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Base Change for Fisher Metrics: Case of the $q-$Gaussian Inverse Distribution

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Abstract—It is known that the Riemannian manifold determined by the family of inverse Gaussian distributions endowed with the Fisher metric has negative constant curvature $\kappa = -\frac{1}{2}$, as does the family of usual Gaussian distributions. In the present paper, firstly we arrive at this result by following a different path, much simpler than the previous ones. We first put the family in exponential form, thus endowing the family with a new set of parameters, or coordinates, θ_1, θ_2; then we determine the matrix of the Fisher metric in terms of these parameters; and finally we compute this matrix in the original parameters. Secondly, we define the Inverse $q-$Gaussian distribution family ($q < 3$), as the family obtained by replacing the usual exponential function by the Tsallis $q-$exponential function in the expression for the Inverse Gaussian distribution, and observe that it supports two possible geometries, the Fisher and the $q-$Fisher geometry. And finally, we apply our strategy to obtain results about the Fisher and $q-$Fisher geometry of the Inverse $q-$Gaussian distribution family, similar to the ones obtained in the case of the Inverse Gaussian distribution family.

Keywords—Base of Changes, Information Geometry, Inverse Gaussian distribution, Inverse q-Gaussian distribution, Statistical Manifolds.

I. Introduction

LET us consider a family of probability distributions of parameters $\xi = (\xi_1, \xi_2, \dots, \xi_n)$ with density function $p(x, \xi)$ and likelihood function $\ell(p) = \log(p)$ where $x = (x_1, \dots, x_n)$ are random variables. In terms of the original parameters $\xi_1, \dots, \xi_n$ of the family, the Fisher metric is given by

$$g_{ij}^F = \int (\partial_i \ell(p)) (\partial_i \ell(p)) p d\mu_{\mathcal{X}} \tag{1}$$

where $\mu_{\mathcal{X}}$ is the measure defined on the space $\mathcal{X}$ where x is defined. Many of the probability distribution families, due to their relevance in the modeling of diverse phenomena, can be put in the form of an exponential family. Putting a family of probability distributions $p(x, \xi)$ in the exponential form $p(x, \xi) = C(x) \exp \left[\sum_{i=1}^{n} \theta_i F_i(x) - \psi(\theta)\right]$, where x_i are random variables and θ_i are the parameters or coordinates known as natural parameters. Turns out to make the study of the family much easier. In particular, its Fisher metric can be easily calculated as $\tilde{g}_{ij}^F = \partial_i \partial_j \psi$, where ∂_k denotes differentiation with respect to θ_k and the notation $\sim$ (tilde) will be used to differentiate the metric when it comes from the exponential family. The parameters $\xi = (\xi_1, \dots, \xi_n)$ and $\theta = (\theta_1, \dots, \theta_n)$ can be regarded as two parametrizations of the same probability distribution family. The coefficients g_{ij}^F and $\tilde{g}_{ij}^F$, of the Fisher metric relative to the coordinate systems $\xi_1, \dots, \xi_n$ and $\theta_1, \dots, \theta_n$, respectively, are related as

$$\tilde{g}_{kl}^F = \sum_{i,j=1}^{n} g_{ij}^F(\theta) \left(\frac{\partial \xi_i}{\partial \theta_k}\right) \left(\frac{\partial \xi_j}{\partial \theta_l}\right), \tag{2a}$$

$$g_{ij}^F = \sum_{k,l=1}^{n} \tilde{g}_{kl}^F(\xi) \left(\frac{\partial \theta_k}{\partial \xi_i}\right) \left(\frac{\partial \theta_l}{\partial \xi_j}\right). \tag{2b}$$

which can be seen as a change of base. For more information on this topic, see [2]. These relations can be expressed in matrix form as $\tilde{g}^F = A g^F A^T$ or in its inverse form $g^F = B \tilde{g}^F B^T$, where $A = \left[\frac{\partial \xi_i}{\partial \theta_j}\right]$ for $i, j = 1, 2, \dots, n$, and $B = A^{-1}$. According to this, if we know the components of the metric $\tilde{g}_{ij}^F$, then it is possible to find the components of this same metric in the coordinates $\xi = (\xi_1, \xi_2, \dots, \xi_n)$ following these steps

i) put the family of probability distributions in exponential family form,
ii) express the parameters θ_i as functions of the parameters ξ_j,
iii) write the matrix $\tilde{g}^F$ as a function of the parameters ξ, and
iv) compute the matrix product defined in equation (see (2a)).

Based on the q index of Tsallis (see [12], [7]), many mathematical and physical concepts have been developed which have allowed for the description of complex physical systems. The q-exponential function defined in (13) has served as a basis for the generalization of the exponential models (see [1]), which are families of density functions of the following form

$$p_q(x, \xi) = C(x) \exp_q \left[\sum_{i=1}^{n} \theta_i F_i(x) - \psi(\theta)\right]. \tag{3}$$

Amari and Ohara (see [1]) defined the q-Fisher metric by means of the Bregman divergence, however, it is possible to define it by means of the equality

$$g_{ij}^{(q)} = E_{\hat{p}} \left[(\partial_i \ell_q)(\partial_j \ell_q)\ p^{q-1}\right], \tag{4}$$

where $\hat{p}$ is known as the q-relative distribution and is defined in (15) and $\ell_q = \log_q(p_q)$. This metric fulfills the properties summarized in the theorem 2, particularly the literal (16c), by means of which a relationship is established between the q-Fisher metric and the Fisher metric. Fig. 1 is intended to clarify how we can go from one representation of the metric to another, according to whether it is expressed in the original parameters or in the natural parameters.

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$$\begin{tikzcd}[column sep=large]
 & \text{Original: } \xi & \text{Natural: } \theta \\
\text{Fisher} & g^F \arrow[r, leftrightarrow, "\tilde{g}^F = B g^F B^T"] \arrow[d, leftrightarrow, "g^F_{ij} = \frac{h_q}{q} g^F_{ij}"'] & \tilde{g}^F \arrow[d, leftrightarrow, "\tilde{g}^F_{ij} = \frac{h_q}{q} \tilde{g}^F_{ij}"] \\
q\text{−Fisher} & g^{(q)} \arrow[r, leftrightarrow, "g^{(q)} = C \tilde{g}^{(q)} C^T"'] & \tilde{g}^{(q)}
\end{tikzcd}$$

Fig. 1 Base change for Fisher metric

According to this Tsallis index q, it is possible to generalize families of distributions, for example, the probability density of a family of q-Gaussian distributions of parameters $\xi = (\mu, \sigma)$ has the form

$$p_q(x, \xi) = \frac{1}{Z_{q,\xi}} \exp_q \left(-\frac{(x-\mu)^2}{(3-q)\sigma^2} \right) \tag{5}$$

where $Z_{q,\xi}$ is the normalization constant (see (19)) and $1 < q < 3$. If $q \to 1$ the probability density of the Gaussian family of distributions is recovered, which we will call usual. In [8], [11] and [3] there is a description of the Riemannian manifold associated to this family. In particular it is shown that its curvature is constant and negative has the form $\kappa = -\frac{q}{3-q} < 0$. In [3], following the steps i)-iv) described above, a coordinatization of this Riemannian manifold is obtained putting the q-Fisher metrics in diagonal form. Whereas, for the case of bivariate Gaussian distributions, which depends on five parameters, two means, two variances and the covariance, the representation is diagonal by blocks (see [5]). This allowed us to conclude that geodesic curves are ellipses centered at the μ axis. In the works that we have been carrying out on the Gaussian distributions, q-Gaussian, Gaussian inverse and q-Gaussian inverse, all with two parameters, we have found that in the original parameters, the Fisher information metric has a representation diagonal. This powerful result facilitates the calculation of the Christoffel symbols in the manifold induced by these families of distributions, as well as the solution of the Euler-Lagrange system that describes the geodesic curves of said system. In our research work in Machine Learning, describing distances is important to optimize the use of kernels or similarity measures in algorithms such as Support Vector Machine (SVM).

Zhang (see [14]) studies the Riemannian manifold determined by the n-th power of the inverse Gaussian distribution and he proves that its curvature is constant and it is given by $\kappa = -\frac{1}{2}$. This led us to the fact that the Fisher metric for the family of inverse Gaussian distributions ($n = 1$), can be put in diagonal form relative to an appropriate coordinatization. It also allowed us to propose a generalization of this family of distributions based on the q index (see (20)) in a different way from that proposed by Zhang. We also give a description of the geometry associated with this information manifold.

II. Information Geometry

Let us consider a manifold M, $x = (x_1, x_2, \ldots, x_n)$ an element in it, θ a vector of parameters in an open $\mathbb{R}^n$. The set S of the probability distributions of the random variable x with respect to the parameter θ is called the exponential model or exponential family if the probability distribution $p(x, \theta)$ is written as

$$S = \left\{ p(x,\theta) \geq 0 : p = C(x) \exp \left[\sum_{i=1}^{n} \theta_i F_i(x) - \psi(\theta) \right] \right\} \tag{6}$$

where $C(x)$ and every $F_i(x)$ for $i = 1, 2, \ldots, n$ are random variables in M and ψ is called the potential function, while $\{\theta_i\}$ is the coordinate system called natural parameters. For this family of distributions Fisher metric is defined as in (1). The manifold M together with the Fisher metric is a Riemannian manifold. For exponential families, Fisher metric depends on the partial derivatives of the second order of the potential function (Hessian manifold) and is given by

$$\tilde{g}^F_{ij} = \partial_i \partial_j \psi \ . \tag{7}$$

The potential function for the exponential families also allows us to obtain the cubic form $C^F_{ijk} = E\left[(\partial_i \ell)(\partial_j \ell)(\partial_k \ell)\right]$ that is related to $\alpha \in \mathbb{R}$ with the α-embeddings and therefore with the Christoffel symbols through

$$\begin{aligned} \Gamma^{(\alpha)}_{ij,k} =& E\left[\left(\partial_i \partial_j \ell + \frac{1-\alpha}{2} (\partial_i \ell)(\partial_j \ell) \right) \partial_k \ell \right] \\ =& E\left[(\partial_i \partial_j \ell)\, \partial_k \ell \right] + \frac{1-\alpha}{2} C^F_{ijk} \ . \end{aligned}$$

Christoffel symbols induce the affine α-connection $g\left(\nabla^{(\alpha)}_{\partial_i} \partial_j, \partial_k\right) = \Gamma^{(\alpha)}_{ij,k}$ where g is a metric defined in M, not necessarily Fisher metric. The triplet (M, g, ∇) is a statistical manifold. When $\alpha = 1$ it is said that the connection is exponential, when $\alpha = -1$ the connection is mixed and if $\alpha = 0$ then $\nabla^{(0)}$ is the connection Levi-Civita for Fisher metric. In terms of the components of metric g, Christoffel symbols and the Levi-Civita connection are $\Gamma_{ij,k} = \frac{1}{2} \sum_{l=1}^{n} \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right) g^{lk}$, where $g^{-1} = \left[g^{ij}\right]$ is the inverse of the metric $g = [g_{ij}]$.

Let g be a metric defined in M; X, Y and Z vector fields such that $Xg(Y, Z) = g\left(\nabla_X Y, Z\right) + g\left(Y, \nabla^*_X Z\right)$ then it is said that the connections ∇ and ∇^* are dual with respect to the g metric. The connection ∇ is flat if and only if ∇^* is flat; in such a case it is said that (M, g, ∇, ∇^*) is a dually flat space (see [1], chapter 3). If ∇ is a flat connection, there is a corresponding coordinate system $\{\theta_i\}$ in M and for ∇^* there is another coordinate system $\{\eta_i\}$ for which the equality $g\left(\frac{\partial}{\partial \eta_i}, \frac{\partial}{\partial \theta_j}\right) = \delta_{ij}$ is fulfilled, it is said that $\{\eta_i\}$ is the dual coordinate system of $\{\theta_i\}$ with respect to the g metric.

Theorem 1. *Let (M, g, ∇, ∇^*) be a dually flat statistical manifold. If $\{\theta_i\}$ is a coordinate system and $\{\eta_i\}$ is the dual coordinate system of $\{\theta_i\}$ then, there are functions ψ and ϕ such that $\frac{\partial \psi}{\partial \theta_i} = \eta_i$ $\frac{\partial \varphi}{\partial \eta_i} = \theta_i$, which satisfy the Legendre Transformation $\psi(p) + \varphi(p) - \sum_{i=1}^{n} \theta_i(p) \eta_i(p) = 0$. Additionally*

$g_{ij} = \frac{\partial^2 \psi}{\partial \theta_i \partial \theta_j}$ *and* $g^{ij} = \frac{\partial^2 \varphi}{\partial \eta_i \partial \eta_j}$ *being* $g^{-1} = \left[g^{ij}\right]$ *the inverse of* g.

III. Inverse Gaussian Distribution

A random variable X has an inverse Gaussian distribution with parameters $\xi = (\mu, \lambda)$ both positive, if the probability density function is

$$p(x;\xi) = \sqrt{\frac{\lambda}{2\pi x^3}} \exp\left(-\frac{\lambda}{2\mu^2 x}(x-\mu)^2\right) \tag{8}$$

where $x > 0$. To demonstrate that the function given in (8) is a density, we can use of the modified Bessel function or with the transformation presented in [13]. The distribution has been used in the description of some aspects of the Brownian movement, finance, time series, etc. (see [4]). In Fig. 2 we can study the behavior of inverse Gaussian distribution, when any of its two parameters are fixed. In the upper part of Fig. 2, we take $\mu = 1$ and the shape parameter λ is varied according to the value specified in each color, we can notice that the maximum moves to the right as λ increases. In the lower part of this graph, we take $\lambda = 1$ and vary μ (see value by color), which means that the maximum does not move but decreases in value. Each density has its maximum value if x assumes the value $\frac{\mu}{2\lambda}\left(\sqrt{9\mu^2 + 4\lambda^2} - 3\mu\right)$. If X is a random variable with such a distribution, the expected value and the variance are given by $E[X] = \mu$ and $Var[X] = \frac{\mu^3}{\lambda}$.

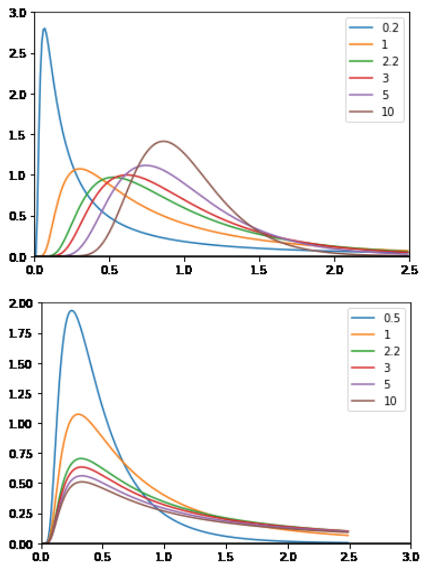

Fig. 2 Inverse Gaussian distributions for $\mu = 1$ (up) and $\lambda = 1$ (down)

We can rewrite the probability density function to have

$$p(x,\xi) = x^{-3/2} \exp\left[-\frac{\lambda}{2\mu^2 x}(x-\mu)^2 + \ln\sqrt{\frac{\lambda}{2\pi}}\right],$$
$$= x^{-3/2} \exp\left[-\frac{\lambda}{2\mu^2}x - \frac{\lambda}{2}x^{-1} - \left(\frac{1}{2}\ln(2\pi) - \frac{1}{2}\ln\lambda - \frac{\lambda}{\mu}\right)\right].$$

According to this last expression, the inverse Gaussian family distribution, is an element of the exponential family with natural parameters $\theta = (\theta_1, \theta_2)$ and potential function given by

$$\theta_1 = \frac{\lambda}{2\mu^2},\ \theta_2 = \frac{\lambda}{2},\ \text{and } \psi(\theta) = \frac{1}{2}\ln\left(\frac{\pi}{\theta_2}\right) - 2\sqrt{\theta_1\theta_2}\,. \tag{9}$$

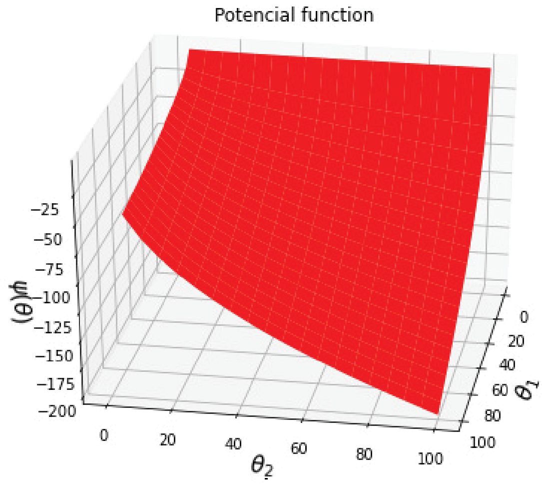

Fig. 3 Potential function for the components θ_1 and θ_2

In the components θ_1 and θ_2, the potential function is convex (see Fig. 3). If we calculate the second order derivatives of the potential function ψ with respect to the components θ_1 and θ_2; we obtain the Fisher metric $\tilde{g}^F$ (see (7))

$$\tilde{g}^F = \begin{bmatrix} \frac{1}{2\theta_1}\sqrt{\frac{\theta_2}{\theta_1}} & -\frac{1}{2\sqrt{\theta_1\theta_2}} \\ -\frac{1}{2\sqrt{\theta_1\theta_2}} & \frac{1}{2\theta_2^2} + \frac{1}{2\theta_2}\sqrt{\frac{\theta_1}{\theta_2}} \end{bmatrix}. \tag{10}$$

In order to obtain Fisher metric g^F with diagonal representation; the components θ_1 and θ_2, given in (9), are derived in function of the original components $\xi = (\mu, \lambda)$ and the matrix (10) is rewritten in this coordinate system ξ, according to (2b) results

$$g^F = \begin{bmatrix} -\frac{\lambda}{\mu^3} & 0 \\ \frac{1}{2\mu^2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{\mu^3}{\lambda} & -\frac{\mu}{\lambda} \\ -\frac{\mu}{\lambda} & \frac{2}{\lambda^2} + \frac{1}{\lambda\mu} \end{bmatrix} \begin{bmatrix} -\frac{\lambda}{\mu^3} & \frac{1}{2\mu^2} \\ 0 & \frac{1}{2} \end{bmatrix}, \tag{11}$$
$$= \begin{bmatrix} \frac{\lambda}{\mu^3} & 0 \\ 0 & \frac{1}{2\lambda^2} \end{bmatrix}.$$

Christoffel symbols associated with Fisher metric g^F defined in (11) are given by: $\Gamma_{11,1} = -\frac{3}{2\mu}$, $\Gamma_{11,2} = -\frac{\lambda^2}{\mu^3}$, $\Gamma_{12,1} = \Gamma_{21,1} = \frac{1}{2\lambda}$, $\Gamma_{22,2} = -\frac{1}{\lambda}$ and zeros for others. Since the components R^1_{212} and R^2_{212} of the metric tensor are $-\frac{1}{4\lambda^2}$ and zero, then these results lead to that the curvature of this information manifold is

$$\kappa = \frac{g^F_{11}R^1_{212} + g^F_{12}R^2_{212}}{\det(g^F)} = -\frac{1}{2}\,. \tag{12}$$

This curvature coincides with that of the statistical manifold induced by the Gaussian family distributions with components $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$.

IV. q-Exponential Family

For the Tsallis entropy index q, the functions q-exponential and q-logarithm are defined, inverses one of the other, through the expressions

$$\exp_q(x) = [1 + (1-q)x]^{\frac{1}{1-q}} \ , \tag{13}$$

$$\log_q(x) = \frac{1}{1-q}\left[x^{1-q} - 1\right] \ . \tag{14}$$

where $q < 3$. Definitions, properties and some generalizations raised by this pair of functions can be found in [12] (Chapter 3), [7] and [9] (Chapter 7). Let us highlight the fact that $\frac{d}{dx}\exp_q(x) = \frac{\exp_q(x)}{1+(1-q)x}$ and $\frac{d}{dx}\log_q(x) = \frac{1}{x^q}$. Fig. 4 shows how the q-exponential and q-logarithm functions tend to the usual exponential and logarithm functions as $q \to 1$.

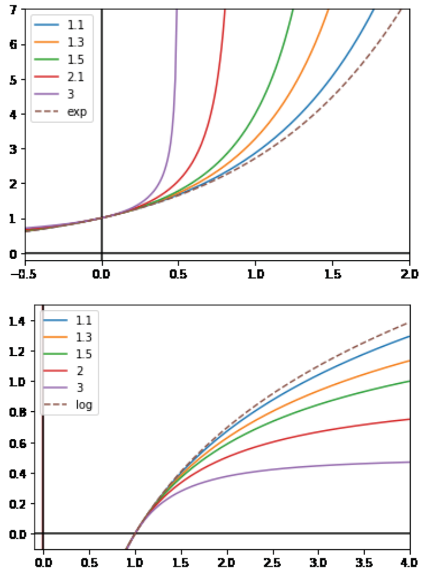

Fig. 4 q-exponential (up) and q-logarithm (down) for some value of q

From the definition of the function q-exponential, Amari (see [1]) generalizes the definition of exponential family given in (6) as

$$S = \left\{ p(x,\theta) \geq 0 : p = C(x)\exp_q\left[\sum_{i=1}^{n}\theta_i F_i(x) - \psi_q(\theta)\right]\right\}$$

where x are random variables and θ natural parameters (see also [10]). The function ψ_q is called q-potential while the likelihood function is given by $\ell_q = \log_q p(x,\theta)$. For the q index, the functional $h_q(\theta)$ is defined as $h_q(\theta) = E\left[p(x,\theta)^q\right]$, from which they are defined: q-relative distribution,

$$\hat{p}_q(x,\theta) = \frac{1}{h_q(\theta)}p(x,\theta)^q \ , \tag{15}$$

and the expected value $E_{\hat{p}}[f] = E_{\hat{p}}\left[f(x)\right]$ relative to $\hat{p}_q$. The q-Fisher metric can be seen as $g_{ij}^{(q)} = E_{\hat{p}}\left[(\partial_i \ell_q)(\partial_j \ell_q) \ \ p^{q-1}\right]$ (see [11] and [3]) for $q < 3$. In Theorem 2 some results are presented for the q-Fisher metric. The last expression in (16c) implies that the q-Fisher metric is conformal with Fisher metric (see [1]).

Theorem 2. *Let $p(x,\xi)$ be a probability distribution belonging to the q-exponential family. The q-Fisher metric is equivalent to*

$$g_{ij}^{(q)} = -\ E_{\hat{p}}\left[\partial_j(\partial_i \ell_q)\right] \ , \tag{16a}$$

$$g_{ij}^{(q)} = \partial_i\partial_j\psi_q \ , \tag{16b}$$

$$g_{ij}^{(q)} = \frac{q}{h_q}g_{ij}^F \quad \textit{where } g_{ij}^F \textit{ is Fisher metric} \ . \tag{16c}$$

where $\hat{p}$ was defined in (15).

Example 1. *A variable X has a q-Gaussian distribution with mean $\mu \in \mathbb{R}$ and variance $\sigma \in \mathbb{R}^+$ and its probability density has the form given in* (5) *where $1 < q < 3$, $\xi = (\mu, \sigma)$ is the set of parameters and $Z_{q,\sigma}$ is the normalization constant that results from solving the integral*

$$Z_{q,\xi} = \int_{-\infty}^{\infty}\exp_q\left(-\frac{(x-\mu)^2}{(3-q)\sigma^2}\right)dx \tag{17}$$

$$= 2\sigma\sqrt{3-q}\int_0^{\infty}\exp_q(-t^2)dt \ , \tag{18}$$

constant that depends on the index q and the parameter σ and that is given by

$$Z_{q,\sigma} = A_q\sigma = \begin{cases} \left[\sqrt{\frac{3-q}{1-q}}B\left(\frac{2-q}{1-q},\frac{1}{2}\right)\right]\ \sigma & \textit{If } q < 1 \\ \sqrt{2\pi}\sigma & \textit{If } q = 1 \\ \left[\sqrt{\frac{3-q}{q-1}}B\left(\frac{3-q}{2(q-1)},\frac{1}{2}\right)\right]\ \sigma & \textit{If } 1 < q < 3 \end{cases} . \tag{19}$$

The functional $h_q(p)$ for a variable with q-Gaussian distribution has the form $h_q(p) = \frac{3-q}{2}Z_{q,\sigma}^{1-q}$ (see [11]). This distribution is an element of the family q-exponential and it induces a Riemannian manifold with curvature $\kappa = -\frac{q}{3-q}$. In [3] you get to the diagonal metric g^F with the base change (2b)*, which allows some of Christoffel symbols to be null and leads to show that the geodesic curves in the manifold induced by this distribution are ellipse arcs centered on the axis μ. Other properties of this family of q-Gaussian distributions can be found in [6].*

V. Inverse q-Gaussian Distribution

According to Tsallis' theory with entropy index q, it is natural to propose as a generalization of the inverse Gaussian distribution, the distribution

$$p(x,\xi) = \frac{x^{-3/2}}{Z_{q,\xi}}\exp_q\left(-\frac{\lambda}{(3-q)\mu^2 x}(x-\mu)^2\right) \tag{20}$$

which we will call the inverse q-Gaussian distribution, where $\xi = (\mu, \lambda)$ are positive parameters, $x \in \mathbb{R}^+$ and $Z_{q,\xi}$ is the normalization constant. In Fig. 5 this distribution is represented for values of q, less than 1 (up) and for $1 < q < 3$ (down). In this figure, horizontally the realizations x of the random variable are represented and vertically the value of the distribution for the set of parameters $\xi = (\mu, \lambda)$.

To find the normalization constant, this integral is solved through the procedure described by Wani (see [13]) as follows

$$Z_{q,\xi} = \int_0^{\infty} x^{-3/2}\exp_q\left(-\frac{\lambda}{(3-q)\mu^2}x + \frac{2\lambda}{(3-q)\mu} - \frac{\lambda}{(3-q)}x^{-1}\right)dx. \tag{21}$$

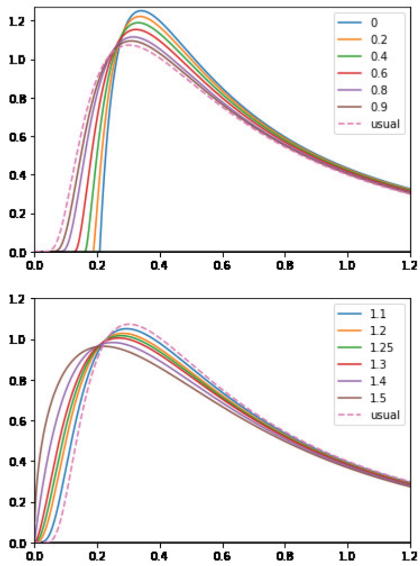

Fig. 5 Graph of inverse q-Gaussian distribution for $\sigma = 1$ and $\mu = 1$, when $q < 1$ (up) and $1 < q < 3$ (down)

To obtain the solution, first we are going to do the substitutions $a^2 = \frac{\lambda}{(3-q)\mu^2}$, $b^2 = \frac{\lambda}{3-q}$ and $x = T^2$ which brings us to the expressions $ab = \frac{\lambda}{(3-q)\mu}$ and $\frac{a}{b} = \frac{1}{\mu}$. The integral (21) can be written as

$$Z_{q,\xi} = 2\int_0^\infty T^{-2}\exp_q\left(-\left(a^2T^2 + \frac{b^2}{T^2}\right) + 2ab\right) dT\ . \tag{22}$$

We make a new substitution $Q = \left(\frac{aT^2-b}{T}\right)^2$ from which equality can be reached $-Q = -\left(a^2T^2 + \frac{b^2}{T^2}\right) + 2ab$. We also have the equation $a^2T^4 - (2ab+Q)T^2 + b^2 = 0$ with the two solutions presented below and for which the term $\frac{dT}{T^2}$.

$$T = \frac{\sqrt{Q+4ab}+\sqrt{Q}}{2a}, \qquad \frac{dT}{T^2} = \frac{\sqrt{Q+4ab}-\sqrt{Q}}{4b\sqrt{Q(Q+4ab)}}dQ\ ,$$
$$T = \frac{\sqrt{Q+4ab}-\sqrt{Q}}{2a}, \qquad \frac{dT}{T^2} = -\frac{\sqrt{Q+4ab}+\sqrt{Q}}{4b\sqrt{Q(Q+4ab)}}dQ\ ,$$

The first result occurs for $T \geq \sqrt{\frac{b}{a}}$ and $T < \sqrt{\frac{b}{a}}$ for the second. By substituting these results in the integral (22) we have the equality

$$\begin{aligned} Z_{q,\xi} = 2\int_\infty^0 \exp_q(-Q)\left[-\frac{\sqrt{Q+4ab}+\sqrt{Q}}{4b\sqrt{Q(Q+4ab)}}\right]dQ + \\ 2\int_0^\infty \exp_q(-Q)\left[\frac{\sqrt{Q+4ab}-\sqrt{Q}}{4b\sqrt{Q(Q+4ab)}}\right]dQ\ , \\ = \frac{1}{b}\int_0^\infty Q^{-1/2}\exp_q(-Q)dQ\ . \end{aligned}$$

Making substitution $Q = t^2$ and since $b = \sqrt{\frac{\lambda}{3-q}}$ then equals $Z_{q,\xi} = \frac{1}{\sqrt{\lambda}}\left[2\sqrt{3-q}\int_0^\infty \exp_q(-t^2)dt\right]$. Expressions (17) and (19) characterize the normalization constant for the family of q-Gaussian distributions, then it can be concluded that

$$Z_{q,\lambda} = \frac{A_q}{\sqrt{\lambda}} = \begin{cases} \left[\sqrt{\frac{3-q}{1-q}}B\left(\frac{2-q}{1-q}, \frac{1}{2}\right)\right]\frac{1}{\sqrt{\lambda}} & \text{If } q < 1 \\ \sqrt{\frac{2\pi}{\lambda}} & \text{If } q = 1 \\ \left[\sqrt{\frac{3-q}{q-1}}B\left(\frac{3-q}{2(q-1)}, \frac{1}{2}\right)\right]\frac{1}{\sqrt{\lambda}} & \text{If } 1 < q < 3 \end{cases}, \tag{23}$$

where A_q is the constant given in (19) for the q-Gaussian distribution. In Fig. 6, where $\lambda = 2$, we can determine that the function $Z_{p,\lambda}$ is increasing and continuous. In addition $\lim\limits_{q\to 1^-} Z_{q,\lambda}$ and $\lim\limits_{q\to 1^+} Z_{q,\lambda}$ converge to $\sqrt{\frac{2\pi}{\lambda}}$.

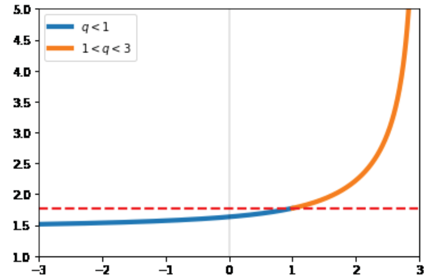

Fig. 6 Function $Z_{q,\lambda}$ for $\lambda = 2$ referred to the inverse q-Gaussian

Let us see now that the this distribution is an element of the q-exponential family. For this we apply the definition of the q-exponential function given in (13) in the definition of probability density of the inverse q-Gaussian distribution (see 20), so

$$p(x,\xi) = \frac{x^{-3/2}}{Z_{q,\lambda}}\left[\left(-\frac{\lambda}{(3-q)\mu^2 x}(x-\mu)^2\right)(1-q)+1\right]^{\frac{1}{1-q}},$$
$$= \frac{\left[\left(-\frac{\lambda Z_{q,\lambda}^{q-1}}{(3-q)\mu^2 x}(x-\mu)^2\right)(1-q) + \left[\frac{Z_{q,\lambda}^{q-1}-1}{1-q}\right](1-q)+1\right]^{\frac{1}{1-q}}}{x^{3/2}}\ .$$

According to the definition of the functions q-logarithm and q-exponential, as well as some algebraic properties, it is concluded that:

$$p(x,\xi) = x^{-3/2}.$$
$$\exp_q\left[-\frac{Z_{q,\lambda}^{q-1}}{3-q}\frac{\lambda}{\mu^2}x - \frac{Z_{q,\lambda}^{q-1}}{3-q}\lambda x^{-1} - \left(-\log_q\left(\frac{1}{Z_{q,\lambda}}\right) - \frac{Z_{q,\lambda}^{q-1}}{3-q}\frac{2\lambda}{\mu}\right)\right],$$

which allows to determine that the family of inverse q-Gaussian distributions is an element of the family q-exponential. Therefore the natural parameters $\theta = (\theta_1, \theta_2)$ are

$$\theta_1 = \frac{Z_{q,\lambda}^{q-1}}{3-q}\frac{\lambda}{\mu^2} \quad \text{and} \quad \theta_2 = \frac{Z_{q,\lambda}^{q-1}}{3-q}\lambda\ . \tag{24}$$

With these components θ_1 and θ_2, we can conclude that $\frac{Z_{q,\lambda}^{q-1}}{3-q}\frac{2\lambda}{\mu} = 2\sqrt{\theta_1\theta_2}$ and also $\frac{1}{Z_{q,\lambda}} = (d_q\theta_2)^{\frac{1}{3-q}}$ where $d_q = \frac{3-q}{A_q^2}$. With these equalities, the q-potential function in the components θ_1, θ_2 is given by

$$\psi_q(\theta_1, \theta_2) = -\log_q(d_q\theta_2)^{\frac{1}{3-q}} - 2\sqrt{\theta_1\theta_2}\ . \tag{25}$$

We can obtain the partial derivative of the potential function on each of the components θ_1 and θ_2, where $\partial_1\psi_q = -\sqrt{\frac{\theta_2}{\theta_1}}$

and $\partial_2\psi_q = -\sqrt{\frac{\theta_1}{\theta_2}} - \frac{d_q^{\frac{1-q}{3-q}}}{(3-q)\theta_2^{\frac{2}{3-q}}}$. And from these results the derivatives of order two are obtained on these same components $\partial_1(\partial_1\psi_q) = \frac{1}{2\theta_1}\sqrt{\frac{\theta_2}{\theta_1}}$, $\partial_1(\partial_2\psi_q) = \partial_2(\partial_1\psi_q) = -\frac{1}{2\sqrt{\theta_1\theta_2}}$ y $\partial_2(\partial_2\psi_q) = \frac{1}{2\theta_2}\sqrt{\frac{\theta_1}{\theta_2}} + \frac{2Z_{q,\lambda}^{q-1}}{(3-q)^2\theta_2^2}$ since $\frac{1}{Z_{q,\lambda}} = (d_q\theta_2)^{\frac{1}{3-q}}$. With partial derivatives of the q-potential function, q-Fisher metric whose representation is not diagonal is

$$\tilde{g}^{(q)} = \begin{bmatrix} \frac{1}{2\theta_1}\sqrt{\frac{\theta_2}{\theta_1}} & -\frac{1}{2\sqrt{\theta_1\theta_2}} \\ -\frac{1}{2\sqrt{\theta_1\theta_2}} & \frac{1}{2\theta_2}\sqrt{\frac{\theta_1}{\theta_2}} + \frac{2Z_{q,\lambda}^{q-1}}{(3-q)^2\theta_2^2} \end{bmatrix} . \tag{26}$$

To obtain the q-Fisher metric with diagonal representation, the components of the natural parameters $\theta = (\theta_1, \theta_2)$ are derived in terms of the given parameters $\xi = (\mu, \lambda)$, it turns out that

$$\begin{aligned} \frac{\partial\theta_1}{\partial\mu} &= -\frac{2Z_{q,\lambda}^{q-1}}{3-q}\frac{\lambda}{\mu^3} , & \frac{\partial\theta_2}{\partial\mu} &= 0 , \\ \frac{\partial\theta_1}{\partial\lambda} &= \frac{Z_{q,\lambda}^{q-1}}{3-q}\frac{1}{\mu^2} , & \frac{\partial\theta_2}{\partial\lambda} &= \frac{Z_{q,\lambda}^{q-1}}{3-q} . \end{aligned}$$

The metric (26) is written in terms of the components μ and λ.

$$\begin{aligned} g^{(q)} &= \begin{bmatrix} -1 & \frac{1}{\mu^2} \\ 0 & \frac{2}{3-q}\frac{1}{\lambda^2} \end{bmatrix} \begin{bmatrix} -\frac{2Z_{q,\lambda}^{q-1}}{3-q}\frac{\lambda}{\mu^3} & \frac{Z_{q,\lambda}^{q-1}}{3-q}\frac{1}{\mu^2} \\ 0 & \frac{Z_{q,\lambda}^{q-1}}{3-q} \end{bmatrix} , \\ &= \begin{bmatrix} \frac{2Z_{q,\lambda}^{q-1}}{3-q}\frac{\lambda}{\mu^3} & 0 \\ 0 & \frac{2Z_{q,\lambda}^{q-1}}{(3-q)^2}\frac{1}{\lambda^2} \end{bmatrix} . \end{aligned}$$

To obtain the Fisher metric, it is necessary to calculate the functional $h_q(p)$ since, according to theorem 2, $g^F = \frac{h_q}{q}g^{(q)}$. This function can be written in the form $h_q(p) = f(q)Z^{1-q}$ where $f(q)$ is a function such that $f(q) \to 1$ as $q \to 1$ and the functional h_q exists if $1 \leq q < 2$. Accordingly, we can write the matrix $g^{(q)}$ and find a diagonal representation of the Fisher metric, in the original parameters $\xi = (\mu, \lambda)$, in the statistical manifold generated by the family of inverse q-Gaussian distributions.

$$g^{(q)} = \begin{bmatrix} \frac{2f(q)}{(3-q)h_q}\frac{\lambda}{\mu^3} & 0 \\ 0 & \frac{2f(q)}{(3-q)^2h(q)}\frac{1}{\lambda^2} \end{bmatrix} , \tag{27}$$

$$g^F = \begin{bmatrix} \frac{2f(q)}{q(3-q)}\frac{\lambda}{\mu^3} & 0 \\ 0 & \frac{2f(q)}{q(3-q)^2}\frac{1}{\lambda^2} \end{bmatrix} . \tag{28}$$

For the metric g^F obtained in (27), The partial derivatives with respect to the components μ and λ, are given by

$$\begin{aligned} \partial_1 g_{11}^F &= -\frac{6f(q)}{q(3-q)}\frac{\lambda}{\mu^4}, & \partial_2 g_{11}^F &= \frac{2f(q)}{q(3-q)}\frac{1}{\mu^3}, \\ \partial_1 g_{22}^F &= 0, & \partial_2 g_{22}^F &= -\frac{4f(q)}{q(3-q)^2}\frac{1}{\lambda^3}. \end{aligned}$$

Based on these results, Christoffel symbols are

$$\begin{aligned} \Gamma_{11,1} &= -\frac{3}{2\mu} & \Gamma_{11,2} &= -\frac{(3-q)}{2}\frac{\lambda^2}{\mu^3} \\ \Gamma_{22,2} &= -\frac{1}{\lambda} & \Gamma_{21,2} &= \Gamma_{12,2} = \Gamma_{22,1} = 0 \\ \Gamma_{21,1} &= \frac{1}{2\lambda} = \Gamma_{12,1} \end{aligned} \tag{29}$$

If $q \to 1$, these results coincide with the Christoffel symbols obtained on the statistical manifold induced by the family of inverse Gaussian distributions. Note that these symbols are not dependent on function $f(q)$. Since $\partial_2\Gamma_{12,1} = -\frac{1}{2\lambda^2}$ and $\partial_2\Gamma_{12,2} = \partial_1\Gamma_{22,1} = \partial_1\Gamma_{22,2} = 0$ then the components R_{212}^1 and R_{212}^2 of the metric tensor are $-\frac{1}{4\lambda^2}$ and zero respectively. As $\det(g^F) = \frac{4f^2(q)}{q^2(3-q)^3}\frac{1}{\lambda\mu^3}$ so

$$\kappa = \frac{g_{11}^F R_{121}^1 + g_{12}^F R_{121}^2}{\det(g^F)} = -\frac{q(3-q)^2}{8f(q)} \tag{30}$$

If $q \to 1$ then $\kappa = -\frac{1}{2}$, which corresponds to the curvature of the manifold of information induced by the family of inverse Gaussian distributions.

To obtain the function $f(q)$ it is necessary to find an expression for the functional $h_q(p)$. For this, the definition of such a functional is applied as $h_q(p) = \int p^q\, dx$ and we follow a similar path to obtaining the normalization constant $Z_{p,\lambda}$ for the family of inverse Gaussian distributions. Defining the constants $a^2 = \frac{\lambda}{(3-q)\mu^2}$, $b^2 = \frac{\lambda}{3-q}$ and then the substitutions: $x = T^2$ at first and then $Q = \left(aT - \frac{b}{T}\right)^2$ in a second moment, leads us to the integral

$$\begin{aligned} h_q(p) = & \frac{(2a)^{2-3q}}{Z_{q,\lambda}^q}\int_0^\infty \frac{\left(\sqrt{Q+4ab}+\sqrt{Q}\right)^{2-3q}}{\sqrt{Q(Q+4ab)}}\left[\exp_q(-Q)\right]^q dQ \\ & + \frac{(2a)^{2-3q}}{Z_{q,\lambda}^q}\int_0^\infty \frac{\left(\sqrt{Q+4ab}-\sqrt{Q}\right)^{2-3q}}{\sqrt{Q(Q+4ab)}}\left[\exp_q(-Q)\right]^q dQ . \end{aligned}$$

Let us take $v = 2-3q$ and the replacement $Q = 4ab\sinh^2(y)$, where $dQ = 8ab\sinh(y)\cosh(y)dy$. Making use of some identities from the usual hyperbolic trigonometry, this substitution will lead us to the integral

$$\begin{aligned} h_q(p) = & \frac{4\mu^v}{Z_{q,\lambda}^q}\int_0^\infty \cosh(vy)\left[\exp_q\left(-4ab\sinh^2(y)\right)\right]^q dy , \\ = & \frac{4\mu^v}{Z_{q,\lambda}^q}\int_0^\infty \cosh(vy)\left[1 - \frac{4(1-q)}{3-q}\frac{\lambda}{\mu}\sinh^2(y)\right]^{\frac{q}{1-q}} dy . \end{aligned}$$

This integral is convergent for $1 \leq q < 2$ and does not have an analytical solution for every value of q.

VI. Conclusion

The methodology proposed by means of the base change given in (2a) and (2b) allows us to obtain the Fisher metric in an easier way, by means of partial derivatives; a process that we exposed in [3] and the present work. On the other hand, the Tsallis index has allowed us to generalize some families as the inverse Gaussian as in (20), generalization that we have not found in recent articles and for which we are studying its applications in finance and the Brownian movement. For the family of inverse distributions q-Gaussian, an analytical expression was obtained for the normalization constant, which is given by $Z_{q,\lambda} = \frac{A_q}{\sqrt{\lambda}}$ where A_q is the same constant that results for the normalization constant $Z_{q,\sigma}$ of q-Gaussian distributions. We also find expressions for Fisher and q-Fisher metrics. And with these we managed to account for the Christoffel symbols and with them the curvature given by $\kappa = -\frac{q(3-q)^2}{8f(q)}$. If we take $q \to 1$, we recover the elements of the information geometry induced by the family of inverse Gaussian distributions.

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