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Prime(Semiprime) Fuzzy h-ideal in Γ-hemiring

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Abstract—The notions of prime(semiprime) fuzzy h-ideal(h-bi-ideal, h-quasi-ideal) in Γ-hemiring are introduced and some of their characterizations are obtained by using "$belongingness(\in)$" and "$quasi-coincidence(q)$". Cartesian product of prime(semiprime) fuzzy h-ideals of Γ-hemirings are also investigated.

Keywords—Γ-hemiring, Fuzzy h-ideals, Prime fuzzy left h-ideal, Prime(semiprime) $(\in, \in \vee q)$-fuzzy left h-bi-ideal(h-ideal, h-quasi-ideal), Cartesian product

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I. Introduction

THE concept of fuzzy set was introduced by Zadeh[17]. Jun and Lee[7] applied the concept of fuzzy sets to the theory of $\Gamma-$rings. The notion of $\Gamma-$semiring was introduced by Rao[12] as a generalization of $\Gamma-$ring as well as of semiring. Ideals of semiring play a crucial role in the structure theory but they do not in general coincide with the usual ring ideals if S is a ring and for this reason, their usage is somewhat limited when we try to obtain some analogues of ring theorem for semirings. Indeed, many results in rings apparently have no analogues in semirings by using only ideals. In this aspect, Henriksen[5] defined a special type of ideals in semirings, which are called k-ideals. A still more restricted class of ideals in hemirings has been given by Iizuka[6]. However a definition of ideal in any commutative semiring S can be given which coincides with Iizuka's definition provided S is a hemiring, and it is called h-ideal. LaTorre[9] investigated h-ideals and k-ideals in hemirings in an effort to obtain analogues of familiar ring theorems. Jun et al[8], Zhan et al[18], considered the fuzzy setting of h-ideals in $\Gamma-$hemiring. As a continuation of this investigation Yin et al[16], Ma et al[10] introduced and studied fuzzy h-bi-ideals, fuzzy h-quasi-ideals, in hemirings. In[2], Bhakat and Das introduced a new type of fuzzy subgroup using the combined notions of $''belongingness(\in)''$ and $''quasi-coinsidence(q)''$. For more results on fuzzy h-bi-ideals, fuzzy h-quasi-ideals in $\Gamma-$hemiring we refer to [15].

In this paper, we consider some properties of prime(semiprime) fuzzy left h-ideals in $\Gamma-$hemirings. The concepts of prime(semiprime) $(\in, \in \vee q)-$fuzzy h-bi-ideals(h-ideals, h-quasi-ideals) in Γ-hemiring are described and some of their characterizations are obtained.

II. Preliminaries

We recall the following preliminaries for subsequent use.

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***Definition** 2.1:* [4] A hemiring [respectively semiring] is a nonempty set S on which operations addition and multiplication have been defined such that the following conditions are satisfied:
(i) (S,+) is a commutative monoid with identity 0.
(ii) (S,.) is a semigroup [respectively monoid with identity 1_S].
(iii) Multiplication distributes over addition from either side.
(iv) 0s=0=s0 for all s∈S.
(v) $1_S \neq 0$
For more preliminaries of semirings(hemirings) we refer to [4].

***Definition** 2.2:* [14] Let S and Γ be two additive commutative semigroups with zero. Then S is called a $\Gamma-$hemiring if there exists a mapping
$S \times \Gamma \times S \rightarrow S$ ((x,α,y) $\mapsto$ xαy for $a, b \in S$ and $\alpha \in \Gamma$) satisfying the following conditions:
(i) $(a + b)\alpha c = a\alpha c + b\alpha c$,
(ii) $a\alpha(b + c) = a\alpha b + a\alpha c$,
(iii) $a(\alpha + \beta)b = a\alpha b + a\beta b$,
(iv) $a\alpha(b\beta c) = (a\alpha b)\beta c$.
(v) $0_S\alpha a = 0_S = a\alpha 0_S$,
(ii)$a0_\Gamma b = 0_S = b0_\Gamma a$
for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$.
For simplification we write 0 instead of 0_S and 0_Γ.

Throughout this paper S denote the Γ-*hemiring.*

***Definition** 2.3:* [17] A fuzzy subset μ of a non-empty set is defined to be a function $\mu : S \rightarrow [0, 1]$.

***Definition** 2.4:* A fuzzy set μ in S is called prime if $\mu(x\gamma y) = \mu(x)$ or $\mu(x\gamma y) = \mu(y)$ for all x,y ∈ S and $\gamma \in \Gamma$.

***Example** 2.5:* Let S= Γ, be the set of non-negative integers. Then S forms a Γ-hemiring. Define a fuzzy subset ζ of S by $\zeta(s) = \begin{cases} 1 \text{ if s is even} \\ 0.1 \text{ otherwise} \end{cases}$. Then ζ is a prime fuzzy h-ideal of S.

***Definition** 2.6:* [14] Let μ be a non empty fuzzy subset of a $\Gamma-$hemiring S (i.e. $\mu(x) \neq 0$ for some $x \in S$). Then μ is called a fuzzy left ideal [fuzzy right ideal] of S if
(i) $\mu(x + y) \geq min[\mu(x), \mu(y)]$ and
(ii) $\mu(x\gamma y) \geq \mu(y)$ [respectively $\mu(x\gamma y) \geq \mu(x)$] for all $x, y \in S$, $\gamma \in \Gamma$.

A fuzzy ideal of a $\Gamma-$hemiring S is a non empty fuzzy subset of S which is a fuzzy left ideal as well as a fuzzy right ideal of S.
Note that if μ is a fuzzy left ideal of a $\Gamma-$hemiring S, then $\mu(0) \geq \mu(x)$ for all x∈ S.

***Example** 2.7:* Let S be the additive commutative semigroup of all non positive integers and Γ be the additive commutative semigroup of all non positive even integers. Then S is a

$\Gamma-$hemiring if $a\gamma b$ denotes the usual multiplication of integers a, γ, b where $a, b \in S$ and $\gamma \in \Gamma$. Let μ be a fuzzy subset of S, defined as follows $\mu(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0.7 & \text{if x is even} \\ 0.1 & \text{if x is odd} \end{cases}$
The fuzzy subset μ of S is a fuzzy ideal of S.

***Definition** 2.8:* [14]A left ideal A of a Γ-hemiring S is called a left h-ideal if for any x, z ∈ S and a, b ∈ A, x + a + z = b + z ⇒ x ∈ A.

A right h-ideal is defined analogously.

***Definition** 2.9:* [14] A fuzzy left ideal μ of a Γ-hemiring S is called a fuzzy left h-ideal if for all a, b, x, z ∈ S, x + a + z = b + z ⇒ $\mu(x) \geq \min\ \{\mu(a), \mu(b)\}$.

A fuzzy right h-ideal is defined similarly.

By a fuzzy h-ideal μ, we mean that μ is both fuzzy left and right h-ideal.

***Example** 2.10:* In Example 2.7, μ is a fuzzy h-ideal also.

***Definition** 2.11:* [13] Let f be a function from a set X to a set Y; μ be a fuzzy subset of X and σ be a fuzzy subset of Y.

Then image of μ under f, denoted by $f(\mu)$, is a fuzzy subset of Y defined by

$$f(\mu)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

The pre-image of σ under f, symbolized by $f^{-1}(\sigma)$, is a fuzzy subset of X defined by

$f^{-1}(\sigma)(x) = \sigma(f(x))\ \forall x \in$ X.

***Definition** 2.12:* [13] Let f be a function from a set X to a set Y and μ be a fuzzy subset of X. Then μ is said to be f-invariant if f(x)=f(y) implies $\mu(x) = \mu(y)$ where x,y ∈ X.

***Theorem** 2.13:* [14] A fuzzy set μ of S is a fuzzy left h-ideal of S if and only if μ_t (the level subset of μ), t ∈ [0,1] is a left h-ideal of S whenever it is non-empty.

***Corollary** 2.14:* [14] Let A be a non-empty subset of a $\Gamma-$hemiring S. Then A is a left h-ideal of S if and only if χ_A(the characteristic function of A) is a fuzzy left h-ideal of S.

***Theorem** 2.15:* [14] Let A be a non-empty subset of a $\Gamma-$hemiring S. Let μ be a fuzzy set in S defined by

$\mu(x) = \begin{cases} \text{s} & \text{if } x \in A \\ \text{t} & \text{otherwise} \end{cases}$ where s>t; s,t∈ [0,1].

Then μ is a fuzzy left h-ideal of S if and only if A is a left h-ideal of S.

***Definition** 2.16:* [11] Let μ and θ be two fuzzy sets of a $\Gamma-$hemiring S. Define h-product of μ and θ by

$$\mu\Gamma_h\theta(x) = \sup_{x+a_1\gamma b_1+z=a_2\delta b_2+z}[\min\{\mu(a_1), \mu(a_2), \theta(b_1), \theta(b_2)\}]$$
$$= 0, \text{if x cannot be expressed as above}$$

for x,z,$a_1, a_2, b_1, b_2 \in$ S and $\gamma, \delta \in \Gamma$.

III. Prime fuzzy left h-ideal

***Definition** 3.1:* [11] A fuzzy left(right) h-ideal ζ of a $\Gamma-$hemiring S is said to be prime if ζ is a non-constant function and for any two fuzzy left(right) h-ideals μ and ν of S, $\mu\Gamma_h\nu \subseteq \zeta$ implies $\mu \subseteq \zeta$ or $\nu \subseteq \zeta$.

***Theorem** 3.2:* A fuzzy set χ_p of a $\Gamma-$hemiring S is a prime fuzzy left(right) h-ideal of if and only if P(≠ S) is a prime left(right) h-ideal of S i.e., for any two left(right) h-ideal A and B of S, AΓB⊆ P implies either A⊆ P or B⊆ P.

***Theorem** 3.3:* [11] A fuzzy subset ζ of a $\Gamma-$hemiring S is a prime fuzzy left(right) h-ideal of S if and only if
(i) $\zeta^0 = \{x \in S : \zeta(x) = \zeta(0)\}$ is a prime left(right) h-ideal of S.
(ii) Im $\zeta = \{\zeta(x) : x \in S\}$ contains exactly two elements.
(iii) $\zeta(0) = 1$.

***Example** 3.4:* Let N_0(=S) be the set of all non-negative integers and E(=Γ) be the set of all non-negative even integers. Then N_0 is a $\Gamma-$hemiring.

Consider the following fuzzy subsets of N_0 :

$$\zeta(n) = \begin{cases} 0.5 & \text{if n is even} \\ 0.2 & \text{otherwise} \end{cases}$$
$$\mu(n) = \begin{cases} 0.7 & \text{if n is even} \\ 0 & \text{otherwise} \end{cases}$$
$$\nu(n) = \begin{cases} 0.3 & \text{if n=3k, for some k } \in N_0 \\ 0 & \text{otherwise} \end{cases}$$

Then ζ, μ and ν are fuzzy left h-ideals of N_0 such that $\mu\Gamma_h\nu \subseteq \zeta$. But $\mu(2) = 0.7 > 0.5 = \zeta(2)$, and $\nu(3) = 0.3 > 0.2 = \zeta(3)$. Thus ζ is not a prime fuzzy left h-ideal of N_0.

***Theorem** 3.5:* If $\{\mu_i : i \in \mathbf{Z}_+\}$ is any collection of non-constant prime(semiprime) fuzzy h-ideals of S such that $\mu_1 \subseteq \mu_2 \subseteq ... \subseteq \mu_n$... then the following statements are true-
(a) $\cup\mu_i$ is a prime(semiprime) fuzzy h-ideal of S.
(b) $\cap\mu_i$ is a prime(semiprime) fuzzy h-ideal of S.

Proof: It follows from Theorem 3.3 that
(i) $1 \in\ Im\ \mu_i$.
(ii) The ideal $I_{\mu_i} = \{x \in S | \mu_i(x) = 1\}$ is prime.
(iii) There exists $\alpha_i \in [0, 1)$ such that $\mu_i(x) = \alpha_i$ for all x∈ $S - I_{\mu_i}$ for all $i \in \mathbf{Z}_+$.
Now, $I_{\mu_1} \subseteq ... \subseteq I_{\mu_n} \subseteq ...$ (since $\mu_1 \subseteq \mu_2 \subseteq ... \subseteq \mu_n$...) and hence $\cup I_{\mu_i}$ and $\cap I_{\mu_i}$ are prime h-ideals of S.
(a) Let μ and θ be any two fuzzy h-ideals of S such that $\mu\Gamma_h\theta \subseteq \cup\mu_i$. Assume that $\cup\mu_i$ is not prime. Then for some x,y ∈S we have $\mu(x) > \cup\mu_i(x)$ and $\theta(y) > \cup\mu_i(y)$ Therefore $\cup\mu_i(x) \neq 1$ and $\cup\mu_i(y) \neq 1$ so that x,y$\notin \cup I_{\mu_i}$ and hence $\cup\mu_i(x) = \cup\mu_i(y) = \sup\{\alpha_i\} = \cup\mu_i(x\gamma y)$. But then $(\mu\Gamma_h\theta)(x\gamma y) \geq \min\{\mu(x), \theta(y)\} \geq \min\{\cup\mu_i(x), \cup\mu_i(y)\} = \cup\mu_i(x\gamma y)$. This contradiction establishes the result.
(b) Straightforward.
For semiprime fuzzy h-ideals the proofs follows very similar. ■

***Theorem** 3.6:* Suppose S and S' be two Γ-hemiring and f be a morphism from S onto S'. If μ and μ' be any prime(semiprime) fuzzy h-ideal of S and S' respectively. Then
(i)f(μ) is a prime(semiprime) fuzzy h-ideal of S', provided μ is f-invariant.
(ii)$f^{-1}(\mu')$ is a prime(semiprime) fuzzy h-ideal of S.

Proof: (i) Let σ' and θ' be any two fuzzy h-ideals of S' such that $\sigma'\Gamma_h\theta' \subseteq f(\mu)$. Then $f^{-1}(\sigma'\Gamma_h\theta') \subseteq f^{-1}(f(\mu))$ ⇒ $f^{-1}(\sigma')\Gamma_h f^{-1}(\theta') \subseteq \mu$. Therefore $f^{-1}(\sigma') \subseteq \mu$ or $f^{-1}(\theta') \subseteq \mu$, since μ is prime. Hence either
$\sigma' = f(f^{-1}(\sigma')) \subseteq f(\mu)$ or $\theta' = f(f^{-1}(\theta')) \subseteq f(\mu)$.
So, $f(\mu)$ is prime fuzzy h-ideal of S'.

(ii) Let σ and θ be any two fuzzy h-ideals of S such that $\sigma\Gamma_h\theta \subseteq f^{-1}(\mu')$. Then $f(\sigma\Gamma_h\theta) \subseteq f(f^{-1}(\mu'))$ ⇒ $f(\sigma)\Gamma_h f(\theta) \subseteq \mu'$. Since μ' is prime either $f(\sigma) \subseteq \mu'$

or $f(\theta) \subseteq \mu'$. Therefore either $f^{-1}(f(\sigma)) \subseteq f^{-1}(\mu')$ or $f^{-1}(f(\theta)) \subseteq f^{-1}(\mu')$. Hence $\sigma \subseteq f^{-1}(\mu')$ or $\theta \subseteq f^{-1}(\mu')$. So, $f^{-1}(\mu')$ is prime fuzzy h-ideal of S.
Similarly, we can prove the result for semiprime fuzzy h-ideal of S. ■

Let S_1 and S_2 be two Γ-hemirings. Now if we define a mapping $(S_1 \times S_2) \times \Gamma \times (S_1 \times S_2) \rightarrow S_1 \times S_2$ by $(x_1, y_1)\alpha(x_2, y_2) = (x_1\alpha x_2, y_1\alpha y_2)$ for all $(x_1, y_1), (x_2, y_2) \in S_1 \times S_2$ and for all $\alpha \in \Gamma$, then the cartesian product $S_1 \times S_2$ becomes a Γ-hemiring.

***Definition** 3.7:* [3] Let μ and σ be two fuzzy subsets of a set X. The the cartesian product of μ and σ is defined by $(\mu \times \sigma)(x, y) = \min\{\mu(x), \sigma(y)\} \forall x, y \in X$.

***Lemma** 3.8:* Let μ and σ be two fuzzy subsets of a set X and $t \in [0, 1]$. Then $(\mu \times \sigma)_t = \mu_t \times \sigma_t$.

Proof: (x,y)$\in \mu_t \times \sigma_t \Leftrightarrow x \in \mu_t$ and $y \in \sigma_t \Leftrightarrow \mu(x) \geq t$ and $\sigma(y) \geq t \Leftrightarrow \min\{\mu(x), \sigma(y)\} \geq t \Leftrightarrow (\mu \times \sigma)(x, y) \geq t \Leftrightarrow (x, y) \in (\mu \times \sigma)_t$. Hence $(\mu \times \sigma)_t = \mu_t \times \sigma_t$. ■

***Proposition** 3.9:* [14] Let μ and σ be two fuzzy left h-ideals(fuzzy right h-ideals, fuzzy h-ideals) of a Γ-hemiring S. Then $\mu \times \sigma$ is a fuzzy left h-ideal(resp. fuzzy right h-ideal, fuzzy h-ideal) of the Γ-hemiring $S \times S$.

***Proposition** 3.10:* Let μ and σ be two prime fuzzy h-ideal of a Γ-hemiring S. Then $\mu \times \sigma$ is a prime fuzzy h-ideal of the Γ-hemiring $S \times S$.

Proof: By Proposition 3.9 $\mu \times \sigma$ is a fuzzy h-ideal of $S \times S$. Let $(a, b), (c, d) \in S \times S$. Then $(\mu \times \sigma)\{(a, b)\gamma(c, d)\}$
$= (\mu \times \sigma)(a\gamma c, b\gamma d) = \min\{\mu(a\gamma c), \sigma(b\gamma d)\}$
$= \min[\max\{\mu(a), \mu(c)\}, \max\{\sigma(b), \sigma(d)\}]$
(since μ and σ are prime fuzzy h-ideals of S)
$= \max[\min\{\mu(a), \sigma(b)\}, \min\{\mu(c), \sigma(d)\}]$
$= \max\{(\mu \times \sigma)(a, b), (\mu \times \sigma)(c, d)\}$.
Hence $(\mu \times \sigma)$ is a prime fuzzy h-ideal of $S \times S$. ■

***Example** 3.11:* Let S be the set of non-positive integers and Γ be the set of non-positive even integers. Then S forms a Γ-hemiring where $a\alpha b$ $(a, b \in S \text{ and } \alpha \in \Gamma)$ denotes the usual multiplication of a, α, and b. Now define two fuzzy subsets μ and σ of S by $\mu(x) = \begin{cases} 1 & \text{if x is even} \\ 0.1 & \text{otherwise} \end{cases}$ and $\sigma(x) = \begin{cases} 1 & \text{if x is even} \\ 0.2 & \text{otherwise} \end{cases}$, then μ and σ are two fuzzy prime h-ideal of S. Now $(\mu \times \sigma)(x, y) = 1$ for all (x,y)$\in S \times S$. Therefore $(\mu \times \sigma)$ is a prime fuzzy h-ideal of the Γ-hemiring $S \times S$.

***Proposition** 3.12:* Let μ and σ be two prime fuzzy h-ideals of a Γ-hemiring S. Then the level subset $(\mu \times \sigma)_t, t \in Im(\mu \times \sigma)$ is a prime h-ideal of the Γ-hemiring $S \times S$.

Proof: By Proposition 3.10 $\mu \times \sigma$ is a prime fuzzy h-ideal of $S \times S$ and so $(\mu \times \sigma)_t$ is an h-ideal of $S \times S$. To show $(\mu \times \sigma)_t$ is prime, suppose for $(x, y), (m, n) \in S \times S$ and $\gamma \in \Gamma$, $(x, y)\gamma(m, n) \in (\mu \times \sigma)_t$. Then $(\mu \times \sigma)\{(x, y)\gamma(m, n)\}$ $\geq t \Rightarrow (\mu \times \sigma)(x\gamma m, y\gamma n) \geq t \Rightarrow \min\{\mu(x\gamma m), \sigma(y\gamma n)\}$ $\geq t \Rightarrow \mu(x\gamma m) \geq t$ or $\sigma(y\gamma n) \geq t \Rightarrow x\gamma m \in \mu_t$ or $y\gamma n \in \sigma_t \Rightarrow x \in \mu_t$ or $m \in \mu_t$ and $y \in \sigma_t$ or $n \in \sigma_t$(since μ_t and σ_t are prime h-ideals of S. Hence $(x, y) \in \mu_t \times \sigma_t$ or $(m, n) \in \mu_t \times \sigma_t$. Since by Lemma 3.8 $(\mu \times \sigma)_t = \mu_t \times \sigma_t$, we deduce that $(x, y) \in \mu_t \times \sigma_t$ or $(m, n) \in \mu_t \times \sigma_t$. Consequently, $(\mu \times \sigma)_t$ is a prime h-ideal of $S \times S$. ■

***Proposition** 3.13:* If the level subset $(\mu \times \sigma)_t, t \in Im(\mu \times \sigma)$ of $\mu \times \sigma$ is a prime h-ideal of $S \times S$ then $(\mu \times \sigma)$ is a prime fuzzy h-ideal of the Γ-hemiring $S \times S$.

Proof: Straightforward. ■

IV. Prime $(\in, \in \vee q)$-fuzzy h-ideals

A fuzzy set F of S of the form $F(y) = \begin{cases} t(\neq 0) & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$ is said to be a fuzzy point with support x and value t and is denoted by x_t.
A fuzzy point x_t is said to belong to (resp. be quasi-coincident with) a fuzzy set F written as $x_t \in$F(resp. x_t qF) if F(x)≥t (resp. F(x)+t>1). If $x_t \in$F or (resp. and) x_tqF, then we write $x_t \in \vee q$(resp. $\in \Lambda q$)F. If F(x)<t (resp. F(x)+t≤1), then we call $x_t \overline{\in}$F (resp. $x_t \overline{q}$F).
The symbol $\overline{\in \vee q}$ means $\in \vee q$ does not hold. Using the notion of "belongingness(∈)" and "quasi-coincidence(q)" of fuzzy points with fuzzy subsets, the concept of (α, β)-fuzzy subgroup where α and β are any two of $\{\in, q, \in \vee q, \in \Lambda q\}$ with $\alpha \neq \ \in \Lambda q$, was introduced in[2]. More results on $\{\in, q, \in \vee q, \in \Lambda q\}$-fuzzy subsemigroup can be found in[1].

***Definition** 4.1:* A fuzzy set F of S is said to be an $(\in, \in \vee q)$-fuzzy h-ideal of S if for all t,r ∈ (0,1], x,y,z ∈ S, $\gamma \in \Gamma$ the following condition hold:
(i) $x_t \in$F, $y_r \in$ F implies $(x + y)_{\min(t,r)} \in \vee q$F
(ii)$x_t \in$F, $y_r \in$ F implies $(x\gamma y)_{\min(t,r)} \in \vee q$F
(iii)$a_t \in$F, $b_r \in$ F implies $x_{\min(t,r)} \in \vee q$F for all a,b,x,z ∈ S with x+a+z=b+z.

***Definition** 4.2:* A fuzzy set F of S is said to be an $(\in, \in \vee q)$-fuzzy h-bi-ideal of S if for all t,r ∈ (0,1], x,y,z ∈ S, $\alpha, \beta, \gamma \in \Gamma$ the following condition hold:
(i) $x_t \in$F, $y_r \in$ F implies $(x + y)_{\min(t,r)} \in \vee q$F
(ii)$x_t \in$F, $y_r \in$ F implies $(x\gamma y)_{\min(t,r)} \in \vee q$F
(iii)$a_t \in$F, $b_r \in$ F implies $x_{\min(t,r)} \in \vee q$F for all a,b,x,z ∈ S with x+a+z=b+z
(iv)$x_t \in$F, $z_r \in$F implies $(x\alpha y\beta z)_{\min(t,r)} \in \vee q$F.

***Definition** 4.3:* A fuzzy set F of S is said to be an $(\in, \in vq)$-fuzzy h-interior-ideal of S if for all t,r ∈ (0,1], x,y,z ∈ S, $\alpha, \beta, \gamma \in \Gamma$ the following condition hold:
(i) $x_t \in$F, $y_r \in$ F implies $(x + y)_{\min(t,r)} \in \vee q$F
(ii)$x_t \in$F, $y_r \in$ F implies $(x\gamma y)_{\min(t,r)} \in \vee q$F
(iii)$a_t \in$F, $b_r \in$ F implies $x_{\min(t,r)} \in \vee q$F for all a,b,x,z ∈ S with x+a+z=b+z
(iv)$x_t \in$F, implies $(y\alpha x\beta z)_t \in \vee q$F.

***Definition** 4.4:* A fuzzy set F of S is said to be an $(\in, \in \vee q)$-fuzzy h-quasi-ideal of S if for all t,r ∈ (0,1], x,y,z ∈ S, $\gamma \in \Gamma$ the following condition hold:
(i) $x_t \in$F, $y_r \in$ F implies $(x + y)_{\min(t,r)} \in \vee q$F
(ii)$a_t \in$F, $b_r \in$ F implies $x_{\min(t,r)} \in \vee q$F for all a,b,x,z ∈ S with x+a+z=b+z.
(iii) $x_t \in (Fo_h\chi_S) \cap (\chi_S o_h F)$ implies $x_t \in \vee q$F

***Definition** 4.5:* An $(\in, \in \vee q)$-fuzzy h-ideal (resp. h-bi-ideal, h-quasi-ideal) F of S is called prime if $(x\gamma y)_t \in$ F implies $x_t \in \vee q$ F or $y_t \in \vee q$ F, for all x,y ∈ S, $\gamma \in \Gamma$ and t∈ (0,1].

Here we prove all the results for fuzzy h-bi-ideals; similar conclusion can be easily made for fuzzy h-ideals and fuzzy h-quasi-ideals.

Remark 4.6: Theorem 3.9 of [15] shows that the conditions of definition 4.2 are equivalent to
(i) For all x,y∈S, $F(x+y) \geq \min\{F(x), F(y), 0.5\}$
(ii) For all x,y∈S, $\gamma \in \Gamma$ $F(x\gamma y) \geq \min\{F(x), F(y), 0.5\}$
(iii) For all a,b,x,z∈S, x+a+z=b+z implies
$F(x) \geq \min\{F(x), F(y), 0.5\}$
(iv) For all x,y,z∈ S, $\alpha, \beta \in \Gamma$,
$F(x\alpha y\beta z) \geq \min\{F(x), F(z), 0.5\}$

Theorem 4.7: An $(\in, \in \vee q)-$fuzzy h-bi-ideal F of S is prime if and only if for all x,y ∈ S and $\gamma \in \Gamma$, $\max\{F(x), F(y)\} \geq \min\{F(x\gamma y), 0.5\}$.

Proof: Let F be a prime $(\in, \in \vee q)$-fuzzy h-bi-ideal of S. If there exist x,y ∈ S, $\gamma \in \Gamma$ such that $\max\{F(x), F(y)\} < \min\{F(x\gamma y), 0.5\}$=t, then $0 < t \leq 0.5$ and $(x\gamma y)_t \in F$ but $x_t \overline{\in}$ F and $y_t \overline{\in}$ F. Since F(x)+t≤ 1, $x_t \overline{q}$ F. Similarly $y_t \overline{q}$ F. Hence we have $x_t \overline{\in vq}$ F and $y_t \overline{\in \vee q}$ F which is a contradiction. Thus for all x,y ∈ S and $\gamma \in \Gamma$, $\max\{F(x), F(y)\} \geq \min\{F(x\gamma y), 0.5\}$.
Conversely, suppose the condition holds.
Let $(x\gamma y)_t \in$ F. Then $F(x\gamma y) \geq t$ and so $\max\{F(x), F(y)\} \geq \min\{F(x\gamma y), 0.5\} \geq \min\{t, 0.5\}$.
Now if t≤0.5, then F(x)≥ t or F(y)≥ t which implies $x_t \in$ F or $y_t \in$ F. Thus $x_t \in \vee q$ F or $y_t \in \vee q$ F.
If t>0.5, then $\max\{F(x), F(y)\} \geq 0.5$. So F(x)≥ 0.5 or F(y)≥ 0.5 which implies F(x)+t≥1 or F(y)+t≥ 0.5 that is $x_t q$ F or $y_t q$ F. Thus $x_t \in \vee q$ F or $y_t \in \vee q$ F. Hence F is prime. ∎

Theorem 4.8: An $(\in, \in \vee q)-$fuzzy h-bi-ideal F of S is prime if and only if $F_t(\neq \phi)$ (the level subset of F) is a prime h-bi ideal of S for all t ∈ (0,0.5].

Proof: Let F be a prime $(\in, \in \vee q)-$fuzzy h-bi-ideal of S and $0 < t < 0.5$. Suppose x,y,z ∈ S and $\alpha, \beta, \gamma \in \Gamma$. If x,y $\in F_t$ then F(x)≥ t and F(y)≥ t. This implies
$F(x+y) \geq \min\{F(x), F(y), 0.5\} = \min\{t, t, 0.5\} = t$
$F(x\gamma y) \geq \min\{F(x), F(y), 0.5\} = \min\{t, t, 0.5\} = t$
$F(x\alpha z\beta y) \geq \min\{F(x), F(y), 0.5\} = \min\{t, t, 0.5\} = t$
which implies $x+y, x\gamma y, x\alpha z\beta y \in F_t$.
Now, let x,z∈ S and a,b $\in F_t$ be such that x+a+z=b+z. Then $F(x) \geq \min\{F(a), F(b), 0.5\} = \min\{t, t, 0.5\} = t$.
So x $\in F_t$. Therefore F_t is an h-bi-ideal of S.
Let $x\gamma y \in F_t$. Now from Theorem 4.7, we have $\max\{F(x), F(y)\} \geq \min\{F(x\gamma y), 0.5\} \geq \min\{t, 0.5\} = t$ and so $F(x) \geq t$ or $F(y) \geq t$ which implies $x \in F_t$ 0r $y \in F_t$. Thus F_t is a prime h-bi-ideal of S.
Conversely, assume that F is a fuzzy subset of S such that $F_t(\neq \phi)$ is a prime h-bi-ideal of S. Then for every $x, y \in S$ and $\gamma \in \Gamma$, we can write $F(x) \geq \min\{F(x), F(y), 0.5\} = t_0$ and $F(y) \geq \min\{F(x), F(y), 0.5\} = t_0$. Hence $x, y \in F_{t_0}$ and so $x + y \in F_{t_0}$.
Thus $F(x+y) \geq \min\{F(x), F(y), 0.5\} = t_0$.
Since $x, y \in F_{t_0} \subseteq S$ and $\gamma \in \Gamma$ $x\gamma y \in F_{t_0}$. Hence $F(x\gamma y) \geq \min\{F(x), F(y), 0.5\} = t_0$.
Now let a,b $\in F_{t_0}$ such that x+a+z=b+z, for x,z∈ S. Then $x \in F_{t_0}$ which implies $F(x) \geq \min\{F(a), F(b), 0.5\}$.
Also for x,z∈ F_{t_0}, z∈ S and $\alpha, \beta \in \Gamma$, $x\alpha z\beta y \in F_{t_0}$ that is $F(x\alpha z\beta y) \geq \min\{F(x), F(y), 0.5\}$.
Thus F is an $(\in, \in \vee q)-$fuzzy h-bi-ideal of S.
Now for prime ideal, suppose $(x\gamma y)_t \in$ F. Then $x\gamma y \in F_t$. Since F_t is prime, $x \in F_t$ or $y \in F_t$ that is $x_t \in F$ or $y_t \in F$. Thus $x_t \in \vee q$ F or $y_t \in \vee q$.
Therefore F must be an prime$(\in, \in vq)-$fuzzy h-bi-ideal of S. ∎

Remark 4.9: The conclusion of the above result can be made for $0.5 < t \leq 1$.

Let F be a fuzzy set of S and t∈ $(0, 1]$. Suppose $Q(F;t) = \{x \in S \mid x_t q F\}$ and $[F]_t = \{x \in S \mid x_t \in \vee q F\}$. Then it is clear that $[F]_t = F_t \cup Q(F;t)$.

Theorem 4.10: A fuzzy set F of S is an $(\in, \in \vee q)-$fuzzy h-bi-ideal of S if and only if $[F]_t(\neq \phi)$ is an h-bi ideal of S for all t ∈ (0,1].

Proof: Let F be an $(\in, \in \vee q)-$fuzzy h-bi-ideal of S and let x,y $\in [F]_t$ for t∈(0,1]. Then $x_t \in \vee q F$ and $y_t \in \vee q F$ that is F(x)≥t or F(x)+t>1 and F(y)≥t or F(y)+t>1. Since F is an $(\in, \in \vee q)-$fuzzy h-bi-ideal of S, we have
$F(x+y) \geq \min\{F(x), F(y), 0.5\}$.
Case1 : F(x)≥ t and F(y)≥t
(i) If t≤0.5, then F(x+y)≥t. Therefore $(x+y)_t$ ∈F.
(ii) If t≥0.5, then $F(x+y) > \min\{F(x), F(y), 0.5\}$
$\geq \min\{t, 0.5\} = 0.5$. Therefore $(x+y)_t qF$.
Case2 : F(x)≥ t and F(y)+t>1
(i) If t≤0.5, then F(x+y)≥ $\min\{F(x), F(y), 0.5\}$
$\geq \min\{t, 1-t, 0.5\}$ =t. Therefore $(x+y)_t$ ∈F.
(ii) If t>0.5, then $F(x+y) > \min\{F(x), F(x), 0.5\}$
$\geq \min\{t, F(y), 0.5\} = \min\{F(y), 0.5\} > \min\{1-t, 0.5\}$
$= 1-t$ which implies F(x+y)+t>1. Therefore $(x+y)_t qF$.
Case3 : F(x)+t>1 and F(y)≥t
The proof is similar as case 2.
Case4 : F(x)+t>1 and F(y)+t>1
(i) If t≤0.5, then F(x+y)≥ $\min\{F(x), F(y), 0.5\}$
$> \min\{1-t, 1-t, 0.5\} = 0.5$ ≥t. Therefore $(x+y)_t$ ∈F.
(ii) If t>0.5, then $F(x+y) > \min\{F(x), F(y), 0.5\}$
$> \min\{1-t, 1-t, 0.5\} = 1-t$ which implies F(x+y)+t>1.
Therefore $(x+y)_t qF$.
Thus in any case, we have $(x+y)_t \in \vee qF$ and so x+y∈ $[F]_t$.
Similarly we can prove that $x\gamma y, x\alpha z\beta y \in [F]_t$, for x,y,z∈ S and $\alpha, \beta, \gamma \in \Gamma$.
Now, let x,z∈ S and a,b∈ $[F]_t$ be such that x+a+z=b+z. Using the above way we can prove x∈ $[F]_t$. Therefore $[F]_t$ is an h-ideal of S.
Conversely, let F be a fuzzy set of S and t∈(0,1] be such that $[F]_t$ is an h-bi-ideal of S. If $F(x+y) < t < \min\{F(x), F(y), 0.5\}$ for some t∈(0,0.5], then F(x)≥t and F(y)≥t that is x,y∈ $F_t \subseteq [F]_t$ which implies x+y∈ $[F]_t$ which is equivalent to say, $x_t \in F$ and $y_t \in$ F imply $(x+y)_{\min(t,t)} \in \vee qF$. Hence we have F(x+y)≥t or F(x+y)+t>1, a contradiction. This proves that for all x,y ∈ S, F(x+y)≥ $\min\{F(x), F(y), 0.5\}$. Similarly we can prove the other conditions. Therefore F is an $(\in, \in \vee q)$-fuzzy bi-ideal of S. ∎

Theorem 4.11: A fuzzy set F of S is prime $(\in, \in \vee q)$-fuzzy h-bi-ideal of S if and only if $[F]_t(\neq \phi)$ is a prime h-bi-ideal of S for all t∈(0,1].

Proof: Let F be a prime$(\in, \in \vee q)$-fuzzy h-bi-ideal of S, then by Theorem 4.10 $[F]_t$ is an h-bi-ideal of S for all

t∈(0,1]. To prove $[F]_t$ is prime,let $x\gamma y \in [F]_t$ for all x,y ∈ S and $\gamma \in \Gamma$. Since $[F]_t = F_t \cup Q(F;t)$, we have $x\gamma y \in F_t$ or $x\gamma y \in Q(F;t)$
Case 1: $x\gamma y \in F_t$
Then $F(x\gamma y) \geq t$.
(i) If t≤ 0.5, then $\max\{F(x), F(y)\} \geq \min\{F(x\gamma y), 0.5\} \geq t$ which implies $x \in F_t \subseteq [F]_t$ or $y \in F_t \subseteq [F]_t$
(ii) If t>0.5,then $\max\{F(x), F(y)\} \geq \min\{F(x\gamma y), 0.5\} > 0.5$ that is $\max\{F(x), F(y)\} + t > 1$, which implies $x \in Q(F;t) \subseteq [F]_t$ or $y \in Q(F;t) \subseteq [F]_t$
Case 2: $x\gamma y \in Q(F;t)$
Then $F(x\gamma y) + t$ >1 and $F(x\gamma y) < t$.
(i) If $F(x\gamma y)$ ≤0.5, then $\max\{F(x), F(y)\} + t \geq \min\{F(x\gamma y), 0.5\} + t = F(x\gamma y) + t > 1$ which implies $F(x) + t > 1$ or $F(y) + t > 1$ that is $x \in Q(F;t) \subseteq [F]_t$ or $y \in Q(F;t) \subseteq [F]_t$.
(ii) If $F(x\gamma y) > 0.5$, then $0.5 < F(x\gamma y) \leq t$. Thus $\max\{F(x), F(y)\} + t \geq \min\{F(x\gamma y), 0.5\} + t = 0.5 + t > 1$ which implies $F(x) + t > 1$ or $F(y) + t > 1$ that is $x \in Q(F;t) \subseteq [F]_t$ or $y \in Q(F;t) \subseteq [F]_t$.
Therefore, $[F]_t$ is a prime h-bi-ideal of S.
Conversely, $[F]_t$ is a prime h-bi-ideal of S for all t∈(0,1]. Then by Theorem 4.10, F is an $(\in, \in vq)-$fuzzy h-bi-ideal of S. Let $(x\gamma y)_t \in F$, then $x\gamma y \in F_t \subseteq [F]_t$. Since $[F]_t$ is prime $x \in [F]_t$ or $y \in [F]_t$. This implies $x_t \in \vee q$ F or $y_t \in \vee q$ F. Therefore F is a $(\in, \in \vee q)-$prime fuzzy h-bi-ideal of S. ■

V. Semiprime$(\in, \in \vee q)$-fuzzy h-ideal

***Definition** 5.1:* An $(\in, \in \vee q)$-fuzzy h-ideal (resp. h-bi-ideal, h-quasi-ideal) F is called semiprime if $(x\gamma x)_t \in F$ implies x∈ $\vee q$F for all x∈ S, $\gamma \in \Gamma$ and t∈(0,1].

***Definition** 5.2:* A fuzzy set F of S is called semiprime if F(x)=F(xγx) for all x∈S and $\gamma \in \Gamma$.

***Theorem** 5.3:* An $(\in, \in \vee q)-$fuzzy h-ideal (resp. h-bi-ideal, h-quasi-ideal)F of S is semiprime if and only if $F(x) \geq \min\{F(x\gamma x), 0.5\}$ for all x∈X and $\gamma \in \Gamma$.

Proof: Similar as Theorem 4.7. ■

***Theorem** 5.4:* An $(\in, \in \vee q)-$fuzzy h-ideal (resp. h-bi-ideal, h-quasi-ideal)F of S is semiprime if and only if $F_t(\neq \phi)$ is a semiprime h-ideal (resp. h-bi-ideal, h-quasi-ideal) of S.

Proof: Similar as Theorem 4.8. ■

***Theorem** 5.5:* A fuzzy set F of S is an $(\in, \in \vee q)-$semiprime fuzzy h-bi-ideal(resp. h-ideal, h-quasi-ideal) of S if and only if $[F]_t(\neq \phi)$ is a semiprime h-bi-ideal(resp. h-ideal, h-quasi-ideal) of S for t∈(0,1].

Proof: Similar as Theorem 4.11. ■

***Proposition** 5.6:* Let μ and σ be two semiprime fuzzy h-ideals of a Γ-hemiring S. Then $\mu \times \sigma$ is a semiprime fuzzy h-ideal of the Γ-hemiring $S \times S$.

Proof: By Proposition 3.9, $\mu \times \sigma$ is a fuzzy h-ideal of $S \times S$. Let $(a, b) \in S \times S$. Then $(\mu \times \sigma)\{(a, b)\gamma(a, b)\} = (\mu \times \sigma)(a\gamma a, b\gamma b) = \min\{\mu(a\gamma a), \sigma(b\gamma b)\}$
$= \min\{\mu(a\gamma a), \sigma(b\gamma b)\} = \min\ \{\mu(a), \sigma(b)\}$ (since μ and σ are semiprime fuzzy h-ideal of $S) = (\mu \times \sigma)(a, b)$. Hence $(\mu \times \sigma)$ is a semiprime fuzzy h-ideal of $S \times S$. ■

***Proposition** 5.7:* Let μ and σ be two semiprime fuzzy h-ideals of a Γ-hemiring S. Then the level subset $(\mu \times \sigma)_t, t \in Im(\mu \times \sigma)$ is a semiprime h-ideal of the Γ-hemiring $S \times S$.

Proof: By Proposition 3.9, $\mu \times \sigma$ is a fuzzy h-ideal of $S \times S$. Let for $(x, y) \in S \times S$ and $\gamma \in \Gamma$, $(x, y)\gamma(x, y) \in (\mu \times \sigma)_t$. Then $(\mu \times \sigma)\{(x, y)\gamma(x, y)\} \geq t \Rightarrow (\mu \times \sigma)(x\gamma x, y\gamma y) \geq t \Rightarrow \min\{\mu(x\gamma x), \sigma(y\gamma y)\} \geq t \Rightarrow \mu(x\gamma x) \geq t$ and $\sigma(y\gamma y) \geq t \Rightarrow x\gamma x \in \mu_t$ and $y\gamma y \in \sigma_t \Rightarrow x \in \mu_t$ and $y \in \sigma_t$(since μ_t and σ_t are semiprime ideals of S. Thus $(x, y) \in \mu_t \times \sigma_t = (\mu \times \sigma)_t$(cf. Lemma 3.8). Hence $(\mu \times \sigma)_t$ is a semiprime ideal of $S \times S$. ■

***Proposition** 5.8:* If the level subset $(\mu \times \sigma)_t, t \in Im(\mu \times \sigma)$ of $\mu \times \sigma$ is a semiprime h-ideal of $S \times S$ then $(\mu \times \sigma)$ is a semiprime fuzzy h-ideal of the Γ-hemiring $S \times S$.

Proof: Straightforward. ■

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