The Proof of Analogous Results for Martingales and Partial Differential Equations Options Price Valuation Formulas Using Stochastic Differential Equation Models in Finance

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***Abstract*—**Valuing derivatives (options, futures, swaps, forwards, etc.) is one uneasy task in financial mathematics. The two ways this problem can be effectively resolved in finance is by the use of two methods (Martingales and Partial Differential Equations (PDEs)) to obtain their respective options price valuation formulas. This research paper examined two different stochastic financial models which are Constant Elasticity of Variance (CEV) model and Black-Karasinski term structure model. Assuming their respective option price valuation formulas, we proved the analogous of the Martingales and PDEs options price valuation formulas for the two different Stochastic Differential Equation (SDE) models. This was accomplished by using the applications of Girsanov theorem for defining an Equivalent Martingale Measure (EMM) and the Feynman-Kac theorem. The results obtained show the systematic proof for analogous of the two (Martingales and PDEs) options price valuation formulas beginning with the Martingales option price formula and arriving back at the Black-Scholes parabolic PDEs and vice versa.

***Keywords*—**Option price valuation, Martingales, Partial Differential Equations, PDEs, Equivalent Martingale Measure, Girsanov Theorem, Feyman-Kac Theorem, European Put Option.

I. Introduction

An option affords the holder the opportunity (right), but not obligation to buy or sell an asset at the predetermined price in the future. Every option has the expiration date, strike price and command a premium which is also called the price of the option. The underlying assets are stocks, foreign currencies, interest rates, stock indices and commodities, etc. In a complete market, some cases in option contracts are resolved by cash, while in other cases, they are resolved by direct purchase of the underlying asset. Take for example, warrants which gives the holder the right but not obligation to buy a number of underlying assets under agreed terms and conditions. If the holder decides to exercise his rights on the warrants, the underlying asset definitely has to be delivered [1].

There are several reasons why investors trade options rather than trade stocks. A key reason for this choice is that it saves transaction costs and helps avoid market restrictions. A trader can use options to take a particular risk position and pay lower transaction costs than the stocks would require. Options are attractive to individuals and institutions wishing to mitigate their exposure to risk. It may be regarded as insurance policies against unfavorable movements in a market economy. In the course of trading options together with its stock portfolios, investors would be able to carefully regulate or fine-tune the risk and return on their respective investments [1].

Valuing or pricing financial derivative products is one of the most common problems in mathematical finance. Apart from approximations by discrete-time models, there are basically two main methods to obtain valuation formula for a given financial derivative. These are pricing by the Martingales method (also known as no arbitrage method) and pricing using the PDEs approach [2]. It is important to state that the two above stipulated approaches for the derivation of valuation formula for derivatives or options (i.e. the Martingale approach and PDE) are equivalent and yield the same result. This can be ascertained through a logical proof using the two stochastic financial models as examples.

In this paper, we examined a method of proving the analogous of the two approaches through the application of Girsanov theorem for determining an EMM and the Feynman-Kac theorem (Theorem 4) to drive home our argument that the different valuation formulas derived from two approaches are analogous. That is to show that the Martingale (No-arbitrage) option price valuation formula produces the Black-Scholes parabolic PDEs and finally we solve the Black-Scholes PDEs by applying the Feynman-Kac Theorem to obtain back the martingale valuation formula.

II. Theoretical Perceptions

Definition 1. (Risk-neutral measure) A probability measure $\mathbb{P}^*$ on Ω is called a risk-neutral measure if it satisfies

$$\mathbb{E}^*[S_t|\mathcal{F}_u] = e^{r(t-u)}S_u, 0 \leq u \leq t$$

where $\mathbb{E}^*$ denotes the expectation under $\mathbb{P}^*$.

Proposition 1. [3] The measure $\mathbb{P}^*$ is risk-neutral if and only if the discounted price process $(X_t)_{t \in \mathbb{R}_+}$ is a martingale under $\mathbb{P}^*$.

Theorem 1. (Girsanov Theorem, [4]): The process $\widetilde{W_t} = W_t - \int_0^t \theta_s \, ds$ is Brownian motion under the measure $\mathbb{Q}$.

Theorem 2. Let $(\phi_t)_{t\epsilon[0,T]}$ be an adapted process satisfying the

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Novikov integrability condition

$$E\left[exp\left(\frac{1}{2}\int_0^T|\phi|^2 dt\right)\right] < \infty$$

and let $\mathbb{Q}$ denote the probability measure defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = exp\left(-\int_0^T \phi_s\, dW_s - \frac{1}{2}\int_0^T \phi_s^2\, ds\right)$$

then

$$\widetilde{W}_t = W_t + \int_0^t \phi_s\, ds, t\, \epsilon\, [0,T],$$

is a standard Brownian motion under $\mathbb{Q}$.

Theorem 3. (Martingale Representation Theorem, [4]) Suppose M_t is an $\mathcal{F}_t$ −martingale where $\{\mathcal{F}_t\}_{t\geq 0}$ is the filteration generated by the n −dimensional standard Brownian motion, $W_t = \left(W_t^{(1)}, \ldots, W_t^{(n)}\right)$. If $E[M_t^2] < \infty$ for all t then there exists a unique n −dimensional adapted stochastic process, ϕ_t such that

$$M_t = M_0 + \int_0^t \phi_s^T dW_t \text{ for all } t \geq 0$$

where ϕ_s^T denotes the transpose of the vector, ϕ_s.

A. Ito Formula for Ito Processes

We now turn to the general expression of Ito's formula which applies to Ito processes of the form

$$X_t = X_0 + \int_0^t \mu_s\, ds + \int_0^t \sigma_s\, dW_s, t\, \epsilon\, \mathbb{R}_t \quad (1)$$

or in differential notation

$$dX_t = \mu_t dt + \sigma_t dW_t$$

where $(\mu_t)_{t\epsilon\mathbb{R}_+}$ and $(\sigma_t)_{t\epsilon\mathbb{R}_+}$ are square-integrable adapted processes.

Lemma 1. (Ito formula for Ito processes) For any Ito process $(X_t)_{t\epsilon\mathbb{R}_+}$ of the form (1) and any $f\epsilon C^{1,2}(\mathbb{R}_+\times\mathbb{R})$ and $Z_t = f(t, X_t)$ we have

$$Z_t = f(0,X_0) + \int_0^t \mu_s \frac{\partial f}{\partial x}(s,X_s)ds + \int_0^t \sigma_s \frac{\partial f}{\partial x}(s,X_s)dB_s + \int_0^t \frac{\partial f}{\partial s}(s,X_s)ds + \frac{1}{2}\int_0^t|\sigma_s|^2\frac{\partial^2 f}{\partial x^2}(s,X_s)ds$$

or in differential form

$$dZ_t = \frac{\partial f}{\partial t}(t,X_t)dt + \frac{\partial f}{\partial x}(t,X_t)dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,X_t)(dX_t)^2 = \left(\frac{\partial f}{\partial t}(t,X_t)dt + \frac{\partial f}{\partial x}(t,X_t)\mu_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,X_t)\sigma_t^2\right)dt + \frac{\partial f}{\partial x}(t,X_t)\sigma_t dW_t \quad (2)$$

Theorem 4. (Feynman-Kac Theorem in one Dimension; [5]) Suppose that x_t follows the stochastic process

$$dx_t = \mu(x_t,t)dt + \sigma(x_t,t)dW_t^{\mathbb{Q}} \quad (3)$$

where $W_t^{\mathbb{Q}}$ is Brownian motion under the measure $\mathbb{Q}$. Let $V(x_t,t)$ be a differentiable function of x_t and t and suppose that $V(x_t,t)$ follows the PDE given by

$$\frac{\partial V}{\partial t} + \mu(x_t,t)\frac{\partial V}{\partial x} + \frac{1}{2}\sigma(x_t,t)^2\frac{\partial^2 V}{\partial x^2} - r(x_t,t)V(x_t,t) = 0 \quad (4)$$

and with boundary condition $V(X_T,T)$. The theorem asserts that $V(x_t,t)$ has the solution

$$V(x_t,t) = E^{\mathbb{Q}}\left[e^{-\int_t^T r(x_u,u)du}V(X_T,T)\middle|\mathcal{F}_t\right] \quad (5)$$

Note that the expectation is taken under the measure $\mathbb{Q}$ that makes the stochastic term in (5) Brownian motion. The generator of the process in (3) is defined as the operator

$$A = \mu(x_t,t)\frac{\partial}{\partial x} + \frac{1}{2}\sigma(x_t,t)^2\frac{\partial}{\partial x^2} \quad (6)$$

So, the PDE in $V(x_t,t)$ is sometimes written

$$\frac{\partial V}{\partial t} + AV(x_t,t) - r(x_t,t)V(x_t,t) = 0 \quad (7)$$

The Feynman-Kac theorem can be used in both directions. That is:

i. If we know that x_t follows the process in (4) and we are given a function $V(x_t,t)$ with boundary condition $V(X_T,T)$, then we can always obtain the solution for $V(x_t,t)$ as (5).
ii. If we know that the solution to $V(x_t,t)$ is given by (5) and that x_t follows the process in (3), then we are assured that $V(x_t,t)$ satisfies the PDE in (7).

III. Methods

A. Analogous of Martingale and PDEs Option Price Valuation Formulas for CEV Model

Let us attempt to prove the equivalence of the Martingales and PDEs approaches for the CEV model [6] using application of Girsanov Theorem (Theorem 1) and the diffusion for discounted option price.

Let us consider the SDE for the CEV model

$$dX_u = rX_u du + \sigma X_u^\alpha dW_u \qquad dB_t = rB_t dt \quad (8)$$

We apply the Girsanov's Theorem so that the process for dX_t becomes

$$dX_u = rX_u du + \sigma X_u^\alpha dW_u^{\mathbb{B}}, \quad (9)$$

The European put has payoff

$$u_T = (K - X_T)^+.$$

So in accordance with the fundamental theorem of arbitrage which guarantees that given a numeraire (asset with a positive price) N_t, each relative asset will be a martingale under the corresponding measure $\mathbb{N}$, and consequently, so will V_t/N_t

since it is a linear combination of martingales. The martingale property of V_t/N_t implies that

$$E^{\mathbb{N}}\left[\frac{V_T}{N_T}\middle|\mathcal{F}_t\right]=\frac{V_t}{N_t} \tag{10}$$

from which the time $-t$ price of the derivative, V_t, is

$$V_t=N_tE^{\mathbb{N}}\left[\frac{V_T}{N_T}\middle|\mathcal{F}_t\right] \tag{11}$$

Hence, from (11), the time $-t$ price of the Put with $u_t=V_t$ and $u_T=V_T$ is

$$u_t=u(t,X_t)=B_tE^{\mathbb{B}}\left[\frac{K-X_T}{B_T}|\mathcal{F}_+\right]$$

where

$$B_t=exp\left(\int_0^t rdu\right)=e^{rt}$$

$$u_t=e^{-r(T-t)}E^{\mathbb{B}}[(K-X_T)^+|\mathcal{F}_t] \tag{12}$$

Next is to find the diffusion for the discounted option price $\frac{u_t}{B_t}$ which is a martingale only when the Black-Scholes PDE is satisfied. Under the Black-Scholes model, the no-arbitrage price of an option u_t is given by

$$u_t=N_tE^{\mathbb{N}}\left[\frac{u_T}{N_T}|\mathcal{F}_t\right] \tag{13}$$

with numeraire $N_t=B_t$

$$u_t=u(X_t,t)=e^{-r(T-t)}E^{\mathbb{B}}[u[X_T,T]|\mathcal{F}_t] \tag{14}$$

where $\mathbb{B}$ is the measure under which the discounted stock price

$$\frac{X_t}{B_t}=e^{-rt}X_t$$

is a martingale. Under $\mathbb{B}$, the discounted option price,

$$Z(X_t,t)=e^{-rt}u(X_t,t),$$

is also a martingale, which we write as

$$Z_t=e^{-rt}u_t$$

for convenience.

By Ito's lemma (Lemma 1), $Z(X_t,t)$ follows the process

$$dZ_t=\frac{\partial Z}{\partial t}dt+\frac{\partial Z}{\partial X}dX_t+\frac{1}{2}\frac{\partial^2 Z}{\partial X^2}(dX_t)^2 \tag{15}$$

We need the Ito multiplication in Table I.

TABLE I
ITO'S MULTIPLICATION TABLE

$\cdot$	dt	dW_t
dt	0	0
dW_t	0	dt

Now by applying the product rule to $Z_t=e^{-rt}u_t$, we have

$$\frac{\partial Z}{\partial t}=-re^{-rt}u_t+e^{-rt}\frac{\partial u}{\partial t} \tag{16}$$

Substituting (16) into (15) and substituting for dX_t from (8) yields

$$dZ_t=e^{-rt}\left[-ru_t+\frac{\partial u}{\partial t}\right]dt+\frac{\partial Z}{\partial X}[rX_tdt+\sigma X_t^{\alpha}dW_t]+\frac{1}{2}\frac{\partial^2 Z}{\partial X^2}\sigma^2X_t^{2\alpha}dt \tag{17}$$

Note that

$$\frac{\partial Z}{\partial X}=e^{-rt}\frac{\partial u}{\partial X}\text{ and }\frac{\partial^2 Z}{\partial X^2}=e^{-rt}\frac{\partial^2 u}{\partial X^2}$$

Substituting for this in (17), we have

$$dZ_t=e^{-rt}\left[-ru_t+\frac{\partial u}{\partial t}\right]dt+rX_t\frac{\partial Z}{\partial X}dt+\frac{\partial Z}{\partial X}\sigma X_t^{\alpha}dW_t+\frac{1}{2}\frac{\partial^2 Z}{\partial X^2}\sigma^2X_t^{2\alpha}dt=e^{-rt}\left[-ru_t+\frac{\partial u}{\partial t}\right]dt+e^{-rt}rX_t\frac{\partial u}{\partial X}dt+e^{-rt}\sigma X_t^{\alpha}\frac{\partial u}{\partial X}dW_t+\frac{1}{2}e^{-rt}\sigma^2X_t^{2\alpha}\frac{\partial^2 u}{\partial X^2}dt$$

$$dZ_t=e^{-rt}\left[-ru_t+\frac{\partial u}{\partial t}+rX_t\frac{\partial u}{\partial X}+\frac{1}{2}\sigma^2X_t^{2\alpha}\frac{\partial^2 u}{\partial X^2}\right]dt+e^{-rt}\left[\sigma X_t^{\alpha}\frac{\partial u}{\partial X}\right]dW_t \tag{18}$$

Now, the expectation under which the martingale price of the option is obtained is taken under the EMM, $\mathbb{B}$, so the process dZ_t must reflect this. Here, we make a change of the measure to $\mathbb{B}$ in (18) to obtain

$$dZ_t=e^{-rt}\left[-ru_t+\frac{\partial u}{\partial t}+rX_t\frac{\partial u}{\partial X}+\frac{1}{2}\sigma^2X_t^{2\alpha}\frac{\partial^2 u}{\partial X^2}\right]dt+e^{-rt}\left[\sigma X_t^{\alpha}\frac{\partial u}{\partial X}\right]\left[dW_t^{\mathbb{B}}-\left(\frac{\mu-r}{\sigma}\right)dt\right] \tag{19}$$

Since we have that $dW_t^{\mathbb{B}}=dW_t+\left(\frac{\mu-r}{\sigma}\right)dt$, which is obtained from the application of Girsanov's Theorem as stipulated in (9), rearranging the terms in (19) we have

$$dZ_t=e^{-rt}\left[-ru_t+\frac{\partial u}{\partial t}+rX_t\frac{\partial u}{\partial X}+\frac{1}{2}\sigma^2X_t^{2\alpha}\frac{\partial^2 u}{\partial X^2}\right]dt+e^{-rt}\left[\sigma X_t^{\alpha}\frac{\partial u}{\partial X}\right]dW_t^{\mathbb{B}} \tag{20}$$

Hence, we have that in (20)

$$Z_t=e^{-rt}u[X_t,t],$$

The discounted time$-t$ option price, is a martingale under $\mathbb{B}$ only when the drift term in (20) is zero. That is when

$$\frac{\partial u}{\partial t}+rx\frac{\partial u}{\partial x}+\frac{1}{2}\sigma^2x^{2\alpha}\frac{\partial^2 u}{\partial x^2}-ru=0 \tag{21}$$

We recognize (20) as the Black-Scholes PDE for the given SDE in (8). Clearly, we can see that the discounted price of the derivative can only be a martingale if the Black-Scholes PDE is satisfied. Hence, the value of u_t that solves the PDE in (20) with the boundary condition $u(T,X_T)=(K-X_T)^+$ is the same value u_t that appears in (12). Consequently, we proceed to prove the equivalence using the application of Feynman-Kac theorem (Theorem 4). Now given the derived PDE for the CEV model in (20), we now seek to solve the PDE by using the Feynman-Kac method to see if we can obtain back the Martingales option price valuation formula for SDE of the CEV model. Hence, we have

$$\frac{\partial u}{\partial t} + rx\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2 x^{2\alpha}\frac{\partial^2 u}{\partial x^2} - ru = 0 \tag{21}$$

with the boundary condition

$$u(T, X_T) = (K - X_T)^+ = \varphi(x)$$

Let $X(t)$ solve the SDE

$$dX = rXdt + \sigma X^\alpha dW$$

with initial condition $x(t) = x$. We define a process $x(t)$ on interval $[t, T]$ as follows:

$$dX = rxdt + \sigma x^\alpha dW, x(t) = x$$

Applying Ito's formula (Lemma 1) on $u(T, X_T)$, we have

$$du = \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dX + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(dX)^2$$
$$(dX)^2 = (rxdt + \sigma x^\alpha dW)(rxdt + \sigma x^\alpha dW)$$
$$(dX)^2 = (rx)^2(dt)^2 + \sigma^2 x^{2\alpha}(dW)^2 + 2rx\sigma x^\alpha(dt.dW)$$

Using the Ito multiplication table (Table I), we have

$$(dX)^2 = \sigma^2 x^{2\alpha}dt$$

Therefore

$$du = \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}(rxdt + \sigma x^\alpha dW) + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\sigma^2 x^{2\alpha}dt = \frac{\partial u}{\partial t}dt + rx\frac{\partial u}{\partial x}dt + \sigma x^\alpha\frac{\partial u}{\partial x}dW + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\sigma^2 x^{2\alpha}dt = \frac{\partial u}{\partial t}dt + rx\frac{\partial u}{\partial x}dt + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\sigma^2 x^{2\alpha}dt + \frac{\partial u}{\partial x}\sigma x^\alpha dW$$

$$du = \underbrace{\left[\frac{\partial u}{\partial t} + rx\frac{\partial u}{\partial x} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\sigma^2 x^{2\alpha}\right]}_{=ru \text{ (in (21))}}dt + \frac{\partial u}{\partial x}\sigma x^\alpha dW \tag{22}$$

So, we now have

$$du = rudt + \frac{\partial u}{\partial x}\sigma x^\alpha dW$$

Integrating from t to T, we have

$$\int_t^T du = \int_t^T rudt + \int_t^T \frac{\partial u}{\partial x}\sigma x^\alpha dW$$

which gives

$$u(x(T), T) - u(x(t), t) = \int_t^T rudt + \int_t^T \frac{\partial u}{\partial x}\sigma x^\alpha dW$$

Substituting the initial and terminal terms, we get

$$\psi(x(T)) - u(x, t) = \int_t^T rudt + \int_t^T \frac{\partial u}{\partial x}\sigma x^\alpha dW$$

where $\psi(x(T)) = (K - X_T)^+$ so that,

$$\psi(x(T)) - u(x, t) = \int_t^T rudt + \int_t^T \frac{\partial u}{\partial x}\sigma x^\alpha dW \tag{23}$$

Taking expectations of both sides, we have

$$E[\psi(x(T)) - u(x, t)] = E\left[\int_t^T rudt\right] + E\left[\int_t^T \frac{\partial u}{\partial x}\sigma x^\alpha dW\right] \tag{24}$$
$$\text{but}, E\left[\int_t^T \frac{du}{dx}\sigma x^\alpha dB\right] = 0$$

We now have

$$E[u(x, t)] = E[\psi(x(T)] - E\left[\int_t^T ru\,dt\right]$$
$$E[u(x, t)] = E\left[\psi(x(T)) - \int_t^T rudt\right] = E^{\mathbb{B}}\left[e^{-\int_t^T rsds}\psi(x(T))\,\middle|F_t\right]$$
$$u(x, t) = E^{\mathbb{B}}\left[e^{-r(T-t)}(K - X_T)^+\,\middle|F_t\right]$$
$$\therefore u(x, t) = e^{-r(T-t)}E^{\mathbb{B}}[(K - X_T)^+]. \tag{25}$$

where $u(x, t) = u(X_t, t)$, which is the same option price valuation formula for the CEV model derived in (12).

III. Analogous of Martingales and PDEs Option Price Valuation Formulas for Black-Karasinski Term Structure Model

Again, we proceed to prove that the two distinct option price valuation formulas derived for Black-Karasinski model [7] are equivalent. First, we use the application of the Girsanov theorem and the diffusion for the discounted option price. We are given that the SDE for the models is as follows:

$$d(logr_u) = \varphi(u)(log\mu(u) - logr_u)du + \sigma(u)dW_u \tag{26}$$
$$dB_t = rB_t dt$$

We apply the Girsanov's Theorem so that the process for dX_t becomes

$$dX_t = \varphi M_t dt + \sigma dW_t^{\mathbb{B}} \tag{27}$$

where $X_u = logr_u$ and $M_u = log\mu(u) - logr_u$

The European Put has payoff

$$u_T = u(X_T, T) = (K - X_T)^+$$

So in accordance with (11), the time $-t$ price of the Put option is

$$u_t = B_t E^{\mathbb{B}}\left[\frac{K - X_T}{B_T}\middle|\mathcal{F}_t\right]$$

where

$$B_t = \exp\left(\int_0^t rd_u\right) = e^{rt}$$
$$\therefore\ u_t = u(X_t, t) = e^{-r(T-t)}E^{\mathbb{B}}[(K - X_T)^+|\mathcal{F}_t] \tag{28}$$

Next is to find the diffusion for the discounted option price $\frac{u_t}{N_t}$ which is a Martingale only when the Black-Scholes PDE is satisfied. Under the Black-Scholes model, the Martingale (no-arbitrage) price of an option or financial derivative u_t is given by

$$u_t = N_t E^{\mathbb{N}}\left[\frac{U_T}{N_T}\middle|\mathcal{F}_t\right]$$

with numeraire $N_t = B_t$

$$u_t = u(X_t, t) = e^{-(T-t)} E^{\mathbb{B}}[u(X_T, T)|\mathcal{F}_t] \quad (30)$$

where $\mathbb{B}$ is the measure under which the discounted stock price $\frac{X_t}{B_t} = e^{-rt}X_t$ is a Martingale. Under $\mathbb{B}$, the discounted option price is $Z(X_t, t) = e^{-rt}u(X_t, t)$ which is also a Martingale, which we write as $Z_t = e^{-rt}u_t$ for convenience.

By Ito's Lemma, $Z(X_t, t)$ follows the process

$$dZ_t = \frac{\partial Z}{\partial t}dt + \frac{\partial Z}{\partial X}dX_t + \frac{1}{2}\frac{\partial^2 Z}{\partial X^2}(dX_t)^2 \quad (31)$$

We need to recall the Ito multiplication given in Table I. Now by applying the product rule to $Z_t = e^{-rt}u_t$, we have

$$\frac{\partial Z}{\partial t} = -re^{-rt}u_t + e^{-rt}\frac{\partial u}{\partial t} \quad (32)$$

Note that

$$\frac{\partial Z}{\partial X} = e^{-rt}\frac{\partial u}{\partial X} \text{ and } \frac{\partial^2 Z}{\partial X^2} = e^{-rt}\frac{\partial^2 u}{\partial X^2}$$

Substituting (32) into (31) and substituting dX_t from (27) yields

$$dZ_t = e^{-rt}\left[-ru_t + \frac{\partial u}{\partial t}\right]dt + \frac{\partial Z}{\partial X}[\varphi M_t dt + \sigma dW_t] + \frac{1}{2}\frac{\partial^2 Z}{\partial X^2}\sigma^2 dt = e^{-rt}\left[-ru_t + \frac{\partial u}{\partial t}\right]dt + \varphi M_t\frac{\partial Z}{\partial X}dt + \sigma\frac{\partial Z}{\partial X}dW_t + \frac{1}{2}\sigma^2\frac{\partial^2 Z}{\partial X^2}dt = e^{-rt}\left[-rU_t + \frac{\partial u}{\partial t}\right]dt + e^{-rt}\varphi M_t\frac{\partial u}{\partial X}dt + e^{-rt}\sigma\frac{\partial u}{\partial X}dW_t + \frac{1}{2}e^{-rt}\sigma^2\frac{\partial^2 u}{\partial X^2}dt$$

$$dZ_t = e^{-rt}\left[-ru_t + \frac{\partial u}{\partial t}\right]dt + e^{-rt}\varphi M_t\frac{\partial u}{\partial X}dt + e^{-rt}\sigma\frac{\partial u}{\partial X}dW_t + \frac{1}{2}e^{-rt}\sigma^2\frac{\partial^2 u}{\partial X^2}dt$$

$$\therefore\ dZ_t = e^{-rt}\left[-ru_t + \frac{\partial u}{\partial t} + \varphi M_t\frac{\partial u}{\partial X} + \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial X^2}\right]dt + e^{-rt}\left[\sigma\frac{\partial u}{\partial X}\right]dW_t) \quad (33)$$

Now, the expectation under which the Martingale price of the option is obtained is taken under the EMM, $\mathbb{B}$. So, the process dZ_t must reflect this. Here, we make a change of measure to $\mathbb{B}$ in (33) to obtain

$$dZ_t = e^{-rt}\left[-ru_t + \frac{\partial u}{\partial t} + \varphi M_t\frac{\partial u}{\partial X} + \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial X^2}\right]dt + e^{-rt}\left[\sigma\frac{\partial u}{\partial X}\right]\left[dW_t^{\mathbb{B}} - \left(\frac{\mu-r}{\sigma}\right)dt\right] \quad (34)$$

$$\widetilde{W}_t = \frac{\mu-r}{\sigma}t + W_t \text{ [Girsanov Theorem]}$$

$$d\widetilde{W}_t - \left[\frac{\mu-r}{\sigma}\right]dt = dW_t$$

$$i.e.\, dW_t = d\widetilde{W}_t - \left[\frac{\mu-r}{\sigma}\right]dt$$

But, $B = e^{rt}$

$$\Rightarrow \widetilde{W}_t = \frac{W_t}{e^{rt}} = \frac{W_t}{B_t} = W_t^{\mathbb{B}}$$

$$\therefore dW_t = dW_t^{\mathbb{B}} - \left[\frac{\mu-r}{\sigma}\right]dt.$$

Since $dW_t^{\mathbb{B}} = dW_t + \left[\frac{\mu-r}{\sigma}\right]dt$ from the application of Girsanov Theorem that produced (27). Rearranging the terms in (33), we have

$$dZ_t = e^{-rt}\left[-ru_t + \frac{\partial u}{\partial t} + \varphi(t)\underbrace{(log\mu(t) - logr_t)}_{M_t\ in\ (27)}\frac{\partial u}{\partial X} + \frac{1}{2}\sigma^2(t)\frac{\partial^2 u}{\partial X^2}\right]dt + e^{-rt}\left[\sigma\frac{\partial U}{\partial X}\right]dW_t^{\mathbb{B}} \quad (35)$$

Hence, we have that in (35)

$$Z_t = e^{-rt}u[X_T, t]$$

The discounted time $-t$ option price is a Martingale under $\mathbb{B}$ only when the drift factor in (35) is zero. That is when

$$\frac{\partial u}{\partial t} + \underbrace{\varphi(t)(log\mu(t) - logr_t)}_{=rx}\frac{\partial u}{\partial X} + \frac{1}{2}\sigma^2(t)\frac{\partial^2 u}{\partial X^2} - ru_t = 0 \quad (36)$$

Again, we recognize (36) as the Black-Scholes PDE for the given SDE in (26). Evidently, we can see that the discounted price of the option can only be a martingale if the Black-Scholes PDE is satisfied. Hence the value of u_t that solves the PDE in (36) with boundary condition $(K - X_T)^+ = 1$ is the same value u_t that appears in (28).

Next is to prove the equivalence of the derived option price valuation formulas for the Black-Karasinski model using the Feyman-Kac method (Theorem 4). The derived PDE valuation formulas for the model are given as

$$\frac{\partial u}{\partial x} + rx\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2(t)\frac{\partial^2 u}{\partial x^2} = ru \quad (37)$$

We define a process $x(t)$ on the interval $[t, T]$ as:

$$dX = rxdt + \sigma(t)dW, x(t) = x$$

Using Ito formula for $u(x, t)$ gives

$$du = \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dX + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(dX)^2 = \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}(rxdt + \sigma(t)dW) + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(rxdt + \sigma(t)dW)(rxdt + \sigma(t)dW)$$

by applying the Ito multiplication table, we have

$$du = \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}(rxdt + \sigma(t)dW) + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\underbrace{(rxdt + \sigma(t)dW)(rxdt + \sigma(t)dW)}_{=\sigma^2(t)dt} = \frac{\partial u}{\partial t}dt + rx\frac{\partial u}{\partial x}dt + \sigma(t)\frac{\partial u}{\partial x}dW + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\sigma^2(t)dt = \underbrace{\left[\frac{\partial u}{\partial t} + rx\frac{\partial u}{\partial x} + + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\sigma^2(t)\right]}_{=ru\ (in\ (37))}dt + \sigma(t)\frac{\partial u}{\partial x}dW$$

We now have

$$du = rudt + \frac{\partial u}{\partial x}\sigma(t)dW$$

Integrating from t to T, we have

$$\int_t^T du = \int_t^T rudt + \int_t^T \frac{\partial u}{\partial x}\sigma(t)dW$$

which gives

$$u(x(T),T) - u(x(t),t) = \int_t^T rudt + \int_t^T \frac{\partial u}{\partial x}\sigma(t)dW$$

Substituting the initial and terminal terms, we get

$$\varphi(x(T)) - u(x,t) = \int_t^T rudt + \int_t^T \frac{\partial u}{\partial x}\sigma(t)dW \qquad (38)$$

where $\varphi(x(T)) = u(X_T, T) = 1$

Taking expectations of both sides of (38), we have

$$E[\varphi(x(T) - u(x,t)] = E\left[\int_t^T rudt\right] + \underbrace{E\left[\int_t^T \frac{\partial u}{\partial x}\sigma(t)dW\right]}_{=0}$$

$$E[u(x,t)] = E[\varphi(x(T))] - E\left[\int_t^T rudt\right]$$

$$u(x,t) = E[1] - E\left[\int_t^T rudt\right] = E\left[e^{-\int_t^T r(X_u,u)du}\varphi(x(T))|\mathcal{F}_t\right] (39)$$

By Ito's formula, the one-dimensional process $X_t = r_t$ and (39) therefore becomes

$$u(x,t) = E\left[e^{-\int_t^T r_s ds}1|\mathcal{F}_t\right] = E\left[e^{-\int_t^T X_s(t,x)ds}\right] (40)$$

It is obvious that (40) is the same option price valuation formula derived in (30) for the Black-Karasinski term structure model.

IV. Analytical Result

It is obvious from the preceding sections that we have been able to postulate and prove Theorem 5 as stated below.

Theorem5. In a Black-Scholes economy, given an SDE model in finance of the form,

$$dX_t = rX_t dt + \sigma X_t dW_t \qquad (41)$$
$$dB_t = rB_t dt.$$

The Martingales and PDEs European options price valuation formulas are said to be equivalent if and only if,

$$e^{-r(T-t)}E^{\mathbb{B}}[\psi(X_T)|\mathcal{F}_t] = \frac{\partial g}{\partial t}(t,x) + rx\frac{\partial g}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 g}{dx^2}(t,x) - rg(t,x), (42)$$
$$x > 0, t \epsilon [0,T].$$

where $\psi(X_T)$ is the payoff function given as $(K - x)^+$ for Puts and $(x-K)^+$for Calls. B_t is the fixed bond and the numeraire that determines the measure $\mathbb{B}$ under which the discounted stock price

$$\tilde{X} = \frac{X_t}{B_t} = e^{-rt}X_t;$$

and the discounted option price

$$\tilde{u}_t = \frac{u(X_t,t)}{e^{-rt}} = e^{-rt}u(X_t,t).$$

are Martingales.

V. Conclusion

Clearly from the above preceding analysis or explicit solutions of two financial models which are the CEV and Black-Karasinski models considered in this paper, we have been able to prove through the applications of Girsanov and Feyman-Kac theorems, that the Martingales and PDEs options price valuation formulas obtained for these two distinct models are analogous.

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