Abstract—The aim of this paper is to introduce the concepts of \((\varepsilon, \varepsilon \lor q)\)-fuzzy subalgebras, \((\varepsilon, \varepsilon \lor q)\)-fuzzy ideals and \((\varepsilon, \varepsilon \lor q)\)-fuzzy quotient algebras of BCI-algebras with operators, and to investigate their basic properties.

Keywords—BCI-algebras with operators, \((\varepsilon, \varepsilon \lor q)\)-fuzzy subalgebras, \((\varepsilon, \varepsilon \lor q)\)-fuzzy ideals, \((\varepsilon, \varepsilon \lor q)\)-fuzzy quotient algebras.

I. INTRODUCTION

The fuzzy set is a generalization of the classical set and was used afterwards by several authors such as Imai [1], Iséki [2] and Xi [3], in various branches of mathematics. Particularly, in the area of fuzzy topology, after the introduction of fuzzy sets by Zadeh [15], much research has been carried out: the concept of fuzzy subalgebras and fuzzy ideals of BCK-algebras, and their some properties.


In this paper, we introduce the concepts of \((\varepsilon, \varepsilon \lor q)\)-fuzzy subalgebras, \((\varepsilon, \varepsilon \lor q)\)-fuzzy ideals and \((\varepsilon, \varepsilon \lor q)\)-fuzzy quotient algebras of BCI-algebras with operators. Moreover, the basic properties were discussed and several results have been obtained.

II. PRELIMINARIES

Some definitions and propositions were recalled which may be needed.

An algebra \(\{X, *, 0\}\) of type \((2, 0)\) is called a BCK-algebra, if for all \(x, y, z \in X\), it satisfies:

1. \((x * y) * (x * z) = (z * x) * y = z * (x * y) = (y * z) * x\),
2. \(x * (x * y) = y = (x * y) * x\),
3. \(x * (x * x) = x\),
4. \(x \leq y \) and \(y \leq x \) imply \(x = y\).

A BCI-algebra is called a BCK-algebra if it satisfies

\[ A(x * y) \supseteq A(x) \land A(y), \forall x, y \in X. \]

Definition 1. [5] \(\{X, *, 0\}\) is a BCI-algebra, a fuzzy subset \(A\) of \(X\) is called a fuzzy ideal of \(X\) if it satisfies:

1. \(A(0) \supseteq A(x), \forall x \in X\),
2. \(A(x) \supseteq A(x * y) \land A(y), \forall x, y \in X\).

Definition 2. [4] \(\{X, *, 0\}\) is a BCI-algebra, a fuzzy subset \(A\) of \(X\) is called a fuzzy subalgebra of \(X\) if it satisfies:

\[ A(x * y) \supseteq A(x) \land A(y), \forall x, y \in X. \]

Definition 3. [12] \(\{X, *, 0\}\) is a BCI-algebra, a fuzzy subset \(A\) of \(X\) of the form

\[ A(y) = \begin{cases} t(\neq 0), & y = x, \\ 0, & y \neq x, \end{cases} \]

is said to be a fuzzy point with support \(x\) and value \(t\), and is
denoted by \( x \).

**Definition 4.** [12] If \( x \) is a fuzzy point, it is said to belong to (resp. be quasi-coincident with) a fuzzy subset \( A \), written as \( x \in A \) (resp. \( x \in_q A \)) if \( A(x) \geq 1 \) (resp. \( A(x) + t > 1 \)). If \( x \in A \) or \( x \in_q A \), then we write \( x \in \mathcal{V}qA \). The symbol \( \in \mathcal{V}q \) (resp. \( \in \mathcal{V}q \)) means \( \in \mathcal{V} \) (resp. \( \in \mathcal{V}q \)) does not hold.

**Definition 5.** [10] \( \langle X, \ast, 0 \rangle \) is a BCI-algebra, a fuzzy set \( A \) of \( X \) is called an \( (e, e \in \mathcal{V}q) \)-fuzzy ideal of \( X \) if for all \( t, r \in (0, 1) \) and \( x, y \in X \), it satisfies:
1. \( x \ast A \Rightarrow 0, \in \mathcal{V}qA \)
2. \( (x \ast y) \ast A \Rightarrow \in \mathcal{V}qA \)

**Definition 6.** [10] A fuzzy set \( A \) is an \( (e, e \in \mathcal{V}q) \)-fuzzy ideal of \( X \) if and only if it satisfies:
1. \( A(0) \supseteq A(x) \ast 0.5, \forall x \in X \)
2. \( A(x) \supseteq A(x \ast y) \wedge A(y) \ast 0.5, \forall x, y \in X \).

**Definition 7.** [7] \( \langle X, \ast, 0 \rangle \) is a BCI-algebra, \( M \) is a non-empty set, if there exists a mapping \( m: X \times X \rightarrow M \times X \) which satisfies
\[
m(x \ast y) = (mx) \ast (my), \forall x, y \in X, m \in M,
\]
then \( M \) is called a left operator of \( X \), and \( X \) is called BCI-algebra with left operator \( M \), or \( M \) is BCI-algebra for short.

**Proposition 1.** [6] Let \( \langle X, \ast, 0 \rangle \) be a BCI-algebra, if \( A \) is an \( (e, e \in \mathcal{V}q) \)-fuzzy ideal of \( X \), and \( x \ast y \leq z \), then
\[
A(x) \supseteq A(y) \wedge A(z) \ast 0.5, \forall x, y, z \in X.
\]

**Definition 8.** [13] Let \( A \) and \( B \) be fuzzy sets of set \( X \), then the direct product \( A \times B \) of \( A \) and \( B \) is a fuzzy subset of \( X \times X \), define \( A \times B \) by
\[
A \times B(x, y) = A(x) \wedge B(y), \forall x, y \in X.
\]

**Definition 9.** [7] Let \( \langle X, \ast, 0 \rangle \) and \( \langle X, \ast', 0 \rangle \) be two \( M \)-BCI-algebras, if for all \( x \in X, m \in M \), \( f(mx) = mf(x) \), and \( f \) is a homomorphism from \( \langle X, \ast, 0 \rangle \) to \( \langle X, \ast', 0 \rangle \), then \( f \) is called a homomorphism with operators.

**Definition 10.** [13] \( \langle X, \ast, 0 \rangle \) is an \( M \)-BCI-algebra, let \( B \) be a fuzzy set of \( X \), and \( A \) be a fuzzy relation of \( B \), if it satisfies:
\[
A_{x}(x, y) = B(x) \wedge B(y), \forall x, y \in X,
\]
then \( A \) is called a strong fuzzy relation of \( B \).

**Definition 11.** [14] If \( \langle X, \ast, 0 \rangle \) is an \( M \)-BCI-algebra, \( A \) is a non-empty subset of \( X \), and \( mx \in A \) for all \( x \in A, m \in M \), then \( A(x, 0) \) is called a \( M \)-subalgebra of \( \langle X, \ast, 0 \rangle \).

In this paper, \( X \) always means a \( M \)-BCI-algebra unless otherwise specified.

### III. \( (e, e \in \mathcal{V}q) \)-Fuzzy Subalgebras of BCI-Algebras with Operators

**Definition 12.** \( \langle X, \ast, 0 \rangle \) is a BCI-algebra, a fuzzy set \( A \) of \( X \) is called a \( M - (e, e \in \mathcal{V}q) \)-fuzzy subalgebra of \( X \) if for all \( t, r \in (0, 1) \) and \( x, y \in X \), it satisfies:
1. \( x \ast A \Rightarrow 0, \in \mathcal{V}qA \)
2. \( x \ast A \Rightarrow (mx) \in \mathcal{V}qA \)

**Proposition 2.** \( \langle X, \ast, 0 \rangle \) is a BCI-algebra, a fuzzy set \( A \) of \( X \) is an \( M - (e, e \in \mathcal{V}q) \)-fuzzy subalgebra of \( X \) if and only if it satisfies:
1. \( A(x \ast y) \supseteq A(x) \wedge A(y) \ast 0.5, \forall x, y \in X \)
2. \( A(mx) \supseteq A(x) \ast 0.5, \forall x \in X \).

**Proof.** Suppose that \( A \) is an \( M - (e, e \in \mathcal{V}q) \)-fuzzy subalgebra of \( X \). (1) Let \( x, y \in X \), suppose that \( A(x) \wedge A(y) \ast 0.5 \), then \( A(x \ast y) \supseteq A(x) \wedge A(y) \ast 0.5 \). If not, then we have \( A(x \ast y) < A(x) \wedge A(y) \ast 0.5, \forall x, y \in X \), which is a contradiction, then whenever \( A(x) \wedge A(y) \ast 0.5 \). We have \( A(x \ast y) \supseteq A(x) \wedge A(y) \ast 0.5 \), then \( (x \ast y)_{t, r} \in A \), which implies that \( A(x \ast y)_{t, r} \subseteq A(x) \wedge A(y) \ast 0.5 \), because if \( A(x \ast y) < 0.5 \), then \( A(x \ast y) + 0.5 > 0.5 \), which is a contradiction, hence
\[
A(x \ast y) \supseteq A(x) \wedge A(y) \ast 0.5, \forall x, y \in X.
\]

(2) Let \( x \in X \) and assume that \( A(x) < 0.5 \). If \( A(mx) < A(x) \), then we have \( A(mx) + t < A(x) \), \( \forall t \in (0, 0.5) \), and we have \( x \in A \) and \( (mx) \subseteq A \), since \( A(mx) + t < A(x) \). If \( A(mx) \subseteq A(x) \), then \( (mx) \subseteq \mathcal{V}qA \), which is a contradiction, hence \( A(mx) \subseteq A(x) \).

Now if \( A(x) \supseteq 0.5 \), then \( x_{0.5} \in A \), thus \( (mx)_{0.5} \in \mathcal{V}qA \), hence \( A(mx) \supseteq 0.5 \), otherwise \( A(mx) + t > 0.5 \), which is a contradiction, consequently, \( A(mx) \supseteq A(x) \). Conversely, assume that \( A \) satisfies condition (1), (2).

(1) Let \( x, y \in X \) and \( t, r \in (0, 0.5) \) be such that \( x \in A \) and \( y \in A \), then \( A(x) \supseteq t_{1} \) and \( A(y) \supseteq t_{2} \). Suppose that \( A(x \ast y) < t_{1} \), if \( A(x) \wedge A(y) < 0.5 \), then \( A(x \ast y) \supseteq A(x) \wedge A(y) \ast 0.5 \), which is a contradiction, so we have \( A(x) \wedge A(y) \supseteq 0.5 \), it follows that...
$A(x \ast y) + t_1 \land t_2 \geq 2.4(x \ast y) \geq 2(A(x) \land A(y) \land 0.5) = 1,$

so that $(x \ast y)_{t \geq 0} \in \mathbb{Q}A$.

(2) Let $x \in X$ and $t \in (0,1]$ be such that $x, y \in A$, then we have $A(x) \geq t$. Suppose that $A(x) < 0.5$, then $A(x) = A(x) \geq 0.5 \geq A(x) \geq t$, this is a contradiction, hence we know that $A(x) \geq 0.5$, and we have

$$A(mx) + t > 2A(mx) > 2(A(x) \land 0.5) = 1,$$

then $(mx) \in \mathbb{Q}A$. Consequently, $A$ is an $M - (\epsilon, \infty)$-fuzzy subalgebra.

**Example 1.** If $A$ is an $M - (\epsilon, \infty)$-fuzzy subalgebra of $X$, then $X_{\lambda}$ is an $M - (\epsilon, \infty)$-fuzzy subalgebra of $X$, define $X_{\lambda}$ by

$$X_{\lambda}: X \rightarrow [0,1], X_{\lambda}(x) = \begin{cases} 1, & x \in A \\ 0, & x \not\in A. \end{cases}$$

**Proof.** (1) For all $x, y \in X$, if $x, y \in A$, then $x \ast y \in A$, then we have

$$X_{\lambda}(x \ast y) = 1 \geq X_{\lambda}(x) \land X_{\lambda}(y) \land 0.5,$$

if there exists at least one which does not belong to $A$ between $x$ and $y$, for example $x \not\in A$, thus

$$X_{\lambda}(x \ast y) \geq 0 = X_{\lambda}(x) \land X_{\lambda}(y) \land 0.5.$$

(2) For all $x, y, m \in M$, if $x \in A$, then $mx \in A$, therefore $X_{\lambda}$ by

$$X_{\lambda}(mx) = 1 \geq X_{\lambda}(x) \land 0.5,$$

if $x \not\in A$, then $X_{\lambda}(mx) \geq 0 = X_{\lambda}(x) \land 0.5$, therefore $X_{\lambda}$ is an $M - (\epsilon, \infty)$-fuzzy subalgebra of $X$.

**Proposition 3.** $A$ is an $M - (\epsilon, \infty)$-fuzzy subalgebra of $X$ if and only if $A(t)$ is an $M - \epsilon$-subalgebra of $X$, where $A(t)$ is a non-empty set, define $X_{\lambda}$ by

$$A = \{ x \in X, \ A(x) \geq t \}, \forall t \in [0,0.5].$$

**Proof.** Suppose $A$ is an $M - (\epsilon, \infty)$-fuzzy subalgebra of $X$, $A(t)$ is a non-empty set, $t \in [0,0.5]$, then we have $A(x \ast y) \geq A(x) \land A(y) \land 0.5$. If $x \in A(t)$, $y \in A(t)$, then $A(x) \geq t$, $A(y) \geq t$, thus

$$A(x \ast y) \geq A(x) \land A(y) \land 0.5 \geq t,$$

then we have $x \ast y \in A$. If $A$ is an $M - (\epsilon, \infty)$-fuzzy subalgebra of $X$, then $A(mx) \geq A(x) \land 0.5 \geq t, \forall x \in X, m \in M$, then we have $mx \in A$. Therefore $A(t)$ is an $M - \epsilon$-subalgebra of $X$.

Conversely, suppose $A$ is an $M - \epsilon$-subalgebra of $X$, then we have $x \ast y \in A$. Let $A(x) = t$, then

$$A(x \ast y) \geq t = A(x) \land A(y) \land 0.5.$$

If $A(t)$ is an $M - \epsilon$-subalgebra of $X$, then we have

$$A(mx) \geq t = A(x) \land A(y) \land 0.5, \forall x \in X, m \in M,$$

therefore $A$ is an $M - (\epsilon, \infty)$-fuzzy subalgebra of $X$.

**Proposition 4.** Suppose $X, Y$ are $M - \text{BCI}$-algebras, $f$ is a mapping from $X$ to $Y$, if $A$ is an $M - (\epsilon, \infty)$-fuzzy subalgebra of the $Y$, then $f^{-1}(A)$ is a $M - (\epsilon, \infty)$-fuzzy subalgebra of $X$.

**Proof.** Let $y \in Y$, suppose $f$ is an epimorphism, and we have $y = f(x), \exists x \in X$. If $A$ is an $M - (\epsilon, \infty)$-fuzzy subalgebra of $Y$, then we have

$$A(x \ast y) \geq A(x) \land A(y) \land 0.5, A(mx) \geq A(x) \land 0.5.$$

For all $x, y \in X, m \in M$, we have

1. $f^{-1}(A)((x \ast y)) = A(f(x) \ast f(y)) \geq A(f(x)) \land A(f(y)) \land 0.5 = f^{-1}(A)(x) \land f^{-1}(A)(y) \land 0.5$;
2. $f^{-1}(A)(mx) = A(f(mx)) = A(mx) \land 0.5$.

Then $f^{-1}(A)$ is an $M - (\epsilon, \infty)$-fuzzy subalgebra of $X$.

**IV. $(\epsilon, \infty)$ - Fuzzy Ideals of BCI-Algebras with Operators**

**Definition 13.** $(X, \ast, 0)$ is a BCI-algebra, a fuzzy set $A$ of $X$ is called an $M - (\epsilon, \infty)$-fuzzy ideal of $X$ if for all $t, r \in [0,1]$ and $x, y \in X$, it satisfies:

1. $x \in A \Rightarrow 0 \in \mathbb{Q}A$,
2. $(x \ast y) \in A$ and $y \in A \Rightarrow x_{t, r} \in \mathbb{Q}A$,
3. $x \in A \Rightarrow (mx) \in \mathbb{Q}A$.

**Proposition 5.** $(X, \ast, 0)$ is a BCI-algebra, a fuzzy set $A$ is an $M - (\epsilon, \infty)$-fuzzy ideal of $X$ if and only if it satisfies:

1. $A(0) \geq A(x) \land 0.5, \forall x \in X$,
2. $A(x) \geq A(x \ast y) \land 0.5, \forall x, y \in X$,
3. $A(mx) \geq A(x) \land 0.5, \forall x \in X$.

**Proof.** Suppose that $A$ is an $M - (\epsilon, \infty)$-fuzzy ideal of $X$.
(1) Let \( x \in X \) and assume that \( A(x) < 0.5 \). If \( A(0) < A(x) \), then we have \( A(0) < A(x), \exists t \in (0,0.5) \), and we have \( x \in A \) and \( A(0) + t < 1 \), since \( A(0) + t < 1 \), we have \( 0 < qA \), it follows that \( \forall qA \), which is a contradiction, then \( A(0) > A(x) \). Now if \( A(0) > 0.5 \), then \( x_0 \in A \), then we have \( 0 \notin qA \), hence \( A(0) > 0.5 \), otherwise, \( A(0) > 0.5 < 0.5 + 0.5 = 1 \), which is a contradiction, consequently, we have \( A(0) > A(x) \land 0.5 \), \( \forall x \in X \).

(2) Let \( (x,y) \in X \) and suppose that \( A(x,y) > A(y) < 0.5 \), then \( A(x) > A(x,y) \land A(y) \), if not, then we have \( A(x) > A(x,y) \land A(y), \exists t \in (0,0.5) \), it follows that \( (x,y) \in A \) and \( y \in A \), but \( x_\infty = x \in qA \), which is a contradiction, hence whenever \( A(x,y) > A(y) < 0.5 \), we have \( A(x) > A(x,y) \land A(y) \). If \( A(x+y) > A(y) \), then \( (x+y) \in A \) and \( y \in A \), which implies that \( x_\infty = x_\infty y _0 < 0 \), therefore \( (x,y) \not\in A \), because if \( A(x) > 0.5 \), then \( A(x) > 0.5 < 0.5 + 0.5 = 1 \), which is a contradiction, then \( A(x) > A(x,y) \land A(y) > 0.5 \), \( \forall x,y \in X \).

(3) Let \( x \in X \) and \( t \in (0,1) \) be such that \( x \in A \), then \( x \in A \), suppose that \( A(0) < 0.5 \), then \( A(0) > A(x) \), which is a contradiction, then we know that \( A(0) > 0.5 \), and we have \( A(0) + t > 2A(0) \geq 2A(x) \), thus \( 0 \in qA \).

(2) Let \( x,y \in X \) and \( t_1,t_2 \in (0,1) \) be such that \( (x,y) \in A \) and \( y \in A \), then \( A(x,y) > t_1 \) and \( A(y) > t_2 \), suppose that \( A(x) < t_1 \land A(y) < 0.5 \), then

\[
A(x) > A(x+y) \land A(y) > 0.5 = A(x+y) \land A(y) > t_1 \land t_2,
\]

This is a contradiction, so we have \( A(x+y) \land A(y) > 0.5 \), it follows that

\[
A(x) + t_1 \land t_2 > 2A(x) \geq 2A(x) \land A(y) \land 0.5 = 1,
\]

so that \( x_\infty y_\infty \in qA \).

(3) Let \( x \in X \) and \( t \in (0,1) \) be such that \( x \in A \), then \( A(x) > t \), suppose that \( (mx) < t \), if \( A(x) > 0.5 \), then \( A(mx) > A(x) \land A(y) > 0.5 \), which is a contradiction, then we know that \( A(x) > 0.5 \), and we have \( A(mx) > A(x) \land A(y) > 0.5 = 1 \), thus \( (mx) \in qA \).

Consequently, \( A \) is an \((e,e_\infty)\)-fuzzy ideal.

Example 2. If \( A \) is an \((e,e_\infty)\)-fuzzy ideal of \( X \), then \( X_\infty \) is an \((e,e_\infty)\)-fuzzy ideal of \( X \). Define \( X_\infty \) by

\[
X_\infty : X \to [0,1], X_\infty (x) = \begin{cases} 1, x \in A & \text{if } x \in A \\ 0, x \notin A & \text{if } x \notin A \end{cases}
\]

Proof. (1) For all \( x,y \in X \), if \( x,y \in A \), then \( x \land y \in A \), thus

\[
X_\infty (0) = 1 \geq X_\infty (x) \land y, 0.5 \in X_\infty (x) \land y, 0.5, X_\infty (x) \land y, 0.5 \in X_\infty (x) \land y, 0.5,
\]

if there exists at least one between \( x \) and \( y \) which does not belong to \( A \) for example \( x \notin A \), thus

\[
X_\infty (0) = 1 \geq X_\infty (x) \land y, 0.5 \in X_\infty (x) \land y, 0.5, X_\infty (x) \land y, 0.5 = 0,
\]

therefore \( X_\infty \) is a \((e,e_\infty)\)-fuzzy ideal of \( X \).

(2) For all \( x \in X \), \( m \in M \), if \( x \in A \), then \( mx \in A \), therefore \( X_\infty (mx) = 1 \geq X_\infty (x) \land y, 0.5 \). If \( x \notin A \), then \( X_\infty (mx) = 0 = X_\infty (x) \land y, 0.5 \), therefore \( X_\infty \) is an \((e,e_\infty)\)-fuzzy ideal of \( X \).

Proposition 6. \( A \) is an \((e,e_\infty)\)-fuzzy ideal of \( X \) and only if \( A \) is an \((e,e_\infty)\)-fuzzy ideal of \( X \), where \( A \) is non-empty set, define \( A \) by

\[
A = \{ x \in X, A(x) \geq t \}, \forall t \in [0,0.5].
\]

Proof. Suppose \( A \) is an \((e,e_\infty)\)-fuzzy ideal of \( X \), \( A \) is a non-empty set, \( t \in [0,0.5] \), then we have \( A(0) \geq A(x) \land 0.5 \leq t \), then we have \( 0 \in A \). If \( x \land y \in A \), then \( A(x+y) \geq t \), \( A(y) \geq t \), thus \( A(x) \geq A(x+y) \land A(y) > 0.5 \), then we have \( x \in A \). For all \( x \in X \), \( m \in M \), if \( A \) is an \((e,e_\infty)\)-fuzzy ideal of \( X \), hence \( A(mx) \geq A(x) \land 0.5 \leq t \), then \( mx \in A \), therefore \( A \) is an \((e,e_\infty)\)-fuzzy ideal of \( X \). Conversely, suppose \( A \) is
an $M$-ideal of $X$, then we have $0 \in A, A(0) \geq 0$. Let $A(x) = 1$, thus $x \in A$, we have $A(0) \geq 0 = A(x)$, suppose there is no $A(x) \geq A(x \cdot y) \wedge A(y) \geq 0.5$, then there exist $x_0, y_0 \in X$, we have $A(x_0) \leq A(x_0 \cdot y_0) \wedge A(y_0) \geq 0.5$, let $t_0 = A(x_0 \cdot y_0) \wedge A(y_0) \geq 0.5$, then $A(x_0) \leq t_0 = A(x_0 \cdot y_0) \wedge A(y_0) \geq 0.5$. If we have $x_0 \in A$, then $A(x_0) \geq t_0$, which is inconsistent with $A(x_0) \leq t_0$. Thus we have $A(x_0) \geq 0.5$. If $A$ is an $M$-ideal of $X$, then we have $A(x \cdot y) \supseteq A(x) \wedge A(y) \geq 0.5, \forall x \in X$. Therefore $A$ is an $M-(e, e, v, q)$-fuzzy ideal of $X$.

**Proposition 7.** Suppose $X, Y$ are $M$-BCI-algebras, $f$ is a mapping from $X$ to $Y$, $A$ is an $M-(e, e, v, q)$-fuzzy ideal of $Y$, then $f^{-1}(A)$ is an $M-(e, e, v, q)$-fuzzy ideal of $X$.

**Proof.** Let $y \in Y$, suppose $f$ is an epimorphism, then we have $y = f(x), \exists x \in X$. If $A$ is an $M-(e, e, v, q)$-fuzzy ideal of $Y$, then we have

$A(0) \geq A(x) \geq 0.5, A(x) \geq A(x \cdot y) \wedge A(y) \geq 0.5, A(x \cdot y) \geq A(x) \wedge A(y) \geq 0.5, \forall x \in X, m \in M$.

For all $x, y \in X, m \in M$, we have

1. $f^{-1}(A)(0) = A(f(0)) = 0 \leq 0.5 = f^{-1}(A)(x) \leq 0.5$,
2. $f^{-1}(A)(x) = 0 \leq 0.5 = f^{-1}(A)(x \cdot y) \wedge f^{-1}(A)(y) \leq 0.5$,
3. $f^{-1}(A)(x) \geq 0 \leq 0.5 = f^{-1}(A)(x \cdot y) \leq 0.5$.

Therefore $f^{-1}(A)$ is an $M-(e, e, v, q)$-fuzzy ideal of $X$.

**V. $(e, e, v, q)$-FUZZY QUOTIENT BCI-ALGEBRAS WITH OPERATORS**

**Definition 14.** Let $A$ be an $M-(e, e, v, q)$-fuzzy ideal of $X$, for all $a \in X$, fuzzy set $A_a$ on $X$ defined as: $A_a : X \rightarrow [0,1]$

$A_a(x) = A(a \cdot x) \wedge A(a \cdot x) \wedge A(a \cdot x) \geq 0.5, \forall x \in X$.

Denote $X/A = \{A_a : a \in X\}$.

**Proposition 8.** Let $A_a, A_b \in X/A$, then $A_a \supseteq A_b$, if and only if $A(a \cdot b) \wedge A(b \cdot a) \geq 0.5 = A(0) \geq 0.5$.

**Proof.** Let $A_a \supseteq A_b$, then we have $A_a(b) = A_b(b)$, thus $A(a \cdot b) \wedge A(b \cdot a) \geq 0.5 = A(a \cdot b) \wedge A(b \cdot a) \geq 0.5 = A(0) \geq 0.5$. Conversely, suppose that $A(a \cdot b) \wedge A(b \cdot a) \geq 0.5 = A(0) \geq 0.5$. For all $x \in X$, since

$(a \cdot x) \wedge (b \cdot a) \geq (a \cdot b) \leq (a \cdot b) \cdot (b \cdot a) \leq b \cdot a$.

It follows from Proposition 1 that

$A(x) \cdot A(a \cdot b) \geq A(a \cdot b) \cdot A(b \cdot a) \geq 0.5$.

Hence

$A_a(x) = A(a \cdot b) \wedge A(a \cdot b) \geq 0.5$.

It follows from Proposition 1 that

$A(a \cdot b) \geq A(a \cdot b) \cdot A(a \cdot b) \geq 0.5$.

Hence

$A_a(x) \supseteq A(x) \cdot A(a \cdot b) \geq 0.5$.

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$A_a(x) \supseteq A(a \cdot b) \cdot A(a \cdot b) \geq 0.5$.
Let \( A \) be an \( M-(\varepsilon, \varepsilon \cup \varnothing) \)-fuzzy ideal of \( X \). The operation "*" of \( R/A \) is defined as: \( \forall A_x, A_y \in R/A, A_x \ast A_y = A_{xy} \). By Proposition 8, the above operation is reasonable.

**Proposition 10.** \( A \) is an \( M-(\varepsilon, \varepsilon \cup \varnothing) \)-fuzzy ideal of \( X \), then \( R/A = \{ R/A \ast A_x \} \) is an \( M - \) BCI-algebra.

**Proof.** For all \( A_1, A_2, A_3 \in R/A \), we have

\[
\left( (A_1 \ast A_2) \ast (A_3 \ast A_4) \right) \ast (A_5 \ast A_6) = A_{(x_1 \ast x_2) \ast (x_3 \ast x_4)} = A_x;
\]

\[
(A_1 \ast (A_2 \ast A_3)) \ast A_5 = A_{(x_1 \ast (x_2 \ast x_3))} = A_x;
\]

\[
A_1 \ast A_2 = A_{xy} = A_x;
\]

if \( A_1 \ast A_2 = A_y, A_3 \ast A_4 = A_x, \) then \( A_{xy} = A_y, A_{xz} = A_x, \) it follows from Proposition 8 that \( A_{(x \ast y) \ast (x \ast y)} = A_0 \), hence \( A_{(x \ast y) \ast (x \ast y)} = A_0 \), then we have \( A_1 = A_y \).

Therefore \( R/A = \{ R/A \ast A_x \} \) is a BCI-algebra. For all \( A_x \in R/A, m \in M \), we define \( m A_x = A_{mx} \). Firstly, we verify that \( m A_x = A_{mx} \) is reasonable. If \( A_x = A_y \), then we verify \( m A_x = m A_y \), that is to verify \( A_{mx} = A_{my} \). We have

\[
A_{(mx \ast my)} \ast 0.5 = A_0 (mx \ast my) \ast 0.5 \leq A_{(x \ast y)} \ast 0.5,
\]

so we have

\[
A_{(mx \ast my)} \ast A_0 (mx \ast my) \ast 0.5 \geq A_{(x \ast y)} \ast A_0 (x \ast y) \ast 0.5 = A_0 \ast A_0 \ast 0.5,
\]

then \( A_{(mx \ast my)} \ast A_0 (mx \ast my) \ast 0.5 = A_0 \ast A_0 \ast 0.5 \), that is \( A_{mx} = A_{my} \).

In addition, for all \( m \in M, A_x, A_y \in R/A \), we have

\[
m (A_1 \ast A_2) = m A_{xy} = A_{mx} \ast A_y;
\]

\[
A_{mx} \ast A_y = m A_x \ast m A_y;
\]

Therefore \( R/A = \{ R/A \ast A_x \} \) is an \( M - \) BCI-algebra.

**Definition 15.** Let \( \mu \) be an \( M-(\varepsilon, \varepsilon \cup \varnothing) \)-fuzzy subalgebra of \( X \), and \( A \) be an \( M-(\varepsilon, \varepsilon \cup \varnothing) \)-fuzzy ideal of \( X \), we define a fuzzy set of \( X/A \) as follows:

\[
\mu(A) : X/A \rightarrow [0, 1], \mu(A) = \sup_{A \ast A} \mu(x) \ast 0.5, \forall A \in X/A.
\]

**Proposition 11.** \( \mu(A) \) is an \( M-(\varepsilon, \varepsilon \cup \varnothing) \)-fuzzy subalgebra of \( X/A \).

**Proof.** For all \( A_x, A_y \in X/A \), we have

\[
\mu(A_x) \ast \mu(A_y) = \mu(A_{xy}) = \sup_{A \ast A} \mu(x) \ast 0.5, \forall A \in X/A.
\]

\[
\mu(A_x) \ast \mu(A_y) = \mu(A_{xy}) = \sup_{A \ast A} \mu(x) \ast 0.5, \forall A \in X/A.
\]

Therefore \( \mu(A) \) is an \( M - \) BCI-algebra.
\[ A \times B(m(x, y)) \geq A \times B(x, y) \wedge 0.5, \forall (x, y) \in X \times X. \]

Therefore, \( A \times B \) is an \( M-(\varepsilon, \varepsilon) \)-fuzzy ideal of \( X \times X \).

**Proposition 13.** Suppose \( A \) and \( B \) are fuzzy sets of \( X \), if \( A \times B \) is an \( M-(\varepsilon, \varepsilon) \)-fuzzy ideal of \( X \times X \), then \( A \) or \( B \) is an \( M-(\varepsilon, \varepsilon) \)-fuzzy ideal of \( X \).

**Proof.** Suppose \( A \) and \( B \) are \( M-(\varepsilon, \varepsilon) \)-fuzzy ideals of \( X \), then for all \((x_i, x_j), (y_i, y_j) \in X \times X\), we have

\[ A \times B(x_i, x_j) \geq A \times B((x_i, x_j) \ast (y_i, y_j)) \wedge A \times B(y_i, y_j) \wedge 0.5 \]

if \( x_i = y_i = 0 \), then

\[ A \times B(0, x_j) \geq A \times B(0, x_j \ast y_j) \wedge A \times B(0, y_j) \wedge 0.5, \]

then we have

\[ A \times B(0, x) = A(0) \wedge B(x) = B(x), \]

thus \( B(x) \geq B(x \ast y_j) \wedge B(y_j) \wedge 0.5 \). If \( A \times B \) is an \( M-(\varepsilon, \varepsilon) \)-fuzzy ideal of \( X \), then

\[ A \times B(m(x, y)) \geq A \times B(x, y) \wedge 0.5, \forall (x, y) \in X \times X, \]

let \( x = 0 \), then

\[ A \times B(m(x, y)) = A \times B(mx, my) = A(mx) \wedge B(my) = B(my) \]

\[ \geq A(x) \wedge B(y \wedge 0.5) = A(0) \wedge B(y) \wedge 0.5 \]

\[ = B(y) \wedge 0.5, \]

then we have

\[ B(my) \geq B(y) \wedge 0.5, \forall y \in X, m \in M. \]

Therefore \( B \) is an \( M-(\varepsilon, \varepsilon) \)-fuzzy ideal of \( X \).

**Proposition 14.** If \( B \) is a fuzzy set, \( A \) is a strong fuzzy relation \( A_B \) of \( B \), then \( B \) is an \( M-(\varepsilon, \varepsilon) \)-fuzzy ideal of \( X \) if and only if \( A_B \) is an \( M-(\varepsilon, \varepsilon) \)-fuzzy ideal of \( X \times X \).

**Proof.** If \( B \) is an \( M-(\varepsilon, \varepsilon) \)-fuzzy ideals of \( X \), then for all \((x, y) \in X \times X\), we have

\[ A_B(0, 0) = B(0) \wedge B(0) \geq B(x) \wedge 0.5 \wedge B(y) \wedge 0.5 \]

\[ = A_B(x, y) \wedge 0.5; \]

for all \((x_i, x_j), (y_i, y_j) \in X \times X\), we have

\[ A_B(x_i, x_j) = B(0) \wedge B(0) \geq B(x_i) \wedge 0.5 \wedge B(y_j) \wedge 0.5 \]

\[ \geq (B(x_i \ast y_j) \wedge B(y_j) \wedge 0.5) \wedge (B(x_j \ast y_j) \wedge B(y_j) \wedge 0.5) \]

\[ = (B(x_i \ast y_j) \wedge B(x_j \ast y_j) \ast (B(y_j) \wedge B(y_j)) \wedge 0.5 \]

\[ = A_B(x_i \ast y_j, x_j \ast y_j) \ast A_B(y_j, y_j) \wedge 0.5 \]

\[ = A_B((x_i \ast y_j, x_j \ast y_j) \ast (y_j, y_j)) \ast A_B(y_j, y_j) \wedge 0.5; \]

for all \((x, y) \in X \times X\), we have

\[ A_B(m(x, y)) = A_B(mx, my) = B(mx) \wedge B(my) \]

\[ \geq B(x) \wedge 0.5 \wedge B(y) \wedge 0.5 = A_B(x, y) \wedge 0.5. \]

Therefore \( A_B \) is an \( M-(\varepsilon, \varepsilon) \)-fuzzy ideal of \( X \times X \). Conversely, suppose \( A_B \) is an \( M-(\varepsilon, \varepsilon) \)-fuzzy ideal of \( X \times X \), then

\[ B(0) \wedge B(0) = A_B(0, 0) \geq A_B(x, y) \wedge 0.5 \leq B(x) \wedge B(y) \wedge 0.5, \]

for all \((x, y) \in X \times X\), we have

\[ B(x) \wedge B(y) \geq B(x \ast y) \wedge B(y) \wedge 0.5, \]

let \( x_i = y_i = 0 \), then

\[ B(x) \wedge B(0) \geq B(x \ast y_j) \wedge B(y_j) \wedge 0.5, \]

if \( A_B \) is an \( M-(\varepsilon, \varepsilon) \)-fuzzy ideal of \( X \times X \), then

\[ A_B(m(x, y)) \geq A_B(x, y), \forall x, y \in X \times X, m \in M, \]

We have

\[ B(mx) \wedge B(my) \geq A_B(mx, my) \geq A_B(x, y) \wedge 0.5 = B(x) \wedge B(y) \wedge 0.5, \]

if \( x = 0 \), then

\[ B(0) \wedge B(my) = A_B(0, my) \geq A_B(0, y) \wedge 0.5 = B(0) \wedge B(y) \wedge 0.5, \]

namely, \( B(my) \geq B(y) \wedge 0.5 \). Therefore \( B \) is an \( M-(\varepsilon, \varepsilon) \)-fuzzy ideal of \( X \).

**REFERENCES**