu_i$ is a prime(semi) fuzzy h-ideal of $S$.

Proof: It follows from Definition 3.3 that

(i) $1 \in \text{Im} \mu_i$.

(ii) The ideal $I_{\mu_i} = \{x \in S | \mu_i(x) = 1\}$ is prime.

(iii) There exists $\alpha_1, \beta_1 \in [0,1]$ such that $\mu_i(x) = \alpha_i$ for all $x \in S = I_{\mu_i}$ for all $i \in \mathbb{Z}$.

Now, $I_{\mu_1} \subseteq \cdots \subseteq I_{\mu_i} \subseteq \cdots (since \mu_1 \subseteq \mu_2 \subseteq \cdots \subseteq \mu_{n_0})$... then the following statements are true.

(a) $\mu_i$ is a prime(semi) fuzzy h-ideal of $S$.

(b) $\cap \mu_i$ is a prime(semi) fuzzy h-ideal of $S$.

Proof: It follows from Definition 3.3 that

(i) $1 \in \text{Im} \mu_i$.

(ii) The ideal $I_{\mu_i} = \{x \in S | \mu_i(x) = 1\}$ is prime.

(iii) There exists $\alpha_1, \beta_1 \in [0,1]$ such that $\mu_i(x) = \alpha_i$ for all $x \in S = I_{\mu_i}$ for all $i \in \mathbb{Z}$.

Now, $I_{\mu_1} \subseteq \cdots \subseteq I_{\mu_i} \subseteq \cdots (since \mu_1 \subseteq \mu_2 \subseteq \cdots \subseteq \mu_{n_0})$... then the following statements are true.

(a) $\mu_i$ is a prime(semi) fuzzy h-ideal of $S$.

(b) $\cap \mu_i$ is a prime(semi) fuzzy h-ideal of $S$.

Proof: It follows from Definition 3.3 that

(i) $1 \in \text{Im} \mu_i$.

(ii) The ideal $I_{\mu_i} = \{x \in S | \mu_i(x) = 1\}$ is prime.

(iii) There exists $\alpha_1, \beta_1 \in [0,1]$ such that $\mu_i(x) = \alpha_i$ for all $x \in S = I_{\mu_i}$ for all $i \in \mathbb{Z}$.

Now, $I_{\mu_1} \subseteq \cdots \subseteq I_{\mu_i} \subseteq \cdots (since \mu_1 \subseteq \mu_2 \subseteq \cdots \subseteq \mu_{n_0})$... then the following statements are true.
or \( f(\theta) \subseteq \mu' \). Therefore either \( f^{-1}(f(\sigma)) \subseteq f^{-1}(\mu') \) or \( f^{-1}(f(\theta)) \subseteq f^{-1}(\mu) \). Hence \( \sigma \leq f^{-1}(\mu) \) or \( \theta \leq f^{-1}(\mu) \). So, \( f^{-1}(\mu) \) is prime fuzzy h-ideal of \( S \). Similarly, we can prove the result for semiprime fuzzy h-ideal of \( S \).

Let \( S_1 \) and \( S_2 \) be two \( \Gamma \)-hemirings. Now if we define a mapping \((S_1 \times S_2) \times \Gamma \rightarrow (S_1 \times S_2) \rightarrow S_1 \times S_2 \) by \((x_1,y_1, \alpha)(x_2, y_2) = (x_1\alpha x_2, y_1\alpha y_2)\) for all \((x_1, y_1), (x_2, y_2) \in S_1 \times S_2\) and for all \(\alpha \in \Gamma\), then the cartesian product \( S_1 \times S_2 \) becomes a \( \Gamma \)-hemiring.\n
**Definition 3.7:** [3] Let \( \mu \) and \( \sigma \) be two fuzzy subsets of a set \( X \). The the cartesian product of \( \mu \) and \( \sigma \) is defined by \((\mu \times \sigma)(x,y) = \min\{\mu(x), \sigma(y)\}\) for all \( x, y \in X \).

**Lemma 3.8:** Let \( \mu \) and \( \sigma \) be two fuzzy subsets of a set \( X \) and \( t \in [0,1] \). Then \((\mu \times \sigma)t = \mu_t \times \sigma_t \).

**Proof:** \((x,y) \in (\mu \times \sigma)t \) if and only if \( x \in \mu_t \) and \( y \in \sigma_t \). Hence, \((\mu \times \sigma)t = \mu_t \times \sigma_t \).

**Proposition 3.9:** [14] Let \( \mu \) and \( \sigma \) be two fuzzy left h-ideals of \( S \). Then \( \mu \times \sigma \) is a fuzzy left h-ideal (resp. fuzzy right h-ideal, fuzzy h-ideal) of the \( \Gamma \)-hemiring \( S \times S \).

**Proposition 3.10:** Let \( \mu \) and \( \sigma \) be two prime fuzzy h-ideal of the \( \Gamma \)-hemiring \( S \times S \). Then \((\mu \times \sigma) t = \mu_t \times \sigma_t \) is a prime fuzzy h-ideal of the \( \Gamma \)-hemiring \( S \times S \).

**Example 3.11:** Let \( S \) be the set of non-negative integers and \( \Gamma \) be the set of non-positive even integers. Then \( S \) forms a \( \Gamma \)-hemiring.

**Proposition 3.12:** Let \( \mu \) and \( \sigma \) be two prime fuzzy h-ideals of the \( \Gamma \)-hemiring \( S \). Then the level subset \((\mu \times \sigma)t \in I_m(\mu \times \sigma)\) is a prime h-ideal of the \( \Gamma \)-hemiring \( S \times S \).

**Proof:** By Proposition 3.10 \( \mu \times \sigma \) is a prime fuzzy h-ideal of \( S \times S \) and so \((\mu \times \sigma)t \) is an h-ideal of \( S \times S \) and \( \gamma \in \Gamma \). Therefore \((\mu \times \sigma)(x,y) = \mu_t \times \sigma_t \) is a prime h-ideal of \( S \times S \). Hence \((\mu \times \sigma)t \) is a prime h-ideal of \( S \times S \).

**Proposition 3.13:** If the level subset \((\mu \times \sigma)t \in I_m(\mu \times \sigma)\) of \( \mu \times \sigma \) is a prime h-ideal of \( S \times S \) then \((\mu \times \sigma)t \) is a prime fuzzy h-ideal of the \( \Gamma \)-hemiring \( S \times S \).

**Proof:** Straightforward.

**IV. PRIME \((\in,\notin)\)-FUZZY H-IDEALS**

A fuzzy set \( F \) of \( S \) is said to be a fuzzy point with support \( x \) and value \( t \) and is denoted by \( x_t \). A fuzzy point \( x_t \) is said to belong to (resp. be quasi-coincident with) a fuzzy set \( F \) written as \( x_t \in F \) or \( x_t \notin F \) if \( F(x) \geq t \) (resp. \( F(x) < t \)). If \( x_\in \notin F \) or (resp. and) \( x_\notin F \), then we write \( x_t \in F \) or \( x_t \notin F \).

**Proposition 3.14:** If \( \alpha \neq \beta \) the fuzzy points with fuzzy subsets, the concept of \((\alpha, \beta)\)-fuzzy subgroup where \( \alpha \) and \( \beta \) are any two of \( [\alpha, \in \notin, q] \) with \( \alpha \neq \beta \) was introduced in [2]. More results on \((\in \notin, q)\)-fuzzy subsemigroup can be found in [1].

**Definition 4.1:** A fuzzy set \( F \) of \( S \) is said to be an \((\in \notin, q)\)-fuzzy h-ideal of \( S \) if for all \( t,r \in [0,1], x,y,z \in S \) with \( x+a+z=b+z \).

**Definition 4.2:** A fuzzy set \( F \) of \( S \) is said to be an \((\in \notin, q)\)-fuzzy h-bi-ideal \( S \) if for all \( t,r \in [0,1], x,y,z \in S \) with \( x+a+z=b+z \).

**Definition 4.3:** A fuzzy set \( F \) of \( S \) is said to be an \((\in \notin, q)\)-fuzzy h-interior-ideal of \( S \) if for all \( t,r \in [0,1], x,y,z \in S \) with \( x+a+z=b+z \).

**Definition 4.4:** A fuzzy set \( F \) of \( S \) is said to be an \((\in \notin, q)\)-fuzzy h-quasi-ideal of \( S \) if for all \( t,r \in [0,1], x,y,z \in S \) with \( x+a+z=b+z \).

**Definition 4.5:** An \((\in \notin, q)\)-fuzzy h-ideal (resp. h-bi-ideal, h-quasi-ideal) \( F \) of \( S \) is called prime if \( (x,y) \in F \) implies \( x_t \in F \) or \( y_t \in F \).
**Remark 4.6:** Theorem 3.9 of [15] shows that the conditions of definition 4.2 are equivalent to
(i) For all \( x, y \in S \), \( F(x + y) \geq \min \{ F(x), F(y), 0.5 \} \)
(ii) For all \( x, y \in S, a, b \in S \), \( x + a + z = b + z \) implies \( F(x) \geq \min \{ F,(a), F(b), 0.5 \} \)
(iii) For all \( x, y, z \in S \), \( \alpha, \beta, \gamma \in \Gamma \), \( x \gamma y \in F \) if and only if \( \alpha + \beta + \gamma = 1 \)
(iv) For all \( x, y \in S \), \( \alpha, \beta, \gamma \in \Gamma \), \( F(x \gamma y) \geq \min \{ F(x), F(y), 0.5 \} \)

**Theorem 4.7:** An \((\varepsilon, \in \vee q)\)-fuzzy h-bi-ideal of \( S \) is prime if and only if for all \( x, y, z \in S \), \( x \gamma y \in F \) implies \( F(x) \geq \min \{ F(x), F(y), 0.5 \} \)

**Proof:** Let \( F \) be a prime set \((\varepsilon, \in \vee q)\)-fuzzy h-bi-ideal of \( S \). If there exist \( x, y \in S \), \( \gamma \in \Gamma \) such that \( F(x \gamma y) = \min \{ F(x), F(y), 0.5 \} \), then \( x \gamma y \in F \) and \( y \gamma x \in F \). Hence we have \( x \gamma y \in F \) and \( y \gamma x \in F \) for all \( x, y \in S \). Therefore \( F \) is prime.

**Theorem 4.8:** An \((\varepsilon, \in \vee q)\)-fuzzy h-bi-ideal of \( S \) is prime if and only if \( F(x \gamma y) \geq \min \{ F(x) + t, F(y) + t, 0.5 \} \) for all \( t \in (0, 0.5) \).

**Proof:** Let \( F \) be a prime set \((\varepsilon, \in \vee q)\)-fuzzy h-bi-ideal of \( S \). Suppose \( x, y \in F \) and \( t \in (0, 0.5) \). Then \( x \gamma y \in F \). Suppose \( x \gamma y \in F \) and \( t \in (0, 0.5) \). Then \( x \gamma y \notin F \). Therefore \( F \) is prime.
\textbf{Proposition 5.8:} If the level subset \((\mu \times \sigma), t \in Im(\mu \times \sigma)\) of \(\mu \times \sigma\) is a semiprime h-ideal of \(S \times S\) then \((\mu \times \sigma)\) is a semiprime fuzzy h-ideal of the \(\Gamma\)-hemiring \(S \times S\).

\textbf{Proof:} By Proposition 3.9, \(\mu \times \sigma\) is a fuzzy h-ideal of \(S \times S\). Let for \((x, y) \in S \times S\) and \(\gamma \in \Gamma\), \((x, y)\gamma(x, y) \in \mu \times \sigma\). Then \(\mu \times \sigma\) is a semiprime fuzzy h-ideal of \(S \times S\) if \((\mu \times \sigma)\) is a semiprime fuzzy h-ideal of \(S \times S\).

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\textbf{References}


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