Abstract—Valuing derivatives (options, futures, swaps, forwards, etc.) is one uneasy task in financial mathematics. The two ways this problem can be effectively resolved in finance is by the use of two methods (Martingales and Partial Differential Equations (PDEs)) to obtain their respective options price valuation formulas. This research paper examined two different stochastic financial models which are Constant Elasticity of Variance (CEV) model and Black-Karasinski term structure model. Assuming their respective option price valuation formulas, we proved the analogous of the Martingales and PDEs options price valuation formulas for the two different Stochastic Differential Equation (SDE) models. This was accomplished by using the applications of Girsanov theorem for defining an Equivalent Martingale Measure (EMM) and the Feynman-Kac theorem. The results obtained show the systematic proof for analogous of the two (Martingales and PDEs) options price valuation formulas beginning with the Martingales option price formula and arriving back at the Black-Scholes parabolic PDEs and vice versa.

Keywords—Option price valuation, Martingales, Partial Differential Equations, PDEs, Equivalent Martingale Measure, Girsanov Theorem, Feynman-Kac Theorem, European Put Option.

I. INTRODUCTION

An option affords the holder the opportunity (right), but not obligation to buy or sell an asset at the predetermined price in the future. Every option has the expiration date, strike price and command a premium which is also called the price of the option. The underlying assets are stocks, foreign currencies, interest rates, stock indices and commodities, etc. In a complete market, some cases in option contracts are resolved by cash, while in other cases, they are resolved by direct purchase of the underlying asset. Take for example, warrants which gives the holder the right but not obligation to buy a number of underlying assets under agreed terms and conditions. If the holder decides to exercise his rights on the warrants, the underlying asset definitely has to be delivered [1].

There are several reasons why investors trade options rather than trade stocks. A key reason for this choice is that it saves transaction costs and helps avoid market restrictions. A trader can use options to take a particular risk position and pay lower transaction costs than the stocks would require. Options are attractive to individuals and institutions wishing to mitigate their exposure to risk. It may be regarded as insurance policies against unfavorable movements in a market economy. In the course of trading options together with its stock portfolios, investors would be able to carefully regulate or fine-tune the risk and return on their respective investments [1].

Valuing or pricing financial derivative products is one of the most common problems in mathematical finance. Apart from approximations by discrete-time models, there are basically two main methods to obtain valuation formula for a given financial derivative. These are pricing by the Martingales method (also known as no arbitrage method) and pricing using the PDEs approach [2]. It is important to state that the two above stipulated approaches for the derivation of valuation formula for derivatives or options (i.e. the Martingale approach and PDE) are equivalent and yield the same result. This can be ascertained through a logical proof using the two stochastic financial models as examples.

In this paper, we examined a method of proving the analogous of the two approaches through the application of Girsanov theorem for determining an EMM and the Feynman-Kac theorem (Theorem 4) to drive home our argument that the different valuation formulas derived from two approaches are analogous. That is to show that the Martingale (No-arbitrage) option price valuation formula produces the Black-Scholes parabolic PDEs and finally we solve the Black-Scholes PDEs by applying the Feynman-Kac Theorem to obtain back the martingale valuation formula.

II. THEORETICAL PERCEPTIONS

Definition 1. (Risk-neutral measure) A probability measure \( P^* \) on \( \Omega \) is called a risk-neutral measure if it satisfies

\[
\mathbb{E}^*[S_u | \mathcal{F}_u] = e^{r(t-u)}S_u, \quad 0 \leq u \leq t
\]

where \( \mathbb{E}^* \) denotes the expectation under \( P^* \).

Proposition 1. [3] The measure \( P^* \) is risk-neutral if and only if the discounted price process \( (X_t)_{t \in \mathbb{R}_+} \) is a martingale under \( P^* \).

Theorem 1. (Girsanov Theorem, [4]): The process \( \tilde{W}_t = W_t - \int_0^t \tilde{\theta}_s \, ds \) is Brownian motion under the measure \( Q \).

Theorem 2. Let \( (\phi_t)_{t \in [0,T]} \) be an adapted process satisfying the
Novikov integrability condition

\[ E \left[ \exp \left( \frac{1}{2} \int_0^T |\phi|^2 \, dt \right) \right] < \infty \]

and let \( Q \) denote the probability measure defined by

\[ \frac{dQ}{dp} = \exp \left( -\int_0^T \phi_t \, dW_t - \frac{1}{2} \int_0^T \phi_t^2 \, ds \right) \]

then

\[ \tilde{W}_t = W_t + \int_0^t \phi_s \, ds, \quad t \in [0, T], \]

is a standard Brownian motion under \( Q \).

**Theorem 3.** (Martingale Representation Theorem, [4])

Suppose \( M_t \) is an \( \mathcal{F}_t \) martingale where \( \{ \mathcal{F}_t \}_{t \geq 0} \) is the filtration generated by the \( n \) dimensional standard Brownian motion, \( W_t = (W_1^{(t)}, \ldots, W_n^{(t)}) \). If \( E[M_T^2] < \infty \) for all \( t \), then there exists a unique \( n \) dimensional adapted stochastic process, \( \phi_t \) such that

\[ M_t = M_0 + \int_0^t \phi_s \, dW_s \]

where \( \phi_t^T \) denotes the transpose of the vector, \( \phi_t \).

**A. Ito Formula for Ito Processes**

We now turn to the general expression of Ito’s formula which applies to Ito processes of the form

\[ X_t = X_0 + \int_0^t \mu_s \, ds + \int_0^t \sigma_s \, dW_s, \quad t \in \mathbb{R}_t \]

(1)

or in differential notation

\[ dX_t = \mu_t \, dt + \sigma_t \, dW_t \]

where \( (\mu_t)_{t \in \mathbb{R}_+} \) and \( (\sigma_t)_{t \in \mathbb{R}_+} \) are square-integrable adapted processes.

**Lemma 1.** (Ito formula for Ito processes) For any Ito process \( (X_t)_{t \in \mathbb{R}_+} \) of the form (1) and any \( f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}) \) and \( Z_t = f(t, X_t) \), we have

\[ Z_t = f(0, X_0) + \int_0^t \mu_s \frac{\partial f}{\partial s} (s, X_s) \, ds + \int_0^t \sigma_s \frac{\partial f}{\partial x} (s, X_s) \, dB_s + \int_0^t \frac{1}{2} \sigma_s^2 \frac{\partial^2 f}{\partial x^2} (s, X_s) \, ds \]

or in differential form

\[ dZ_t = \frac{\partial f}{\partial t} (t, X_t) \, dt + \frac{\partial f}{\partial x} (t, X_t) \, dX_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} (t, X_t) (dX_t)^2 \]

(2)

Theorem 4. (Feynman-Kac Theorem in one Dimension; [5])

Suppose that \( X_t \) follows the stochastic process

\[ dX_t = \mu(x_t, t) \, dt + \sigma(x_t, t) \, dW_t^Q \]

(3)

where \( W_t^Q \) is Brownian motion under the measure \( Q \). Let \( V(x_t, t) \) be a differentiable function of \( x_t \) and \( t \) and suppose that \( V(x_t, t) \) follows the PDE given by

\[ \frac{\partial V}{\partial t} + \mu(x_t, t) \frac{\partial V}{\partial x} + \frac{1}{2} \sigma^2(x_t, t) \frac{\partial^2 V}{\partial x^2} - r(x_t, t) V(x_t, t) = 0 \]

(4)

and with boundary condition \( V(X_T, T) \). The theorem asserts that \( V(x_t, t) \) has the solution

\[ V(x_t, t) = E^Q \left[ e^{-\int_0^T r(x_s, s) \, ds} V(X_T, T) \right] \]

(5)

Note that the expectation is taken under the measure \( Q \) that makes the stochastic term in (5) Brownian motion. The generator of the process in (3) is defined as the operator

\[ A = \mu(x_t, t) \frac{\partial}{\partial x} + \frac{1}{2} \sigma^2(x_t, t) \frac{\partial^2}{\partial x^2} \]

(6)

So, the PDE in \( V(x_t, t) \) is sometimes written

\[ \frac{\partial V}{\partial t} + AV(x_t, t) - r(x_t, t) V(x_t, t) = 0 \]

(7)

The Feynman-Kac theorem can be used in both directions. That is:

i. If we know that \( x_t \) follows the process in (4) and we are given a function \( V(x, t) \) with boundary condition \( V(X_T, T) \), then we can always obtain the solution for \( V(x_t, t) \) as (5).

ii. If we know that the solution to \( V(x_t, t) \) is given by (5) and that \( x_t \) follows the process in (3), then we are assured that \( V(x_t, t) \) satisfies the PDE in (7).

### III. METHODS

**A. Analogous of Martingale and PDEs Option Price Valuation Formulas for CEV Model**

Let us attempt to prove the equivalence of the Martingales and PDEs approaches for the CEV model [6] using application of Girsanov Theorem (Theorem 1) and the diffusion for the CEV model 

\[ X_t = X_0 + \int_0^t \mu_s \, ds + \int_0^t \sigma_s \, dW_s \]

and with boundary condition \( V(X_T, T) \). The theorem asserts that \( V(x_t, t) \) has the solution

\[ V(x_t, t) = E^Q \left[ e^{-\int_0^T r(x_s, s) \, ds} V(X_T, T) \right] \]

(5)

Note that the expectation is taken under the measure \( Q \) that makes the stochastic term in (5) Brownian motion. The generator of the process in (3) is defined as the operator

\[ A = \mu(x_t, t) \frac{\partial}{\partial x} + \frac{1}{2} \sigma^2(x_t, t) \frac{\partial^2}{\partial x^2} \]

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ii. If we know that the solution to \( V(x_t, t) \) is given by (5) and that \( x_t \) follows the process in (3), then we are assured that \( V(x_t, t) \) satisfies the PDE in (7).

\[ dX_t = rX_t \, du + \sigma X_t^\alpha \, dW_t \]

\[ dB_t = rB_t \, dt \]

We apply the Girsanov’s Theorem so that the process for \( dX_t \) becomes

\[ dX_t = rX_t \, du + \sigma X_t^\alpha \, dW_t^\alpha \]

(9)

The European put has payoff

\[ u_T = (K - X_T)^+ \]

So in accordance with the fundamental theorem of arbitrage which guarantees that given a numeraire (asset with a positive price) \( N_t \), each relative asset will be a martingale under the corresponding measure \( \mathbb{N} \), and consequently, so will \( V_t/N_t \).
since it is a linear combination of martingales. The martingale property of $V_t/N_t$ implies that

$$E^H_r \left[ \frac{V_T}{N_T} F_t \right] = \frac{V_t}{N_t}$$

(10)

from which the time $-t$ price of the derivative, $V_t$, is

$$V_t = N_tE^H_r \left[ \frac{V_T}{N_T} F_t \right]$$

(11)

Hence, from (11), the time $-t$ price of the Put with $u_t = V_t$ and $u_T = V_T$ is

$$u_t = u(t, X_t) = B_tE^B \left[ \frac{K - X_T}{X_t} F_t \right]$$

where

$$B_t = \exp \left( \int_0^t rdu \right) = e^{rt}$$

Next, to find the diffusion for the discounted option price $E^B_r \left[ \frac{u(T, X_T)}{X_T} F_r \right]$ which is a martingale only when the Black-Scholes PDE is satisfied. Under the Black-Scholes model, the no-arbitrage price of an option $u_t$ is given by

$$u_t = N_tE^H_r \left[ \frac{u}{N} F_t \right]$$

(13)

with numeraire $N_t = B_t$.

$$u_t = u(X_t, t) = e^{-r(T-t)}E^B[u(X_T, T)|F_t]$$

(14)

where $\mathbb{B}$ is the measure under which the discounted stock price

$$\frac{X_t}{N_t} = e^{-rT}X_t$$

is a martingale. Under $\mathbb{B}$, the discounted option price,

$$Z(X_t, t) = e^{-ru}u(X_t, t),$$

is also a martingale, which we write as

$$Z_t = e^{-ru}u_t$$

for convenience.

By Ito’s lemma (Lemma 1), $Z(X_t, t)$ follows the process

$$dZ_t = \frac{\partial Z}{\partial t} dt + \frac{\partial Z}{\partial x} dx_t + \frac{1}{2} \frac{\partial^2 Z}{\partial x^2} (dx_t)^2$$

(15)

We need the Ito multiplication in Table 1.

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$dt$</td>
<td>0</td>
</tr>
<tr>
<td>$dW_t$</td>
<td>0</td>
</tr>
</tbody>
</table>

We then apply the result to $Z_t = e^{-ru}u_t$, we have

$$\frac{\partial Z}{\partial t} = -re^{-ru}u_t + e^{-ru} \frac{\partial u}{\partial t}$$

(16)

Substituting (16) into (15) and substituting for $dX_t$ from (8) yields

$$dZ_t = e^{-ru} \left[ -ru + \frac{\partial u}{\partial t} dt + \frac{\partial u}{\partial x} rX_t dt + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} \sigma^2 X_t^2 dt \right]$$

(17)

Note that

$$\frac{\partial Z}{\partial x} = e^{-ru} \frac{\partial u}{\partial x}$$

Substituting this for in (17), we have

$$dZ_t = e^{-ru} \left[ -ru + \frac{\partial u}{\partial t} dt + \frac{\partial u}{\partial x} rX_t dt + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} \sigma^2 X_t^2 dt \right]$$

(18)

Now, the expectation under which the martingale price of the option is obtained is taken under the EMM, $\mathbb{B}$, so the process $dZ_t$ must reflect this. Here, we make a change of the measure to $\mathbb{B}$ in (18) to obtain

$$dZ_t = e^{-ru} \left[ -ru + \frac{\partial u}{\partial t} dt + \frac{\partial u}{\partial x} rX_t dt + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} \sigma^2 X_t^2 dt \right]$$

(19)

Since we have that $dW_t^B = dW_t + \left( \frac{c-r}{\sigma} \right) dt$, which is obtained from the application of Girsanov’s Theorem as stipulated in (9), rearranging the terms in (19) we have

$$dZ_t = e^{-ru} \left[ -ru + \frac{\partial u}{\partial t} dt + \frac{\partial u}{\partial x} rX_t dt + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} \sigma^2 X_t^2 dt \right]$$

(20)

Hence, we have that in (20)

$$Z_t = e^{-ru}u[X_T, t],$$

The discounted time $-t$ option price, is a martingale under $\mathbb{B}$ only when the drift term in (20) is zero. That is when

$$\frac{\partial u}{\partial t} + r \frac{\partial u}{\partial x} + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} - ru = 0$$

(21)

We recognize (20) as the Black-Scholes PDE for the given SDE in (8). Clearly, we can see that the discounted price of the derivative can only be a martingale if the Black-Scholes PDE is satisfied. Hence, the value of $u_t$ that solves the PDE in (20) with the boundary condition $u(T, X_T) = (K - X_T)^2$ is the same value $u_t$ that appears in (12). Consequently, we proceed to
prove the equivalence using the application of Feynman-Kac theorem (Theorem 4). Now given the derived PDE for the CEV model in (20), we now seek to solve the PDE by using the Feynman-Kac method to see if we can obtain back the Martingales option price valuation formula for SDE of the CEV model. Hence, we have
\[
\frac{\partial u}{\partial t} + rx \frac{\partial u}{\partial x} + \frac{1}{2} \sigma^2 x^2 a \frac{\partial^2 u}{\partial x^2} - ru = 0 \tag{21}
\]
with the boundary condition
\[
u(T, X_T) = (K - X_T)^+ = \varphi(x) \]
Let \(X(t)\) solve the SDE
\[
dX = rx dt + \sigma x^a dW
\]
with initial condition \(x(t) = x\). We define a process \(x(t)\) on interval \([t, T]\) as follows:
\[
dX = rx dt + \sigma x^a dW, x(t) = x
\]
Applying Ito’s formula (Lemma 1) on \(u(T, X_T)\), we have
\[
(du)^2 = (dx^2 + \sigma x^a dW)^2 + (\sigma x^a)^2 (dW)^2 + 2 \sigma x^a x^a (dt dW)
\]
Using the Ito multiplication table (Table I), we have
\[
(du)^2 = \sigma^2 x^2 a^2 dt
\]
Therefore
\[
du = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} (rx dx + \sigma x^a dW) + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} \sigma^2 x^{2a} dt = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} (rx dx + \sigma x^a dW) + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} \sigma^2 x^{2a} dt + \frac{\partial u}{\partial x} (rx dx + \sigma x^a dW)
\]
\[
du = \left[ \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} (rx dx + \sigma x^a dW) + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} \sigma^2 x^{2a} dt \right] + \frac{\partial u}{\partial x} (rx dx + \sigma x^a dW) \tag{22}
\]
So, we now have
\[
du = ru dt + \frac{\partial u}{\partial x} \sigma x^a dW
\]
Integrating from \(t\) to \(T\), we have
\[
\int_t^T du = \int_t^T ru dt + \int_t^T \frac{\partial u}{\partial x} \sigma x^a dW
\]
which gives
\[
u(x(T), T) - u(x(t), t) = \int_t^T ru dt + \int_t^T \frac{\partial u}{\partial x} \sigma x^a dW
\]
Substituting the initial and terminal terms, we get
\[
\psi(x(T)) - u(x, t) = \int_t^T ru dt + \int_t^T \frac{\partial u}{\partial x} \sigma x^a dW
\]
where \(\psi(x(T)) = (K - X_T)^+\) so that,
\[
\psi(x(T)) - u(x, t) = \int_t^T ru dt + \int_t^T \frac{\partial u}{\partial x} \sigma x^a dW \tag{23}
\]
Taking expectations of both sides, we have
\[
E[\psi(x(T)) - u(x, t)] = E \left[ \int_t^T ru dt \right] + E \left[ \int_t^T \frac{\partial u}{\partial x} \sigma x^a dW \right] \tag{24}
\]
but, \(E \left[ \int_t^T \frac{\partial u}{\partial x} \sigma x^a dB \right] = 0 \)
We now have
\[
E[u(x, t)] = E[\psi(x(T))] = E \left[ \int_t^T ru dt \right] + E \left[ \int_t^T \frac{\partial u}{\partial x} \sigma x^a dW \right]
\]
\[
\therefore u(x, t) = e^{-r(T-t)} (K - X_T)^+ \tag{25}
\]
where \(u(x, t) = u(X_t, t)\), which is the same option price valuation formula for the CEV model derived in (12).

III. ANALOGOUS OF MARTINGALES AND PDES OPTION PRICE VALUATION FORMULAS FOR BLACK-KARASINSKI TERM STRUCTURE MODEL

Again, we proceed to prove that the two distinct option price valuation formulas derived for Black-Karasinski model [7] are equivalent. First, we use the application of the Girsanov theorem and the diffusion for the discounted option price. We are given that the SDE for the models is as follows:
\[
d(\log r_t) = \varphi (u) (\log (\mu(u) - \log r_t)) du + \sigma (u) dW_t \tag{26}
\]
\[
dB_t = r B_t dt
\]
We apply the Girsanov’s Theorem so that the process for \(dX_t\) becomes
\[
dX_t = \varphi M_t dt + \sigma W_t \tag{27}
\]
where \(X_t = \log r_t\) and \(M_t = \log (\mu(u) - \log r_t)\)
The European Put has payoff
\[
u_t = u(X_t, T) = (K - X_T)^+
\]
So in accordance with (11), the time \(-t\) price of the Put option is
\[
u_t = B_t E^B \left[ \frac{K-X_T}{B_T} \right] | F_t \tag{28}
\]
where
\[
B_t = \exp \left( \int_t^T r du \right) = e^{rt}
\]
\[
\therefore u_t = u(X_t, t) = e^{-r(T-t)} B_t E^B [(K - X_T)^+] | F_t
\]
Next is to find the diffusion for the discounted option price \(\frac{u_t}{B_t}\) which is a Martingale only when the Black-Scholes PDE is satisfied. Under the Black-Scholes model, the Martingale (no-
arbitrage) price of an option or financial derivative \( u_t \) is given by

\[
u_t = N_t E^B \left[ \frac{U_T}{N_T} F_T \right]
\]

with numeraire \( N_t = B_t \)

\[
u_t = u(X_t, t) = e^{-(\tau - t)}E^B[u(X_T, T)|F_T](30)
\]

where \( B \) is the measure under which the discounted stock price \( \frac{X_t}{B_t} = e^{-\tau t}X_t \) is a Martingale. Under \( B \), the discounted option price is \( Z(X_t, t) = e^{-\tau t}u(X_t, t) \) which is also a Martingale, which we write as \( Z_t = e^{-\tau t}u_t \) for convenience.

By Ito’s Lemma, \( Z(X_t, t) \) follows the process

\[
dZ_t = \frac{\partial Z}{\partial t} dt + \frac{\partial Z}{\partial X} dX_t + \frac{1}{2} \frac{\partial^2 Z}{\partial x^2} (dX_t)^2
\]

We need to recall the Ito multiplication given in Table I. Now by applying the product rule to \( Z_t = e^{-\tau t}u_t \), we have

\[
\frac{\partial Z}{\partial t} = -e^{-\tau t}u_t + e^{-\tau t} \frac{\partial u}{\partial t}(32)
\]

Note that

\[
\frac{\partial Z}{\partial x} = -e^{-\tau t} \frac{\partial u}{\partial x} \text{ and } \frac{\partial^2 Z}{\partial x^2} = e^{-\tau t} \frac{\partial^2 u}{\partial x^2}(32)
\]

Substituting (32) into (31) and substituting \( dX_t \) from (27) yields

\[
dZ_t = e^{-\tau t} \left[ -ru_t + \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} \right] \frac{\partial u}{\partial x} dt + \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 u}{\partial x^2} (dX_t)^2
\]

\[
= e^{-\tau t} \left[ -ru_t + \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} \right] \frac{\partial u}{\partial x} dt + \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 u}{\partial x^2} (dX_t)^2
\]

\[
\Rightarrow dZ_t = e^{-\tau t} \left[ -ru_t + \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} \right] \frac{\partial u}{\partial x} dt + \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 u}{\partial x^2} (dX_t)^2
\]

Now, the expectation under which the Martingale price of the option is obtained is taken under the EMM, \( B \). So, the process \( dZ_t \) must reflect this. Here, we make a change of measure to \( B \) in (33) to obtain

\[
\tilde{W}_t = \frac{\mu - r}{\sigma} t + W_t \quad \text{[Girsanov Theorem]}
\]

\[
d\tilde{W}_t = \frac{\mu - r}{\sigma} dt = dW_t
\]

i.e. \( dW_t = d\tilde{W}_t - \frac{\mu - r}{\sigma} dt \)

But, \( B = e^{rt} \)

\[
\Rightarrow \tilde{W}_t = \frac{w_t}{e^{rt}} = \frac{W_t}{B_t} = W_t^B
\]

\[
\Rightarrow dW_t = dW_t^B - \frac{\mu - r}{\sigma} dt
\]

Since \( dW_t^B = dW_t + \left[ \frac{\mu - r}{\sigma} \right] dt \) from the application of Girsanov Theorem that produced (27). Rearranging the terms in (33), we have

\[
dZ_t = e^{-\tau t} \left[ -ru_t + \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} \right] \frac{\partial u}{\partial x} dt + \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 u}{\partial x^2} (dX_t)^2
\]

\[
\Rightarrow dZ_t = e^{-\tau t} \left[ -ru_t + \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} \right] \frac{\partial u}{\partial x} dt + \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 u}{\partial x^2} (dX_t)^2
\]

Hence, we have that in (35)

\[
Z_t = e^{-\tau t}u_t[X_T, t]
\]

The discounted time \(-t\) option price is a Martingale under \( B \) only when the drift factor in (35) is zero. That is when

\[
\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 u}{\partial x^2} = 0 \quad (36)
\]

Again, we recognize (36) as the Black-Scholes PDE for the given SDE in (26). Evidently, we can see that the discounted price of the option can only be a martingale if the Black-Scholes PDE is satisfied. Hence the value of \( u_t \) that solves the PDE in (36) with boundary condition \((K - X_T)^+ = 1 \) is the same value \( u_t \) that appears in (28).

Next is to prove the equivalence of the derived option price valuation formulas for the Black-Karasinski model using the Feynman-Kac method (Theorem 4). The derived PDE valuation formulas for the model are given as

\[
\frac{\partial u}{\partial x} + ru + \frac{\partial^2 u}{\partial x^2} = 0 \quad (37)
\]

We define a process \( x(t) \) on the interval \([t, T]\) as:

\[
dx = rxdt + \sigma(t)dW(t)
\]

Using Ito formula for \( u(x, t) \) gives

\[
du = \frac{\partial u}{\partial t} dt + \frac{\partial u}{\partial x} dx + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} (dx)^2
\]

\[
= \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} (rxdt + \sigma(t)dW(t)) + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} (dx)^2
\]

by applying the Ito multiplication table, we have

\[
du = \frac{\partial u}{\partial t} dt + \frac{\partial u}{\partial x} (rxdt + \sigma(t)dW(t)) + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} (dx)^2
\]

\[
= \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} (rxdt + \sigma(t)dW(t)) + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} (dx)^2
\]

\[
\frac{\partial u}{\partial x} (rxdt + \sigma(t)dW(t)) + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} (dx)^2
\]

\[
\Rightarrow u_t = \frac{\partial u}{\partial x} (rxdt + \sigma(t)dW(t)) + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} (dx)^2
\]

\[
\Rightarrow u_t = \frac{\partial u}{\partial x} (rxdt + \sigma(t)dW(t)) + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} (dx)^2
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\[
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\]

\[
\Rightarrow u_t = \frac{\partial u}{\partial x} (rxdt + \sigma(t)dW(t)) + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} (dx)^2
\]
We now have
\[ du = r(t) dt + \sigma(t) dW_t \]

Integrating from \( t \) to \( T \), we have
\[ \int_t^T du = \int_t^T r(t) dt + \int_t^T \sigma(t) dW_t \]
which gives
\[ u(x(T), T) - u(x(t), t) = \int_t^T r(t) dt + \int_t^T \sigma(t) dW_t \]

Substituting the initial and terminal terms, we get
\[ \varphi(x(T)) = u(x, t) + \int_t^T r(t) dt + \int_t^T \sigma(t) dW_t \tag{38} \]

where \( \varphi(x(T)) = u(X_p, T) = 1 \)

Taking expectations of both sides of (38), we have
\[ E[\varphi(x(T)) - u(x, t)] = E\left[ \int_t^T r(t) dt \right] + E\left[ \int_t^T \sigma(t) dW_t \right] \]

By Ito’s formula, the one-dimensional process \( X_t = r_t \) and (39) therefore becomes
\[ u(x, t) = E\left[ e^{-\int_t^T r_s ds} 1 \right] = E\left[ e^{-\int_t^T x(s, t) ds} \right] \tag{40} \]

It is obvious that (40) is the same option price valuation formula derived in (30) for the Black-Karasinski term structure model.

IV. ANALYTICAL RESULT

It is obvious from the preceding sections that we have been able to postulate and prove Theorem 5 as stated below.

**Theorem 5.** In a Black-Scholes economy, given an SDE model in finance of the form,
\[ dX_t = r X_t dt + \sigma X_t dW_t \]
\[ dB_t = r B_t dt \]

The Martingales and PDEs European options price valuation formulas are said to be equivalent if and only if,
\[ e^{-r(T-t)} E[B_t \varphi(x(T)) | F_t] = \epsilon_{tt} X_t + \epsilon_{x} x \varphi(x(T)) | F_t] + \frac{1}{2} \sigma^2 x \varphi(x(T)) | F_t] - \epsilon_{g} g(t) \varphi(x(T)) | F_t] \tag{42} \]

where \( \varphi(x(T)) \) is the payoff function given as \((K - x)^+ \) for Puts and \((x - K)^+ \) for Calls. \( B_t \) is the fixed bond and the numeraire that determines the measure \( \mathbb{B} \) under which the discounted stock price

and the discounted option price
\[ \tilde{u}_t = \frac{u(x, t)}{e^{-rt}} = e^{-rt} u(x, t) \]
are Martingales.

V. CONCLUSION

Clearly from the above preceding analysis or explicit solutions of two financial models which are the CEV and Black-Karasinski models considered in this paper, we have been able to prove through the applications of Girsanov and Feynman-Kac theorems, that the Martingales and PDEs options price valuation formulas obtained for these two distinct models are analogous.

REFERENCES


