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Identification of Configuration Space Singularities with Local Real Algebraic Geometry

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Abstract—We address the question of identifying the configuration space singularities of linkages, i.e., points where the configuration space is not locally a submanifold of Euclidean space. Because the configuration space cannot be smoothly parameterized at such points, these singularity types have a significantly negative impact on the kinematics of the linkage. It is known that Jacobian methods do not provide sufficient conditions for the existence of CS-singularities. Herein, we present several additional algebraic criteria that provide the sufficient conditions. Further, we use those criteria to analyze certain classes of planar linkages. These examples will also show how the presented criteria can be checked using algorithmic methods.

Keywords—Linkages, configuration space singularities, real algebraic geometry, analytic geometry, computer algebra.

I. Introduction

The configuration space X of a linkage can generally be represented by a real algebraic set embedded in Euclidean space $\mathbb{R}^n$ [1], [2]. A configuration space singularity (CS-singularity) is then a point $p \in X$, where X is not locally a submanifold of $\mathbb{R}^n$ [3]-[5]. The usual approach to finding CS-singularities is to look for points where the Jacobian matrix of the polynomial constraint equations loses rank (i.e., has rank smaller than $n-d$, where $d = \dim X$). In [6, p. 227] three problems regarding this method were identified:

- (P1) X is not equidimensional. No points on a lower-dimensional component of X will have full rank, even if they are actually (lower-dimensional) manifold points. For example, if $X \subset \mathbb{R}^3$ is defined by $xz = 0$ and $yz = 0$, then the Jacobian matrix will lose rank on any point along the z-axis.
- (P2) X is not reduced, i.e., the ideal that defines X is not radical. In this case, the rank is smaller than $n-d$, even if X is smooth. For example, consider $X \subset \mathbb{R}^2$ defined by $x^2 = 0$ in $\mathbb{R}^2$. The Jacobian matrix is zero on X, which is simply the (smooth) y-axis.
- (P3) Even if X is reduced and equidimensional, X can still be a manifold at rank-deficient points because some analytic components might not be visible in real space (see Appendix A). For example, consider $X \subset \mathbb{R}^2$ defined by $f(x,y) = y^3 + 2\,y\,x^2 - x^4 = 0$. Then, X is equidimensional (even irreducible) and reduced. The gradient of f loses rank at the origin, but X is a smooth manifold (Fig. 1).

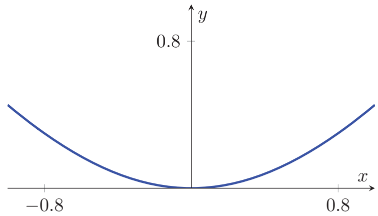

Fig. 1 Zero set of $y^3 + 2xy^2 - x^4$

Recently, an overconstrained 6R-chain was reported [7], [3], which has a smooth configuration space X, but there exists a point p in X, where the Jacobian matrix of the constraint equations is rank-deficient. The problem in this case was (P2). The fat point of the (nonreduced) configuration space disappears when the radical of the ideal of the constraint equations is calculated. We will see later that (P2) will only be a problem for overconstrained mechanism because the constraint equations will generate a radical ideal if they form a complete intersection which is not singular at all points (Theorem 1).

In this paper we review some techniques to handle problems (P1), (P2), (P3) and describe their application to the configuration spaces of certain planar linkages. Our approach to (P3) will be novel and most of the theory for it is presented in Appendix A.

The presented methods are algorithmic and can be adapted to a large class of linkages. We will use the open source computer algebra system `Singular` [8] for all computations. The `Singular`-code used is presented in Listings 1-4 of Appendix B.

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II. The Four Bar

We begin with an analysis of the configuration space of one of the most comprehensively studied planar linkages, the planar four-bar [2], [9]. This is the first example examined in [6]. Refer to [**?**] for a deeper analysis of the blow-up process used and [11, Theorem 1.6] for an investigation of the local topology at the singular points.

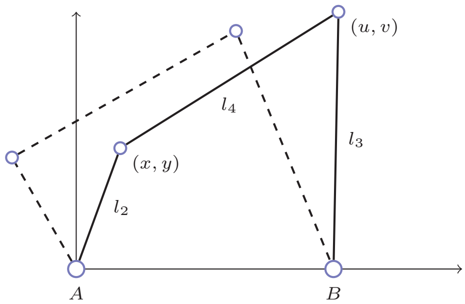

Fig. 2 Four-bar Linkage

The configuration space X of the four-bar linkage is defined as the set of all possible assembly configurations. It can be represented by the real zero set of the ideal $I = \langle p_1, p_2, p_3 \rangle \leq \mathbb{R}[x, y, u, v]$, wherein

$$p_1 = x^2 + y^2 - l_2^2, \quad (1)$$
$$p_2 = (u-2)^2 + v^2 - l_3^2, \quad (2)$$
$$p_3 = (u-x)^2 + (v-y)^2 - l_4^2, \quad (3)$$

are the length-constraints in Euclidean coordinates. $l_2, l_3, l_4 > 0$ are the parameters of the four-bar. Note that we fixed $l_1 = |AB| = 2$. Any other length l_1 is handled by scaling the parameters l_3, l_3, l_4.

A. Dimension of I

We will assume $l_2 \neq 2$, $l_4 \neq 2$. The other three cases

$$l_2 = 2, l_4 \neq 2, \quad l_2 \neq 2, l_4 = 2, \quad l_2 = l_4 = 2$$

can be analyzed in a similar way. To determine $\dim I$, we calculate a Gröbner basis of I with respect to the product ordering `(dp(2),dp(2))` on $(v, y), (u, x)$ [12, Example 1.2.8], see Listing 2. Because p_1, p_2, p_3 depend on the parameters l_2, l_3, l_4, we need to avoid dividing by elements of $\mathbb{Q}(l_2, l_3, l_4)\backslash\mathbb{Q}$ in all Gröbner basis calculations because they could have a value of zero for valid parameters l_2, l_3, l_4. In `Singular` we can achieve this by setting `option(intStrategy)` and `option(contentSB)`.

With the command `std(I)` in `Singular` (see Listing 2), we obtain six polynomials $g_1, \ldots, g_6$ with the leading terms

$$\mathrm{LT}(g_1) = -16\, u^2\, x, \qquad \mathrm{LT}(g_4) = y^2, \quad (4)$$
$$\mathrm{LT}(g_2) = (-2\, l_2^2 + 8)\, v\, u, \qquad \mathrm{LT}(g_5) = 2\, v\, y, \quad (5)$$
$$\mathrm{LT}(g_2) = -2\, v\, x^2, \qquad \mathrm{LT}(g_6) = v^2. \quad (6)$$

According to Exercise 2.3.8 of [12], $G = \{g_1, \ldots, g_6\}$ is a Gröbner basis of I as long as $l_2 \neq \pm 2$ which we had assumed. Hence, we can calculate the dimension of I using

$$\dim I = \dim \langle \mathrm{LM}(G) \rangle \quad (7)$$
$$= \dim \langle u^2\, x, v\, u, v\, x^2, y^2, v\, y,\, v^2 \rangle. \quad (8)$$

The ideal in (8) is a monomial ideal; according to [13, Prop. 9.1.3] and a combinatorial consideration, its dimension must be 1. Consequently, we have $\dim I = 1$ and $\mathrm{ht}\, I = 3$. Because $I = \langle p_1, p_2, p_3 \rangle$ can be generated by three elements, $R := \mathbb{R}[x, y, u, v]/I$ must be equidimensional Cohen-Macaulay according to [14, Proposition 18.13] and [15, 10.133]. This addresses problem (P1).

B. Singular Locus

As stated in [6], there exist singular points in X if and only if the Grashof condition

$$\pm l_2 \pm l_3 \pm l_4 = 2.$$

is fulfilled. We restrict this examination to the case of $l_2 - l_3 + l_4 = 2$, which means $l_3 = l_2 + l_4 - 2 > 0$. The other cases can be handled analogously. Because $\dim I = 1$, and I is equidimensional, the singular points of X are the real zeros of the ideal S, which is generated by I and all the 3-minors of the Jacobian of (p_1, p_2, p_3). Now, we perform a Gröbner basis computation with `Singular` (Listing 2) and obtain $S = \langle s_1, s_2, s_3, s_4 \rangle$, where

$$s_1 = q_1(l_2, l_4)\, x + c_1(l_2, l_4), \quad (9)$$
$$s_2 = q_2(l_2, l_4)\, u + r_2(l_2, l_4)\, x + c_2(l_2, l_4), \quad (10)$$
$$s_3 = q_3(l_2, l_4)\, y, \quad (11)$$
$$s_4 = q_4(l_2, l_4)\, v + f(l_2, l_4, x, y, u). \quad (12)$$

We need to examine the coefficients $q_i \in \mathbb{Q}[l_2, l_5]$ of the leading monomials of all s_i to ensure that $\{s_1, s_2, s_3, s_4\}$ is a Gröbner basis of S. Our calculation in Listing 2 shows that after simplification over $\mathbb{Q}$

$$q_1(l_2, l_4) = l_4^2 \cdot (l_4 - 2) \cdot (l_2 + l_4) \cdot (l_2 + l_4 - 2)^2 \cdot (l_2 + 2)^3 \cdot l_2 \cdot (3\, l_2 - 8), \quad (13)$$
$$q_2(l_2, l_4) = l_4^2 \cdot (l_2 + 2\, l_4 - 2) \cdot (l_2 + l_4 - 2)^2 \cdot (l_2 + 2)^2 \cdot (l_2 - 2)^2 \quad (14)$$
$$q_3(l_2, l_4) = l_4^2 \cdot (l_2 + l_4 - 2)^2 \cdot (l_2 + 2), \quad (15)$$
$$q_4(l_2, l_4) = l_2^2 \cdot (l_2 + 2) \cdot (l_2 - 2). \quad (16)$$

Considering our assumptions

$$l_4, l_2 > 0, \quad l_2 + l_4 - 2 = l_3 > 0, \quad l_2 \neq 2,\ l_3 \neq 2$$

and assuming additionally $l_2 \neq \frac{8}{3}$ (we can check the case $l_2 = \frac{8}{3}$ analogously), we recognize that none of the q_i vanishes. Hence, $\{s_1, s_2, s_3, s_4\}$ is a Gröbner basis of S for all valid parameters l_2, l_4. Then, clearly $\dim J = 0$ and with the following theorem, we deduce that I must be a radical ideal. This solves (P2).

Theorem 1. *Let $Y \subset \mathbb{C}^n$ be the complex zero set of $f_1, \ldots, f_c \in \mathbb{R}[x_1, \ldots, x_n]$, with $\dim Y = n - c$, and let Z be the complex zero set of all c-minors of the Jacobian $D := [\partial_j f_i]_{ij}$. If $\dim Y \cap Z < \dim Y$, e.g. the codimension of Z in Y is ≥ 1, then Y is reduced, i.e. $\langle f_1, \ldots, f_c \rangle$ is a radical ideal.*

Proof: This is almost the statement of [14, Theorem 18.15] because $R = \mathbb{R}[x_1, \ldots, x_n]/\langle f_1, \ldots, f_c \rangle$ is a complete intersection ring and Cohen-Macaulay in particular. We only need to be careful to check that J has height ≥ 1 in R, where J is the ideal generated by the c-Minors of D in R. This follows from the fact that the codimension of Y in X is ≥ 1, which means $\mathrm{ht}\, J + I > \mathrm{ht}\, I$ and from the fact that R is equidimensional, which means all associated primes of I have the same height. ∎

We set $p = (l_2, 0, l_2 + l_4, 0) \in \mathbb{R}^4$. Because p is a zero of (s_1, s_2, s_3, s_4), which can be checked by substitution, p is the only singularity of X.

C. Manifold Points

Finally, we can verify that p is a non-manifold point of X, see Definition 1. We use Theorem 2 and Theorem 3 of Appendix A. Thus, we need to show, that I_p' from (36) is a real ideal.

Since $\dim X = 1$ we can apply Theorem 4 and need to calculate the normalization of X. One way to do this is using the normalization algorithm of `Singular` [16]. However, it is difficult to check the validity of the calculations in each step of the algorithm for the considered values of l_2, l_4. It is still possible to analyze the situation for generic values of l_2, l_4 but we seek a statement for all admissible parameter values.

As a different approach, we calculate the blow up $\pi\colon \tilde{X}_{\mathbb{C}} \to X_{\mathbb{C}}$ at p. $\tilde{X}_{\mathbb{C}}$ will be smooth and must be the normalization of $X_{\mathbb{C}}$, (see also the discussion after Theorem 4).

We begin by moving p to the origin and obtaining $I_p = \langle p'_1, p'_2, p'_3\rangle$, with

$$\begin{aligned} p'_1 &= p_1(x + l_2, y, u + l_2 + l_4, v) \\ &= x^2 + y^2 + 2l_2 x, \end{aligned} \tag{17}$$

$$\begin{aligned} p'_2 &= p_2(\ldots) \\ &= u^2 + v^2 + (2l_2 + 2l_4 - 4)u, \end{aligned} \tag{18}$$

$$\begin{aligned} p'_3 &= p_3(\ldots) \\ &= x^2 + y^2 - 2xu + u^2 - 2yv + v^2 - 2l_4 x + 2l_4 u. \end{aligned} \tag{19}$$

Now, we perform the blow up at the origin with the command `blowUp` in `Singular`. Similar to the description in subsection B, for every step of the algorithm, we check whether the Gröbner basis computations stay valid for the considered values of l_2, l_4 (see Listing 3 with a manual computation of the blow up).

After executing `blowUp`, we obtain two relevant charts. On the first chart, we have the zero set $\tilde{X}_1$ of 9 polynomials $f_1, \ldots, f_9 \in \mathbb{R}[\hat{x}, y, \hat{u}, \hat{v}] =: S$ and the blow up map $\varphi\colon R \to S/\langle f_1, \ldots, f_9\rangle$ defined by $\varphi(x) = \hat{x}\, y$, $\varphi(y) = y$, $\varphi(u) = \hat{u}\, y$, $\varphi(v) = \hat{v}\, y$. If we calculate a pseudo Gröbner basis of the ideal K generated by $\langle f_1, \ldots, f_9\rangle$ and all 3-minors of the Jacobian of $(f_1, \ldots, f_9)$, we obtain only one constant $c(l_2, l_4)$. By factorizing, we can easily see that c will not vanish for all examined values of l_2, l_4; thus, the zero set of K is empty, and $\tilde{X}_1$ is consequently nonsingular.

Now, we set $J := \langle f_1, \ldots, f_9\rangle$. To apply Theorem 4, we need to confirm that all maximal ideals lying over $\varphi(\mathfrak{m}) = \langle y\rangle$ are real, where $\mathfrak{m} = \langle x, y, u, v\rangle$. Thus, we calculate a pseudo Gröbner basis G of $J + \langle y\rangle$. After checking that the leading coefficients of G will not vanish for all investigated values of l_2, l_4, we can make some elements of G monic and obtain the following Gröbner basis of $J + \langle y\rangle$:

$$g_1 = (2\,l_2)\,\hat{x}, \tag{20}$$

$$g_2 = (l_2 - 2)\,\hat{u} + (-l_2 + 2)\,\hat{x}, \tag{21}$$

$$g_3 = y, \tag{22}$$

$$\begin{aligned} g_4 &= \hat{v}^2 + \frac{-2l_2 - 2l_4 + 4}{l_2 - 2}\,\hat{v} + \hat{u}^2 + \frac{-2l_2 - 2l_4 + 4}{l_2 - 2}\,\hat{u}\hat{x} \\ &+ \frac{l_2^2 + 2l_2 l_4 - 2l_2 + l_4^2 - 2l_4}{l_2^2 - 2l_2}\hat{x}^2 + \frac{l_2^2 + 2l_2 l_4 - 2l_2 + l_4^2 - 2l_4}{l_2^2 - 2l_2}. \end{aligned} \tag{23}$$

Next, we can substitute $\hat{x} = 0$ and $\hat{u} = 0$ from g_1, g_2 into g_4 and multiply with $(l_2^2 - 2\,l_2)$. We then have the following:

$$\begin{aligned} &g'(\hat{v}) = (l_2^2 - 2\,l_2)\,\hat{v}^2 \\ &- l_2\,(2\,l_2 - 2\,l_4 + 4)\hat{v} + (l_2^2 + 2l_2 l_4 - 2l_2 + l_4^2 - 2l_4). \end{aligned} \tag{24}$$

Here, g' is a quadratic equation in $\hat{v}$ with the discriminant

$$8\,l_2\,l_4\,(l_2 + l_4 - 2) = 8\,l_2\,l_3\,l_4 > 0. \tag{25}$$

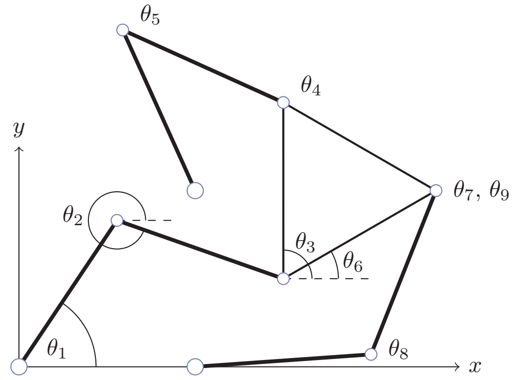

Fig. 3 3RRR-mechanism

Consequently, both maximal ideals lying over $\varphi(\mathfrak{m})$ are real because they belong to real points. It can be analogously confirmed that, in the other chart $\tilde{X}_2$ there are no points lying over the origin (see Listing 3). Therefore, all points lying over the origin are real. Thus, I'_p is real according to Theorem 4, and it follows that $(l_2, 0, l_2 + l_4, 0)$ is a non-manifold point of X.

III. The Planar 3RRR-Mechanism

Another mechanism analyzed in [6] is the 3RRR-linkage, shown in Fig. 3, with parameters as indicated Listing 4 from Appendix B (see also [17] for a beautiful genericity result for the 3RRR-linkage). Because the configuration space X will be three-dimensional, we cannot apply Theorem 4. However, we can use Theorem 5. We want to demonstrate this on the 3RRR-linkage. The same argument can be adapted to many other linkages.

Similar to [6], we use the notations $\cos(\theta_i) = c_i$, $\sin(\theta_i) = s_i$. Then, X is the set of real zeros of the ideal $I = \langle p_1, \ldots, p_{15}\rangle \le \mathbb{R}[\{c_i, s_i \mid i = 1, \ldots, 9\}]$, where

$$p_1 = c_1 + c_2 + c_3 + c_4 + c_5 - 1, \tag{26}$$

$$p_2 = s_1 + s_2 + s_3 + s_4 + s_5 - 1, \tag{27}$$

$$p_3 = c_1 + c_2 + c_6 + c_7 + c_8 - 1, \tag{28}$$

$$p_4 = s_1 + s_2 + s_6 + s_7 + s_8, \tag{29}$$

$$p_5 = c_6 + c_9 - c_3, \tag{30}$$

$$p_6 = s_6 + s_9 - s_3, \tag{31}$$

$$p_{6+i} = c_i^2 + s_i^2 - 1, \quad i = 1, \ldots, 9. \tag{32}$$

We can confirm with `Singular`, that $\dim I = 3$ (Listing 4). On the other hand, the ideal S generated by I and the 15-minors of the Jacobian of $(p_1, \ldots, p_{15})$ is zero-dimensional. Unfortunately, the calculation of a Gröbner base of S is unfeasible. However, as shown in [6], we can verify $\dim S = 0$ by analyzing $\dim I + J_k$, (see Listing 4). Here, $J_1, \ldots, J_r$ denote the ideals given by the factorizing Gröbner base algorithm (`facstd(J)` in `Singular`), applied to the ideal J generated by the 15-minors of the Jacobian. Then, $\dim S \le \max_k \dim I + J_k$.

Furthermore, we know that the coordinate ring $R = \mathbb{R}[\{c_i, s_i\}]/I$ is a Cohen-Macaulay ring because $\operatorname{ht} I = 15$ and I is generated by 15-elements. It follows that

(i) I is equidimensional according to the Unmixedness Theorem and is radical [14, Corollary. 18.14, Theorem 18.15].
(ii) The singular locus of $X_{\mathbb{C}}$ is zero-dimensional. This is a consequence of $\dim S = 0$, (i) and the general Jacobian criterion [12, Thm. 5.7.1].
(iii) R is a normal ring, [14, Theorem 18.15].
(iv) All components of $X_{\mathbb{C}}$ are disjoint. This follows from Hartshorne's Connectedness Theorem [14, Theorem 18.13], which states that the singular intersection of components will have $\operatorname{codim} \leq 1$ in $X_{\mathbb{C}}$. However, this is a contradiction to (ii).

Now, we conclude with (iii) and (iv): At any point of X, the local ring is the local ring of a normal and irreducible $\mathbb{R}$-variety. Thus, we can apply Theorem 5. We see that I'_p from (36) is real for any point p of X iff p is not isolated in the set of nonsingular points of X. However, the singular locus is zero-dimensional by (ii). Consequently, a point is isolated in the set of nonsingular points of X if and only if it is isolated in X.

We can thus see that any singularity p of X is either isolated in X or is a non-manifold point. Since X is given as the configuration space of a linkage, we can usually check that $p \in X$ is not isolated by analyzing the linkage in the configuration corresponding to p.

For example, consider the following singular configuration p of the investigated 3RRR-linkage. One of the legs can rotate freely; thus, p is not isolated in X, and the configuration space will not be a manifold at this configuration:

Variable	Value
(c_1, s_1)	$\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
(c_2, s_2)	$(1, 0)$
(c_3, s_3)	$\left(-\frac{3}{2}, -\frac{1}{2}\right)$
(c_4, s_4)	$(0, 1)$
(c_5, s_5)	$(0, 1)$
(c_6, s_6)	$\left(-\frac{3}{2}, \frac{1}{2}\right)$
(c_7, s_7)	$(0, 1)$
(c_8, s_8)	$(0, -1)$
(c_9, s_9)	$(0, -1)$

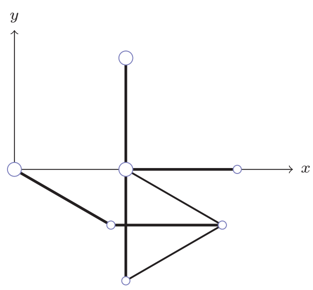

Fig. 4 Singular configuration of the 3RRR-mechanism

In summary, we can state that every singularity of the considered 3RRR-linkage is a CS-singularity iff it is not an isolated configuration. Of course, it is unclear without a further analysis whether there are any configurations isolated in X.

IV. Conclusion

We presented some effective algebraic criteria for deciding whether a point of the configuration space is a CS-singularity and used them analyzing the class of all four bars and one special 3RRR-manipulator. Clearly, all calculations conducted are subject to the limitations of Grbner base methods. However, all dimensional arguments can also be made using numerical methods (`bertini`, etc.) because we always considered ideal dimensions, which is simply the dimension of the corresponding complex zero set.

Appendix A
Local Real Algebraic Geometry

In the following, $I = \langle f_1, \ldots, f_c \rangle$ is an ideal generated by the polynomials $f_i \in \mathbb{R}[\bar{x}]$, where $\bar{x} = (x_1, \ldots, x_n)$. We write X, $(X_{\mathbb{C}})$ for the set of real (complex) zeros of the polynomials in I. In addition, let $\mathbb{R}\{\bar{x}\}$ be the ring of convergent power series with real coefficients. We want to identify non-manifold points in X; thus, we need to make this notion precise:

Definition 1. Let $X \subset \mathbb{R}^n$. A point $p \in X$ is called a **manifold point** of X if there exists an open neighborhood U of p and $U \cap X$ is an embedded C^∞-submanifold of $\mathbb{R}^n$. Equivalently, X is locally the graph of a C^∞-mapping in some of the Cartesian coordinates. "X can be smoothly parameterized locally".

See [18] for our notations regarding manifolds and smooth mappings. To address problem (P3) of the introduction, we first analyze the curve X of Fig. 1, which is the zero set of $f(x,y) = y^3 + 2\,x^2\,y - x^4$. Using the quadratic formula, we can decompose f in the ring of analytic function germs (convergent power series) at $(0,0)$:

$$\begin{aligned} f(x,y) &= y^3 + 2\,x^2\,y - x^4 \\ &= \left(x^2 - y\left(1 + \sqrt{1+y}\right)\right)\left(x^2 - y\left(1 - \sqrt{1+y}\right)\right) \end{aligned} \tag{33}$$

Now, $y\left(1 - \sqrt{1+y}\right)$ is negative for values of y close to 0, hence, the zero set of $g(x,y) = x^2 - y\,\left(1 + \sqrt{1+y}\right)$ coincides with the zero set of f in a neighborhood of (0,0) (in fact, everywhere) and $(\partial_x g, \partial_y g) \neq (0,0)$ at the origin. Thus, X is a smooth manifold at $(0,0)$. Note that this does not occur over the complexes because $x^2 - y\left(1 - \sqrt{1+y}\right)$ has nonempty zero set over $\mathbb{C}$. The difficulty over the real numbers arises because parts of analytic branches (or analytic components for higher dimensional sets) can be "invisible" in real space.

Fortunately, to show to show that a singular point of a real algebraic set is a non-manifold point, it is sufficient to check whether the analytic vanishing ideal is generated by the ideal of the algebraic set:

Theorem 2. *Let the origin be a singularity of X, where X is the zero set of $f_1, \ldots, f_c$. If*

$$\begin{aligned} I' &:= \langle f_1, \ldots, f_c \rangle \cdot \mathbb{R}\{\bar{x}\} \\ &= \left\{ g \in \mathbb{R}\{\bar{x}\} \;\middle|\; \begin{array}{l} \exists\, U \ni 0 \text{ neighborhood s.t., } g \text{ def-} \\ \text{ined on } U \text{ and } g \equiv 0 \text{ on } X_{\mathbb{R}} \cap U \end{array} \right\} \end{aligned} \tag{34}$$

the origin is a non-manifold point of X.

Proof: Suppose the origin is a manifold point of X. Any smooth function parameterizing X is analytic because it is a Nash [19, p. 55] function. Now, using the Weierstrass division theorem and Nagata's Jacobian criterion [20], we can show that $\mathbb{R}\{\bar{x}\}/I'$ is a regular local ring. Then, $\mathbb{R}[\bar{x}]/I$ localized at $\langle x \rangle$ is regular and the origin is thus nonsingular, which is a contradiction. ■

As one of the great achievements of real algebraic geometry a sufficient condition for (34) was found the form of the real analytic Nullstellensatz.

Theorem 3 (R. Risler [21])**.** *Equation* (34) *is true iff I' is* ***real****, i.e.:*

$$g_1^2 + \ldots + g_k^2 \in I' \Rightarrow g_i \in I', \text{ for all } i, \tag{35}$$

for any $k \geq 1$, $g_1, \ldots, g_k \in \mathbb{R}\{\bar{x}\}$.

Using Theorem 2 and Theorem 3, we can show that a singular point p of X is a non-manifold point. The "only" thing we need to confirm is that I'_p is real, where I_p is the translated ideal I such that p will be the origin, i.e.,

$$\begin{aligned} I_p &= \{f(x+p) \mid f \in I\}, \\ I'_p &= I_p \cdot \mathbb{R}\{\bar{x}\}. \end{aligned} \tag{36}$$

The question of when I'_p is real was studied by G. Efroymson [22]. Using his results, we can prove the following result:

Theorem 4. *Let I be a radical ideal with $\dim I = 1$. Moreover, let $R = \mathbb{R}[\bar{x}]/I$ be the coordinate ring of X and let $\varphi\colon R \to \overline{R}$ be the normalization of R. Then, I' is real iff every maximal ideal lying over $\varphi(\langle \bar{x} \rangle)$ is real.*

Proof: Because $\dim R = 1$, every localization of $\overline{R}$ at prime ideals is regular. Thus, the statement of the theorem follows from the characterization of local reality in [22] and a result such as [20, Prop. VI.4.4]. Note that in [22] the ring of formal power series is considered and not the ring of formal power series. However, the extension of I to the ring of formal power series is real iff I' is real [20, Prop. V.4.9]. ■

The criterion of Theorem 4 is quite technical but effective because there are several algorithm to compute the normalization of R. See Listing 1 in Appendix B for a demonstration.

If $\dim R = 1$, then $\overline{R}$ is the coordinate ring of a smooth complex algebraic set $Y_{\mathbb{C}}$. Hence, there is a more intuitive way to think about Theorem 4. The regular map $\pi\colon Y_{\mathbb{C}} \to X_{\mathbb{C}}$ induced by $\varphi\colon R \to \overline{R}$ is a desingularization of $X_{\mathbb{C}}$. Then, I' is real iff all points in the fiber $\pi^{-1}(0)$ are real.

In general, it is difficult to analyze I' using computational algebra since we cannot symbolically encode any infinite power series. This is a problem if $\dim X > 1$. However, we can employ the following criterion proven by G. Efroymson to check if I' is real:

Theorem 5 (Efroymson [22])**.** *Let I be a real prime and $\mathbb{R}[\bar{x}]/I$ localized at $\langle \bar{x} \rangle$ be integrally closed. Then, I' is real iff the origin is contained in the Euclidean closure of the nonsingular points of X.*

In Section III an application of Theorem 5 is shown. This criterion is ideally suited for algebraic sets arising as configuration spaces of linkages because we can apply some geometric reasoning.

We want to conclude with a word of caution: There are singularities of algebraic sets that are non-manifold points but can still be C^k, for a finite k. Consider for example the zero set X of $y^3 - x^4$. The origin is a non-manifold point of X. Nevertheless, X is C^1 at that point (but not C^2).

Appendix B
Singular Code

Our first listing investigates the example described in the introduction:

Listing 1 smooth curve

```
ring r=0,(x,y)
ideal I = y^3 + 2*y*x^2 - x^4;
def nor = normal(I);
def S = nor[1][1];
setring S;
ideal M = norid + ideal(x,y);
primdecGTZ(M);
```

which produces the output:

```
[1]:
[1]:
_[1]=T(2)
_[2]=y
_[3]=x
_[4]=-T(2)^2+T(1)-2
[2]: -- same
[2]:
[1]:
_[1]=T(2)^2+2
_[2]=y
_[3]=x
_[4]=-T(2)^2+T(1)-2
[2]: --same
```

We can see that there are two maximal ideals lying over $\langle x, y \rangle$ in the normalization, but only one is real. Hence, I' is not real. Compare this to the output when f is exchanged with $g(x,y) = y^2 - x^2 - x^3$.

The next listing calculates the singularities of the four-bar configuration space. See Subsection II.B. for an explanation.

Listing 2 four bar singularities

```
LIB "primdec.lib";
LIB "normal.lib";
LIB "realrad.lib";

ring r=(0,l2,l4),(v,y,u,x),(dp(2),dp(2));

poly l3 = l2 + l4 - 2;
poly p1 = x^2 + y^2 - l2^2;
poly p2 = (u - 2)^2 + v^2 - l3^2;
poly p3 = (u - x)^2 + (v - y)^2 - l4^2;

ideal I = p1,p2,p3;

matrix mv=jacob(I);
ideal S = minor(mv,3) + I;

//for calculations with parameters
option(contentSB);
option(intStrategy);

std(I); //pseudo groebner basis of I
ideal Sg = std(S);Sg;
substitute(Sg,x,l2,y,0,u,l2 + l4,v,0);
poly f1 = leadcoef(Sg[1]);
poly f2 = leadcoef(Sg[2]);
```

```
poly f3 = leadcoef(Sg[3]);
poly f4 = leadcoef(Sg[4]);

ring s=0,(l2,l4),dp;
factorize(simplify(imap(r,f1),1));
factorize(simplify(imap(r,f2),1));
factorize(simplify(imap(r,f3),1));
factorize(simplify(imap(r,f4),1));
```

Listing 3 determines the blow up at the singularity and calculates the fiber over the origin. Compare subsection II.C.

Listing 3 four bar blow up

```
LIB "primdec.lib";
LIB "resolve.lib";
LIB "poly.lib";

ring r=(0,l2,l4),(x,y,u,v),dp;
poly l3 = l2 + l4 - 2;
poly p1 = x^2 + y^2 - l2^2;
poly p2 = (u - 2)^2 + v^2 - l3^2;
poly p3 = (u - x)^2 + (v - y)^2 - l4^2;
ideal I = p1,p2,p3;
map phi = r, x + l2, y, u + l2 + l4,v;
ideal J = phi(I);

//ring with different term ordering
ring r2=(0,l2,l4),(x,u,y,v),(dp(2),dp(2));
ideal J = imap(r,J);
option(contentSB);
option(intStrategy);

list blow=blowUp(J,maxideal(1));
//first chart
def Q = blow[1];
setring Q; sT;
//sT nonsingular
std(minor(jacob(sT),3) + sT);
//calculate points in fiber
ideal fiber_points= sT,x(3);
fiber_points = std(fiber_points);
ideal fp_sub =
substitute(fiber_points,y(3),0,y(2),0,x(3),0);
poly f = fp_sub[4];
f*4;

//second chart
//first chart
def Q2 = blow[2];
setring Q2; sT;
//calculate points in fiber
ideal fiber_points= sT,x(1);
fiber_points = std(fiber_points);


//manual blow up
ring s = (0,l2,l4),(x,y,u,v,xs,ys,us,vs),dp;
ideal J = imap(r,J);
ideal Js = substitute(J,x,xs*y,u,us*y,v,vs*y);
ideal Jf = Js/y;

//saturation manually
ring s2 = (0,l2,l4),(t,vs,y,us,xs),
(dp(1),dp(2),dp(2));
ideal Jf = imap(s,Jf);
ideal H = t*Jf +ideal((1-t)*y);
H = std(H);
ideal H2 = H[1..7]; H2 = H2/y;
ideal H2s = simplify(H2,1+2);
ideal fiber_points = H2s,y;
//std(fiber_points);
```

The final listing calculates the dimensions of the configuration space and the singular locus of the 3RRR-linkage, as described in Section III. Please note, that listing 4 takes approximately two minutes to complete on a system with a 2.6 GHz CPU.

Listing 4 3RRR dimension calculation

```
ring r = 0,(c1,s1,c2,s2,c3,s3,c4,
s4,c5,s5,c6,s6,c7,s7,c8,s8,c9,s9),dp;

number l1,l2,l3,l4,l5,l6,x1,y1,x2 =
1,1,1,1,1,1,1,1,1;

poly loop1x = l1*c1 + l2*c2 + c3 + l3*c4 + l4*c5-x1;
poly loop1y = l1*s1 + l2*s2 + s3 + l3*s4 + l4*s5-y1;
poly loop2x = l1*c1 + l2*c2 + c6 + l5*c7 + l6*c8-x2;
poly loop2y = l1*s1 + l2*s2 + s6 + l5*s7 + l6*s8;

poly drx = c6 + c9 - c3;
poly dry = s6 + s9 - s3;

poly circ1 = c1^2 + s1^2 - 1;
poly circ2 = c2^2 + s2^2 - 1;
poly circ3 = c3^2 + s3^2 - 1;
poly circ4 = c4^2 + s4^2 - 1;
poly circ5 = c5^2 + s5^2 - 1;
poly circ6 = c6^2 + s6^2 - 1;
poly circ7 = c7^2 + s7^2 - 1;
poly circ8 = c8^2 + s8^2 - 1;
poly circ9 = c9^2 + s9^2 - 1;

ideal I = loop1x,loop1y,loop2x,loop2y,drx,dry;
ideal J = circ1,circ2,circ3,circ4,
circ5,circ6,circ7,circ8,circ9;
ideal K = I + J;
"dim configuration space";
dim(std(K));
matrix m = jacob(K);
ideal S = minor(m,15);
//list L = minAssGTZ(S);
list L = facstd(S);
//ideal sing = K + L[11];

for (int i=1; i<size(L); i++) {
ideal sing = K + L[i];
i; dim(std(sing)); }
```

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