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($\in,\in\vee q$)-Fuzzy Subalgebras and Fuzzy Ideals of BCI-Algebras with Operators

Yuli Hu, Shaoquan Sun

***Abstract*—**The aim of this paper is to introduce the concepts of ($\in,\in\vee q$)-fuzzy subalgebras, ($\in,\in\vee q$)-fuzzy ideals and ($\in,\in\vee q$)-fuzzy quotient algebras of BCI-algebras with operators, and to investigate their basic properties.

***Keywords*—**BCI-algebras with operators, ($\in,\in\vee q$)-fuzzy subalgebras, ($\in,\in\vee q$)-fuzzy ideals, ($\in,\in\vee q$)-fuzzy quotient algebras.

I. Introduction

THE fuzzy set is a generalization of the classical set and was used afterwards by several authors such as Imai [1], Iseki [2] and Xi [3], in various branches of mathematics. Particularly, in the area of fuzzy topology, after the introduction of fuzzy sets by Zadeh [15], much research has been carried out: the concept of fuzzy subalgebras and fuzzy ideals of BCK-algebras, and their some properties.

BCK-algebras and BCI-algebras are two important classes of logical algebras, which were introduced by Imai and Iseki [1], [2]. In 1991, Xi [3] applied the fuzzy sets to BCK-algebras and discussed some properties about fuzzy subalgebras and fuzzy ideals. From then on, fuzzy BCK/BCI-algebras have been widely investigated by some researchers. Jun et al. [4], [5] raised the notions of fuzzy positive implicative ideals and fuzzy commutative ideals of BCK-algebras. Ming and Ming [12] introduced the neighbourhood structure of a fuzzy point in 1980; Jun et al. [6] introduced the concept of $(\in,\in\vee q)$-ideals of BCI-algebras. In 1993, Zheng [7] defined operators in BCK-algebras and introduced the concept of BCI-algebras with operators and gave some isomorphism theorems of it. Then, Liu [9] introduced the university property of direct products of BCI-algebras. In 2002, Liu [8] introduced the notion of the fuzzy quotient algebras of BCI-algebras. In 2004, Jun [10] introduced the (α,β)-fuzzy ideals of BCK/BCI-algebras and established the characterizations of $(\in,\in\vee q)$-fuzzy ideals. Next, Pan [13] introduced fuzzy ideals of sub-algebra and fuzzy H-ideals of sub-algebra. In 2011, Liu and Sun [11] introduced the concept of generalized fuzzy ideals of BCI-algebra and investigated some basic properties. In 2017, we [14] also introduced the fuzzy subalgebras and fuzzy ideals of BCI-algebras with operators.

In this paper, we introduce the concepts of $(\in,\in\vee q)$-fuzzy subalgebras, $(\in,\in\vee q)$-fuzzy ideals and $(\in,\in\vee q)$-fuzzy quotient algebras of BCI-algebras with operators. Moreover, the basic properties were discussed and several results have been obtained.

Yuli Hu is with the College of Mathematics and Physics, Qingdao University of Science and Technology, Qingdao, China (phone: 178-06259520; e-mail:1198260194@qq.com).

Shaoquan Sun is with the College of Mathematics and Physics, Qingdao University of Science and Technology, Qingdao, China (phone: 185-61681686; e-mail: qdsunsaoquan@163.com).

II. Preliminaries

Some definitions and propositions were recalled which may be needed.

An algebra $\langle X;*,0\rangle$ of type (2,0) is called a BCI-algebra, if for all $x,y,z\in X$, it satisfies:

$(1)\left((x*y)*(x*z)\right)*(z*y)=0;$

$(2)\left(x*(x*y)\right)*y=0;$

$(3)\,x*x=0;$

$(4)\,x*y=0$ and $y*x=0$ imply $x=y$.

We can define $x*y=0$ if and only if $x\le y$, and the above conditions can be written as:

1. $(x*y)*(x*z)\le z*y;$
2. $x*(x*y)\le y;$
3. $x\le x;$
4. $x\le y$ and $y\le x$ imply $x=y$.

A BCI-algebra is called a BCK-algebra if it satisfies $0*x=0$.

Definition 1. [5] $\langle X;*,0\rangle$ is a BCI-algebra, a fuzzy subset A of X is called a fuzzy ideal of X if it satisfies:

(1)$A(0)\ge A(x),\forall x\in X,$

(2)$A(x)\ge A(x*y)\wedge A(y),\forall x,y\in X.$

Definition 2. [4] $\langle X;*,0\rangle$ is a BCI-algebra, a fuzzy subset A of X is called a fuzzy subalgebra of X if it satisfies:

$$A(x*y)\ge A(x)*A(y),\forall x,y\in X.$$

Definition 3. [12] $\langle X;*,0\rangle$ is a BCI-algebra, a fuzzy subset A of X of the form

$$A(y)=\begin{cases}t(\ne 0), y=x,\\ 0, y\ne x,\end{cases}$$

is said to be a fuzzy point with support x and value t, and is denoted by x_t.

Definition 4. [12] If x_t is a fuzzy point, it is said to belong to (resp.be quasi-coincident with) a fuzzy subset A, written as $x_t \in A$ (resp. $x_t qA$) if $A(x) \geq t$ (resp. $A(x)+t>1$). If $x_t \in A$ or $x_t qA$, then we write $x_t \in \vee qA$. The symbol $\overline{\in \vee q}$ (resp. $\overline{\in}$ or $\overline{q}$) means $\in \vee q$ (resp. $\in$ or q) does not hold.

Definition 5. [10] $\langle X;*,0\rangle$ is a BCI-algebra, a fuzzy set A of X is called an $(\in,\in\vee q)$-fuzzy ideal of X if for all $t,r\in(0,1]$ and $x,y\in X$, it satisfies:

1. $x_t \in A \Rightarrow 0_t \in \vee qA$,
2. $(x*y)_t \in A$ and $y_r \in A \Rightarrow x_{t\wedge r} \in \vee qA$.

Definition 6. [10] A fuzzy set A is an $(\in,\in\vee q)$-fuzzy ideal of X if and only if it satisfies:

$(1)\, A(0) \geq A(x)\wedge 0.5, \forall x\in X$,

$(2)\, A(x) \geq A(x*y)\wedge A(y)\wedge 0.5, \forall x,y\in X$.

Definition 7. [7] $\langle X;*,0\rangle$ is a BCI-algebra, M is a non-empty set, if there exists a mapping $(m,x)\rightarrow mx$ from $M\times X$ to X which satisfies

$$m(x*y)=(mx)*(my), \forall x,y\in X, m\in M,$$

then M is called a left operator of X, X is called BCI-algebra with left operator M, or $M-$BCI-algebra for short.

Proposition 1. [6] Let $\langle X;*,0\rangle$ be a BCI-algebra, if A is an $(\in,\in\vee q)$-fuzzy ideal of it, and $x*y\leq z$, then

$$A(x)\geq A(y)\wedge A(z)\wedge 0.5, \forall x,y,z\in X.$$

Definition 8. [13] Let A and B be fuzzy sets of set X, then the direct product $A\times B$ of A and B is a fuzzy subset of $X\times X$, define $A\times B$ by

$$A\times B(x,y)=A(x)\wedge B(y), \forall x,y\in X.$$

Definition 9. [7] Let $\langle X;*,0\rangle$ and $\langle \bar{X};*,0\rangle$ be two $M-$BCI-algebras, if for all $x\in X, m\in M$, $f(mx)=mf(x)$, and f is a homomorphism from $\langle X;*,0\rangle$ to $\langle \bar{X};*,0\rangle$, then f is called a homomorphism with operators.

Definition 10. [13] $\langle X;*,0\rangle$ is an $M-$BCI-algebra, let B be a fuzzy set of X, and A be a fuzzy relation of B, if it satisfies:

$$A_B(x,y)=B(x)\wedge B(y), \forall x,y\in X,$$

then A is called a strong fuzzy relation of B.

Definition 11. [14] If $\langle X;*,0\rangle$ is an $M-$BCI-algebra, A is a non-empty subset of X, and $mx\in A$ for all $x\in A, m\in M$, then $\langle A;*,0\rangle$ is called a $M-$subalgebra of $\langle X;*,0\rangle$.

In this paper, X always means a $M-$BCI-algebra unless otherwise specified.

III. $(\in,\in\vee q)$-Fuzzy Subalgebras of BCI-Algebras with Operators

Definition 12. $\langle X;*,0\rangle$ is a BCI-algebra, a fuzzy set A of X is called a $M-(\in,\in\vee q)$-fuzzy subalgebra of X if for all $t,r\in(0,1]$ and $x,y\in X$, it satisfies:

1. $x_t\in A$ and $y_r\in A\Rightarrow (x*y)_{t\wedge r}\in\vee qA$,
2. $x_t\in A\Rightarrow (mx)_t\in\vee qA$.

Proposition 2. $\langle X;*,0\rangle$ is a BCI-algebra, a fuzzy set A of X is an $M-(\in,\in\vee q)$-fuzzy subalgebra of X if and only if it satisfies:

$(1)\, A(x*y)\geq A(x)\wedge A(y)\wedge 0.5, \forall x,y\in X$,

$(2)\, A(mx)\geq A(x)\wedge 0.5, \forall x\in X$.

Proof. Suppose that A is an $M-(\in,\in\vee q)$-fuzzy subalgebra of X. (1) Let $x,y\in X$, suppose that $A(x)\wedge A(y)<0.5$, then $A(x*y)\geq A(x)\wedge A(y)$, if not, then we have $A(x*y)<t<A(x)\wedge A(y), \exists t\in(0,0.5)$; it follows that $x_t\in A$ and $y_t\in A$, but $(x*y)_{t\wedge t}=(x*y)_t\overline{\in\vee q}A$, which is a contradiction, then whenever $A(x)\wedge A(y)<0.5$. We have $A(x*y)\geq A(x)\wedge A(y)$. If $A(x)\wedge A(y)\geq 0.5$, then $(x)_{0.5}\in A$ and $y_{0.5}\in A$, which implies that $(x*y)_{0.5}=(x*y)_{0.5\wedge 0.5}\in\vee qA$, therefore $A(x*y)\geq 0.5$, because if $A(x*y)<0.5$, then $A(x*y)+0.5<0.5+0.5=1$, which is a contradiction, hence

$$A(x*y)\geq A(x)\wedge A(y)\wedge 0.5, \forall x,y\in X.$$

(2) Let $x\in X$ and assume that $A(x)<0.5$. If $A(mx)<A(x)$, then we have $A(mx)<t<A(x), \exists t\in(0,0.5)$, and we have $x_t\in A$ and $(mx)_t\overline{\in}A$, since $A(mx)+t<1$, we have $(mx)_t\overline{q}A$; it follows that $(mx)_t\overline{\in\vee q}A$, which is a contradiction, hence $A(mx)\geq A(x)$. Now if $A(x)\geq 0.5$, then $x_{0.5}\in A$, thus $(mx)_{0.5}\in\vee qA$, hence $A(mx)\geq 0.5$, otherwise $A(mx)+0.5<0.5+0.5=1$, which is a contradiction, consequently, $A(mx)\geq A(x)\wedge 0.5, \forall x\in X$.

Conversely, assume that A satisfies condition (1), (2).

(1) Let $x,y\in X$ and $t_1,t_2\in(0,1]$ be such that $x_{t_1}\in A$ and $y_{t_2}\in A$, then $A(x)\geq t_1$ and $A(y)\geq t_2$. Suppose that $A(x*y)<t_1\wedge t_2$, if $A(x)\wedge A(y)<0.5$, then $A(x*y)\geq A(x)\wedge A(y)\wedge 0.5=A(x)\wedge A(y)\geq t_1\wedge t_2$, this is a contradiction, so we have $A(x)\wedge A(y)\geq 0.5$, it follows that

$$A(x*y)+t_1\wedge t_2 > 2A(x*y)\geq 2\big(A(x)\wedge A(y)\wedge 0.5\big)=1,$$

so that $(x*y)_{t_1\wedge t_2}\in\vee qA.$

(2) Let $x\in X$ and $t\in(0,1]$ be such that $x_t\in A,$ then we have $A(x)\geq t.$ Suppose that $A(mx)<t,$ if $A(x)<0.5,$ then $A(mx)\geq A(x)\wedge 0.5 = A(x)\geq t,$ this is a contradiction, hence we know that $A(x)\geq 0.5,$ and we have

$$A(mx)+t > 2A(mx)\geq 2\big(A(x)\wedge 0.5\big)=1,$$

then $(mx)_t\in\vee qA.$ Consequently, A is an $M-(\in,\in\vee q)$-fuzzy subalgebra.

Example 1. If A is an $M-(\in,\in\vee q)$-fuzzy subalgebra of X, then X_A is an $M-(\in,\in\vee q)$-fuzzy subalgebra of X, define X_A by

$$X_A: X\to[0,1], X_A(x)=\begin{cases}1, x\in A\\ 0, x\notin A.\end{cases}$$

Proof. (1) For all $x,y\in X,$ if $x,y\in A,$ then $x*y\in A,$ then we have

$$X_A(x*y)=1\geq X_A(x)\wedge X_A(y)\wedge 0.5,$$

if there exists at least one which does not belong to A between x and y, for example $x\notin A,$ thus

$$X_A(x*y)\geq 0 = X_A(x)\wedge X_A(y)\wedge 0.5.$$

(2) For all $x\in X$, $m\in M$, if $x\in A,$ then $mx\in A,$ therefore

$$X_A(mx)=1\geq X_A(x)\wedge 0.5,$$

if $x\notin A,$ then $X_A(mx)\geq 0 = X_A(x)\wedge 0.5,$ therefore X_A is an $M-(\in,\in\vee q)$-fuzzy subalgebra of X.

Proposition 3. A is an $M-(\in,\in\vee q)$-fuzzy subalgebra of X if and only if A_t is an $M-$subalgebra of X, where A_t is a non-empty set, define X_A by

$$A_t=\{x\mid x\in X, A(x)\geq t\},\forall t\in[0,0.5].$$

Proof. Suppose A is an $M-(\in,\in\vee q)$-fuzzy subalgebra of X, A_t is a non-empty set, $t\in[0,0.5],$ then we have $A(x*y)\geq A(x)\wedge A(y)\wedge 0.5.$ If $x\in A_t, y\in A_t,$ then $A(x)\geq t, A(y)\geq t,$ thus

$$A(x*y)\geq A(x)\wedge A(y)\wedge 0.5\geq t,$$

then we have $x*y\in A_t.$ If A is an $M-(\in,\in\vee q)$-fuzzy subalgebra of X, then $A(mx)\geq A(x)\wedge 0.5\geq t,\forall x\in X, m\in M,$ then we have $mx\in A_t.$ Therefore A_t is an $M-$subalgebra of X. Conversely, suppose A_t is an $M-$subalgebra of X, then we have $x*y\in A_t$. Let $A(x)=t,$ then

$$A(x*y)\geq t = A(x)\geq A(x)\wedge A(y)\wedge 0.5.$$

If A_t is an $M-$subalgebra of X, then we have

$$A(mx)\geq t = A(x)\geq A(x)\wedge 0.5,\forall x\in X, m\in M,$$

therefore A is an $M-(\in,\in\vee q)$-fuzzy subalgebra of X.

Proposition 4. Suppose X,Y are $M-$BCI-algebras, f is a mapping from X to Y, if A is an $M-(\in,\in\vee q)$-fuzzy subalgebra of the Y, then $f^{-1}(A)$ is a $M-(\in,\in\vee q)$-fuzzy subalgebra of X.

Proof. Let $y\in Y,$ suppose f is an epimorphism, and we have $y=f(x),\exists x\in X.$ If A is an $M-(\in,\in\vee q)$-fuzzy subalgebra of Y, then we have

$$A(x*y)\geq A(x)\wedge A(y)\wedge 0.5, A(mx)\geq A(x)\wedge 0.5.$$

For all $x,y\in X, m\in M,$ we have

(1) $f^{-1}(A)(x*y)=A\big(f(x)*f(y)\big)\geq A\big(f(x)\big)\wedge A\big(f(Y)\big)\wedge 0.5$
$= f^{-1}(A)(x)\wedge f^{-1}(A)(y)\wedge 0.5;$

(2) $f^{-1}(A)(mx)=A\big(f(mx)\big)=A\big(mf(x)\big)$
$\geq A\big(f(x)\big)\wedge 0.5 = f^{-1}(A)(x)\wedge 0.5.$

Then $f^{-1}(A)$ is an $M-(\in,\in\vee q)$-fuzzy subalgebra of X.

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Definition 13. $\langle X;*,0\rangle$ is a BCI-algebra, a fuzzy set A of X is called an $M-(\in,\in\vee q)$-fuzzy ideal of X if for all $t,r\in(0,1]$ and $x,y\in X$, it satisfies:

1. $x_t\in A\Rightarrow 0_t\in\vee qA,$
2. $(x*y)_t\in A$ and $y_r\in A\Rightarrow x_{t\wedge r}\in\vee qA,$
3. $x_t\in A\Rightarrow (mx)_t\in\vee qA.$

Proposition 5. [13] $\langle X;*,0\rangle$ is a BCI-algebra, a fuzzy set A is an $M-(\in,\in\vee q)$-fuzzy ideal of X if and only if it satisfies:

$(1)\, A(0)\geq A(x)\wedge 0.5,\forall x\in X,$

$(2)\, A(x)\geq A(x*y)\wedge A(y)\wedge 0.5,\forall x,y\in X,$

$(3)\, A(mx)\geq A(x)\wedge 0.5,\forall x\in X.$

Proof. Suppose that A is an $M-(\in,\in\vee q)$-fuzzy ideal of X.

(1) Let $x \in X$ and assume that $A(x) < 0.5$. If $A(0) < A(x)$, then we have $A(0) < t < A(x), \exists t \in (0, 0.5)$, and we have $x_t \in A$ and $0_t \bar{\in} A$, since $A(0) + t < 1$, we have $0_t \bar{q} A$, it follows that $0_t \overline{\in \vee q} A$, which is a contradiction, then $A(0) \geq A(x)$. Now if $A(x) \geq 0.5$, then $x_{0.5} \in A$, then we have $0_{0.5} \in \vee q A$, hence $A(0) \geq 0.5$, otherwise, $A(0) + 0.5 < 0.5 + 0.5 = 1$, which is a contradiction, consequently,

$$A(0) \geq A(x) \wedge 0.5, \forall x \in X.$$

(2) Let $x, y \in X$ and suppose that $A(x * y) \wedge A(y) < 0.5$, then $A(x) \geq A(x * y) \wedge A(y)$, if not, then we have $A(x) < t < A(x * y) \wedge A(y), \exists t \in (0, 0.5)$, it follows that $(x * y)_t \in A$ and $y_t \in A$, but $x_{t \wedge t} = x_t \overline{\in \vee q} A$, which is a contradiction, hence whenever $A(x * y) \wedge A(y) < 0.5$, we have $A(x) \geq A(x * y) \wedge A(y)$. If $A(x * y) \wedge A(y) \geq 0.5$, then $(x * y)_{0.5} \in A$ and $y_{0.5} \in A$, which implies that $x_{0.5} = x_{0.5 \wedge 0.5} \in \vee q A$, therefore $A(x) \geq 0.5$, because if $A(x) < 0.5$, then $A(x) + 0.5 < 0.5 + 0.5 = 1$, which is a contradiction, then

$$A(x) \geq A(x * y) \wedge A(y) \wedge 0.5, \forall x, y \in X.$$

(3) Let $x \in X$ and assume that $A(x) < 0.5$. If $A(mx) < A(x)$, then we have $A(mx) < t < A(x), \exists t \in (0, 0.5)$, and we have $x_t \in A$ and $(mx)_t \bar{\in} A$, since $A(mx) + t < 1$, we have $(mx)_t \bar{q} A$, it follows that $(mx)_t \overline{\in \vee q} A$, which is a contradiction, then $A(mx) \geq A(x)$. Now if $A(x) \geq 0.5$, then $x_{0.5} \in A$, thus $(mx)_{0.5} \in \vee q A$, hence $A(mx) \geq 0.5$, otherwise $A(mx) + 0.5 < 0.5 + 0.5 = 1$, which is a contradiction, consequently, $A(mx) \geq A(x) \wedge 0.5, \forall x \in X$. Conversely, suppose that A satisfies (1), (2), (3) of the Proposition 5, then we have

(1) Let $x \in X$ and $t \in (0,1]$ be such that $x_t \in A$, then we have $A(x) > t$, suppose that $A(0) < t$, if $A(x) < 0.5$, then $A(0) \geq A(x) \wedge 0.5 = A(x) \geq t$, which is a contradiction, then we know that $A(x) \geq 0.5$, and we have $A(0) + t > 2A(0) \geq 2(A(x) \wedge 0.5) = 1$, thus $0_t \in \vee q A$.

(2) Let $x, y \in X$ and $t_1, t_2 \in (0,1]$ be such that $(x * y)_{t_1} \in A$ and $y_{t_2} \in A$, then $A(x * y) \geq t_1$ and $A(y) \geq t_2$, suppose that $A(x) < t_1 \wedge t_2$, if $A(x * y) \wedge A(y) < 0.5$, then

$$A(x) \geq A(x * y) \wedge A(y) \wedge 0.5 = A(x * y) \wedge A(y) \geq t_1 \wedge t_2,$$

This is a contradiction, so we have $A(x * y) \wedge A(y) \geq 0.5$, it follows that

$$A(x) + t_1 \wedge t_2 > 2A(x) \geq 2(A(x * y) \wedge A(y) \wedge 0.5) = 1,$$

so that $x_{t_1 \wedge t_2} \in \vee q A$.

(3) Let $x \in X$ and $t \in (0,1]$ be such that $x_t \in A$, then $A(x) \geq t$, suppose that $A(mx) < t$, if $A(x) < 0.5$, then $A(mx) \geq A(x) \wedge 0.5 = A(x) \geq t$, which is a contradiction, then we know that $A(x) \geq 0.5$, and we have $A(mx) + t > 2A(mx) \geq 2(A(x) \wedge 0.5) = 1$, thus $(mx)_t \in \vee q A$. Consequently, A is an $M - (\in, \in \vee q)$-fuzzy ideal.

Example 2. If A is an $M - (\in, \in \vee q)$-fuzzy ideal of X, then X_A is an $M - (\in, \in \vee q)$-fuzzy ideal of X, define X_A by

$$X_A : X \to [0,1], X_A(x) = \begin{cases} 1, x \in A \\ 0, x \notin A. \end{cases}$$

Proof. (1) For all $x, y \in X$, if $x, y \in A$, then $x * y \in A$, thus

$$X_A(0) = 1 \geq X_A(x) \wedge 0.5,$$
$$X_A(x) = 1 \geq X_A(x * y) \wedge X_A(y) \wedge 0.5,$$

if there exists at least one between x and y which does not belong to A, for example $x \notin A$, thus

$$X_A(0) = 1 \geq X_A(x) \wedge 0.5,$$
$$X_A(x) \geq X_A(x * y) \wedge X_A(y) \wedge 0.5 = 0,$$

therefore X_A is a $(\in, \in \vee q)$-fuzzy ideal of X.

(2) For all $x \in X$, $m \in M$, if $x \in A$, then $mx \in A$, therefore $X_A(mx) = 1 \geq X_A(x) \wedge 0.5$. If $x \notin A$, then $X_A(mx) \geq 0 = X_A(x) \wedge 0.5$, therefore X_A is an $M - (\in, \in \vee q)$-fuzzy ideal of X.

Proposition 6. A is an $M - (\in, \in \vee q)$-fuzzy ideal of X if and only if A_t is an $M -$ideal of X, where A_t is non-empty set, define A_t by

$$A_t = \{x \mid x \in X, A(x) \geq t\}, \forall t \in [0, 0.5].$$

Proof. Suppose A is an $M - (\in, \in \vee q)$-fuzzy ideal of X, A_t is non-empty set, $t \in [0, 0.5]$, then we have $A(0) \geq A(x) \wedge 0.5 \geq t$, then we have $0 \in A_t$. If $x * y \in A_t, y \in A_t$, then $A(x * y) \geq t, A(y) \geq t$, thus $A(x) \geq A(x * y) \wedge A(y) \wedge 0.5 \geq t$, then we have $x \in A_t$. For all $x \in X$, $m \in M$, if A is an $M - (\in, \in \vee q)$-fuzzy ideal of X, hence $A(mx) \geq A(x) \wedge 0.5 \geq t$, thus $mx \in A_t$, therefore A_t is an $M -$ideal of X. Conversely, suppose A_t is an $M-$ideal of X, then we have $0\in A_t, A(0)\geq t.$ Let $A(x)=t,$ thus $x\in A_t$, we have $A(0)\geq t=A(x),$ suppose there is no $A(x)\geq A(x*y)\wedge A(y)\wedge 0.5,$ then there exist $x_0, y_0\in X$, we have $A(x_0)<A(x_0*y_0)\wedge A(y_0)\wedge 0.5,$ let $t_0=A(x_0*y_0)\wedge A(y_0)\wedge 0.5,$ then $A(x_0)<t_0=A(x_0*y_0)\wedge A(y_0)\wedge 0.5,$ if $x_0*y_0\in A_{t_0}, y_0\in A_{t_0}$, then we have $x_0\in A_{t_0},$ then $A(x_0)\geq t_0,$ which is inconsistent with $A(x_0)<t_0=A(x_0*y_0)\wedge A(y_0)\wedge 0.5,$ then we have $A(x)\geq A(x*y)\wedge A(y)\wedge 0.5.$ If A_t is an $M-$ideal of X, then we have $A(mx)\geq t=t\wedge 0.5=A(x)\wedge 0.5, \forall x\in X, m\in M,$ therefore A is an $M-(\in,\in\vee q)$-fuzzy ideal of X.

Proposition 7. Suppose X, Y are $M-$BCI-algebras, f is a mapping from X to Y, A is an $M-(\in,\in\vee q)$-fuzzy ideal of Y, then $f^{-1}(A)$ is an $M-(\in,\in\vee q)$- fuzzy ideal of X.

Proof. Let $y\in Y$, suppose f is an epimorphism, then we have $y=f(x), \exists x\in X.$ If A is an $M-(\in,\in\vee q)$-fuzzy ideal of Y, then we have

$$A(0)\geq A(x)\wedge 0.5,$$
$$A(x)\geq A(x*y)\wedge A(y)\wedge 0.5,$$
$$A(mx)\geq A(x)\wedge 0.5.$$

For all $x, y\in X, m\in M$, we have

(1) $f^{-1}(A)(0)=A(f(0))=A(0)\geq A(f(x))\wedge 0.5=f^{-1}(A)(x)\wedge 0.5;$

(2) $f^{-1}(A)(x)=A(f(x))\geq A(f(x)*f(y))\wedge A(f(y))\wedge 0.5$
$=A(f(x*y))\wedge A(f(y))\wedge 0.5=f^{-1}(A)(x*y)\wedge f^{-1}(A)(y)\wedge 0.5;$

(3) $f^{-1}(A)(mx)=A(f(mx))=A(mf(x))$
$\geq A(f(x))\wedge 0.5=f^{-1}(A)(x)\wedge 0.5.$

Therefore $f^{-1}(A)$ is an $M-(\in,\in\vee q)$-fuzzy ideal of X.

V. $(\in,\in\vee q)$ - Fuzzy Quotient BCI-Algebras with Operators

Definition 14. Let A be an $M-(\in,\in\vee q)$-fuzzy ideal of X, for all $a\in X$, fuzzy set A_a on X defined as: $A_a: X\rightarrow[0,1]$

$$A_a(x)=A(a*x)\wedge A(x*a)\wedge 0.5, \forall x\in X.$$

Denote $X/A=\{A_a: a\in X\}$.

Proposition 8. Let $A_a, A_b\in X/A$, then $A_a=A_b$ if and only if $A(a*b)\wedge A(b*a)\wedge 0.5=A(0)\wedge 0.5.$

Proof. Let $A_a=A_b$, then we have $A_a(b)=A_b(b),$ thus

$$A(a*b)\wedge A(b*a)\wedge 0.5=A(b*b)\wedge A(b*b)\wedge 0.5=A(0)\wedge 0.5,$$

that is $A(a*b)\wedge A(b*a)\wedge 0.5=A(0)\wedge 0.5.$ Conversely, suppose that $A(a*b)\wedge A(b*a)\wedge 0.5=A(0)\wedge 0.5.$ For all $x\in X$, since

$$(a*x)*(b*x)\leq a*b, (x*a)*(x*b)\leq b*a.$$

It follows from Proposition 1 that

$$A(a*x)\geq A(b*x)\wedge A(a*b)\wedge 0.5,$$
$$A(x*a)\geq A(x*b)\wedge A(b*a)\wedge 0.5.$$

Hence

$$\begin{aligned}A_a(x)&=A(a*x)\wedge A(x*a)\wedge 0.5\\&\geq A(b*x)\wedge A(x*b)\wedge A(a*b)\wedge A(b*a)\wedge 0.5\\&=A(b*x)\wedge A(x*b)\wedge A(0)\wedge 0.5\\&=A(b*x)\wedge A(x*b)\wedge 0.5=A_b(x),\end{aligned}$$

that is $A_a\geq A_b$. Similarly, for all $x\in X$, since

$$(b*x)*A(a*x)\leq b*a, (x*b)*A(x*a)\leq a*b.$$

It follows from Proposition 1 that

$$A(b*x)\geq A(a*x)\wedge A(b*a)\wedge 0.5,$$
$$A(x*b)\geq A(x*a)\wedge A(a*b)\wedge 0.5.$$

Hence

$$\begin{aligned}A_b(x)&=A(b*x)\wedge A(x*b)\wedge 0.5\\&\geq A(a*x)\wedge A(x*a)\wedge A(b*a)\wedge A(a*b)\wedge 0.5\\&=A(a*x)\wedge A(x*a)\wedge A(0)\wedge 0.5\\&=A(a*x)\wedge A(x*a)\wedge 0.5=A_a(x),\end{aligned}$$

that is $A_b\geq A_a$. Therefore, $A_a=A_b$. We complete the proof.

Proposition 9. Let $A_a=A_{a'}, A_b=A_{b'}$, then $A_{a*b}=A_{a'*b'}$.

Proof. Since

$$\begin{aligned}&((a*b)*(a'*b'))*(a*a')=((a*b)*(a*a'))*(a'*b')\\&\leq(a'*b)*(a'*b')\leq b'*b,\\&((a'*b')*(a*b))*(b*b')=((a'*b')*(b*b'))*(a*b)\\&\leq(a'*b)*(a*b)\leq a'*a.\end{aligned}$$

Hence

$$A((a*b)*(a'*b'))\geq A(a*a')\wedge A(b'*b)\wedge 0.5,$$
$$A((a'*b')*(a*b))\geq A(b*b')\wedge A(a'*a)\wedge 0.5.$$

Therefore

$$\begin{aligned}&A((a*b)*(a'*b'))\wedge A((a'*b')*(a*b))\wedge 0.5\\&=A(a*a')\wedge A(a'*a)\wedge 0.5\wedge A(b*b')\wedge A(b'*b)\wedge 0.5\wedge 0.5\\&=A(0)\wedge 0.5,\end{aligned}$$

it follows from Proposition 8. that $A_{a*b}=A_{a'*b'}$. We completed the proof.

Let A be an $M-(\in,\in\vee q)$-fuzzy ideal of X. The operation "*" of R/A is defined as: $\forall A_a, A_b \in R/A, A_a * A_b = A_{a*b}$. By Proposition 8, the above operation is reasonable.

Proposition 10. A is an $M-(\in,\in\vee q)$-fuzzy ideal of X, then $R/A=\{R/A;*,A_0\}$ is an $M-$BCI-algebra.

Proof. For all $A_x, A_y, A_z \in R/A$, we have

$$((A_x * A_y)*(A_x * A_z))*(A_z * A_y) = A_{((x*y)*(x*z))*(z*y)} = A_0;$$
$$(A_x *(A_x * A_y))* A_y = A_{(x*(x*y))*y} = A_0;$$
$$A_x * A_x = A_{x*x} = A_0;$$

if $A_x * A_y = A_0, A_y * A_x = A_0$, then $A_{x*y} = A_0, A_{y*x} = A_0$, it follows from Proposition 8 that $A(x*y)=A(0), A(y*x)=A(0)$, hence $A(x*y)\wedge A(y*x)\wedge 0.5 = A(0)\wedge 0.5$, then we have $A_x = A_y$. Therefore $R/A=\{R/A;*,A_0\}$ is a BCI-algebra. For all $A_x \in R/A, m\in M$, we define $mA_x = A_{mx}$. Firstly, we verify that $mA_x = A_{mx}$ is reasonable. If $A_x = A_y$, then we verify $mA_x = mA_y$, that is to verify $A_{mx} = A_{my}$. We have

$$A(mx*my)\wedge 0.5 = A(m(x*y))\wedge 0.5 \geq A(x*y)\wedge 0.5,$$
$$A(my*mx)\wedge 0.5 = A(m(y*x))\wedge 0.5 \geq A(y*x)\wedge 0.5,$$

so we have

$$A(mx*my)\wedge A(my*mx)\wedge 0.5 \geq A(x*y)\wedge A(y*x)\wedge 0.5 = A(0)\wedge 0.5,$$

then $A(mx*my)\wedge A(my*mx)\wedge 0.5 = A(0)\wedge 0.5$, that is $A_{mx} = A_{my}$. In addition, for all $m\in M, A_x, A_y \in R/A$, we have

$$m(A_x * A_y) = mA_{x*y} = A_{m(x*y)}$$
$$= A_{(mx)*(my)} = A_{mx} * A_{my} = mA_x * mA_y.$$

Therefore $R/A=\{R/A;*,A_0\}$ is an $M-$BCI-algebra.

Definition 15. Let μ be an $M-(\in,\in\vee q)$-fuzzy subalgebra of X, and A be an $M-(\in,\in\vee q)$-fuzzy ideal of X, we define a fuzzy set of X/A as follows:

$$\mu/A: X/A \to [0,1],\quad \mu/A(A_t) = \sup_{A_x=A_t} \mu(x)\wedge 0.5, \forall A_t \in X/A.$$

Proposition 11. μ/A is an $M-(\in,\in\vee q)$-fuzzy subalgebea of X/A.

Proof. For all $A_x, A_y \in X/A$, we have

$$\mu/A(A_x * A_y) = \mu/A(A_{x*y}) = \sup_{A_z=A_{x*y}} \mu(z)\wedge 0.5$$
$$\geq \sup_{A_s=A_x, A_t=A_y} \mu(s*t)\wedge 0.5 \geq \sup_{A_s=A_x, A_t=A_y} \mu(s)\wedge\mu(t)\wedge 0.5$$
$$= \sup_{A_s=A_x,} \mu(s)\wedge \sup_{A_t=A_y,} \mu(t)\wedge 0.5$$
$$= \mu/A(A_x)\wedge \mu/A(A_y)\wedge 0.5.$$

For all $m\in M, A_x \in R/A$, we have

$$\mu/A(A_{mx}) = \sup_{A_{mz}=A_{mx}} \mu(mz)\wedge 0.5$$
$$\geq \sup_{A_z=A_x} \mu(z)\wedge 0.5 = \mu/A(A_x)\wedge 0.5.$$

Therefore μ/A is an $M-(\in,\in\vee q)$-fuzzy subalgebra of X/A.

VI. Direct Products of $(\in,\in\vee q)$- Fuzzy Ideals of BCI-algebras with Operators

Proposition 12. Suppose A and B are $M-(\in,\in\vee q)$-fuzzy ideals of X, then $A\times B$ is an $M-(\in,\in\vee q)$-fuzzy ideal of $X\times X$.

Proof. (1) Let $(x,y)\in X\times X$, then

$$A\times B(0,0) = A(0)\wedge B(0) \geq A(x)\wedge 0.5\wedge B(y)\wedge 0.5$$
$$= A(x)\wedge B(y)\wedge 0.5 = A\times B(x,y)\wedge 0.5,$$

then $A\times B(0,0)\geq A\times B(x,y)\wedge 0.5, \forall (x,y)\in X\times X$;

(2) For all $(x_1,x_2),(y_1,y_2)\in X\times X$, we have

$$A\times B((x_1,x_2)*(y_1,y_2))\wedge A\times B(y_1,y_2)\wedge 0.5$$
$$= A\times B(x_1*y_1, x_2*y_2)\wedge A\times B(y_1,y_2)\wedge 0.5$$
$$= (A(x_1*y_1)\wedge B(x_2*y_2))\wedge A(y_1)\wedge B(y_2)\wedge 0.5$$
$$= (A(x_1*y_1)\wedge A(y_1))\wedge (B(x_2*y_2)\wedge B(y_2))\wedge 0.5$$
$$\leq A(x_1)\wedge B(x_2) = A\times B(x_1,x_2),$$

then for all $(x_1,x_2),(y_1,y_2)\in X\times X$, we have

$$A\times B(x_1,x_2) \geq A\times B((x_1,x_2)*(y_1,y_2))\wedge A\times B(y_1,y_2)\wedge 0.5;$$

(3) For all $(x,y)\in X\times X$, we have

$$A\times B(m(x,y)) = A\times B(mx,my) = A(mx)\wedge B(my)$$
$$\geq A(x)\wedge 0.5\wedge B(y)\wedge 0.5 = A(x)\wedge B(y)\wedge 0.5$$
$$= A\times B(x,y)\wedge 0.5,$$

then we have

$$A\times B\left(m\left(x,y\right)\right)\geq A\times B\left(x,y\right)\wedge 0.5,\forall\left(x,y\right)\in X\times X.$$

Therefore $A\times B$ is an $M-(\in,\in\vee q)$-fuzzy ideal of $X\times X$.

Proposition 13. Suppose A and B are fuzzy sets of X, if $A\times B$ is an $M-(\in,\in\vee q)$-fuzzy ideal of $X\times X$, then A or B is an $M-(\in,\in\vee q)$-fuzzy ideal of X.

Proof. Suppose A and B are $M-(\in,\in\vee q)$-fuzzy ideals of X, then for all $(x_1,x_2),(y_1,y_2)\in X\times X$, we have

$$\begin{aligned}A\times B(x_1,x_2)&\geq A\times B\left((x_1,x_2)*(y_1,y_2)\right)\wedge A\times B(y_1,y_2)\wedge 0.5\\&=A\times B\left((x_1*y_1),(x_2*y_2)\right)\wedge A\times B(y_1,y_2)\wedge 0.5,\end{aligned}$$

if $x_1=y_1=0,$ then

$$A\times B(0,x_2)\geq A\times B(0,x_2*y_2)\wedge A\times B(0,y_2)\wedge 0.5,$$

then we have

$$A\times B(0,x)=A(0)\wedge B(x)=B(x),$$

thus $B(x_2)\geq B(x_2*y_2)\wedge B(y_2)\wedge 0.5.$ If $A\times B$ is an $M-(\in,\in\vee q)$-fuzzy ideal of X, then

$$A\times B\left(m(x,y)\right)\geq A\times B(x,y)\wedge 0.5,\forall(x,y)\in X\times X,$$

let $x=0$, then

$$\begin{aligned}&A\times B\left(m(x,y)\right)=A\times B(mx,my)=A(mx)\wedge B(my)=B(my)\\&\geq A(x)\wedge B(y)\wedge 0.5=A(0)\wedge B(y)\wedge 0.5\\&=B(y)\wedge 0.5,\end{aligned}$$

then we have

$$B(my)\geq B(y)\wedge 0.5,\forall y\in X,m\in M.$$

Therefore B is an $M-(\in,\in\vee q)$- fuzzy ideal of X.

Proposition 14. If B is a fuzzy set, A is a strong fuzzy relation A_B of B, then B is an $M-(\in,\in\vee q)$-fuzzy ideal of X if and only if A_B is an $M-(\in,\in\vee q)$- fuzzy ideal of $X\times X$.

Proof. If B is an $M-(\in,\in\vee q)$-fuzzy ideals of X, then for all $(x,y)\in X\times X$, we have

$$\begin{aligned}A_B(0,0)&=B(0)\wedge B(0)\geq B(x)\wedge 0.5\wedge B(y)\wedge 0.5\\&=A_B(x,y)\wedge 0.5;\end{aligned}$$

for all $(x_1,x_2),(y_1,y_2)\in X\times X$, we have

$$\begin{aligned}&A_B(x_1,x_2)=B(x_1)\wedge B(x_2)\\&\geq\left(B(x_1*y_1)\wedge B(y_1)\wedge 0.5\right)\wedge\left(B(x_2*y_2)\wedge B(y_2)\wedge 0.5\right)\\&=\left(B(x_1*y_1)\wedge B(x_2*y_2)\right)\wedge\left(B(y_1)\wedge B(y_2)\right)\wedge 0.5\\&=A_B(x_1*y_1,x_2*y_2)\wedge A_B(y_1,y_2)\wedge 0.5\\&=A_B\left((x_1,x_2)*(y_1,y_2)\right)\wedge A_B(y_1,y_2)\wedge 0.5;\end{aligned}$$

for all $(x,y)\in X\times X$, we have

$$\begin{aligned}&A_B\left(m(x,y)\right)=A_B(mx,my)=B(mx)\wedge B(my)\\&\geq B(x)\wedge 0.5\wedge B(y)\wedge 0.5=A_B(x,y)\wedge 0.5.\end{aligned}$$

Therefore A_B is an $M-(\in,\in\vee q)$-fuzzy ideal of $X\times X$. Conversely, suppose A_B is an $M-(\in,\in\vee q)$-fuzzy ideal of $X\times X$, for all $(x_1,x_2)\in X\times X$, we have

$$B(0)\wedge B(0)=A_B(0,0)\geq A_B(x,x)\wedge 0.5=B(x)\wedge B(x)\wedge 0.5,$$

for all $(x_1,x_2),(y_1,y_2)\in X\times X$, we have

$$\begin{aligned}&B(x_1)\wedge B(x_2)=A_B(x_1,x_2)\\&\geq A_B\left((x_1,x_2)*(y_1,y_2)\right)\wedge A_B(y_1,y_2)\wedge 0.5\\&=A_B(x_1*y_1,x_2*y_2)\wedge A_B(y_1,y_2)\wedge 0.5\\&=\left(B(x_1*y_1)\wedge B(x_2*y_2)\right)\wedge\left(B(y_1)\wedge B(y_2)\right)\wedge 0.5\\&=\left(B(x_1*y_1)\wedge B(y_1)\right)\wedge\left(B(x_2*y_2)\wedge B(y_2)\right)\wedge 0.5,\end{aligned}$$

let $x_2=y_2=0$, then

$$B(x_1)\wedge B(0)\geq\left(B(x_1*y_1)\wedge B(y_1)\right)\wedge B(0)\wedge 0.5,$$

if A_B is an $M-(\in,\in\vee q)$-fuzzy ideal of $X\times X$, then

$$A_B\left(m(x,y)\right)\geq A_B(x,y),\forall x,y\in X\times X,m\in M,$$

We have

$$B(mx)\wedge B(my)=A_B(mx,my)\geq A_B(x,y)\wedge 0.5=B(x)\wedge B(y)\wedge 0.5,$$

if $x=0$, then

$$B(0)\wedge B(my)=A_B(0,my)\geq A_B(0,y)\wedge 0.5=B(0)\wedge B(y)\wedge 0.5,$$

namely, $B(my)\geq B(y)\wedge 0.5.$ Therefore B is an $M-(\in,\in\vee q)$-fuzzy ideal of X.

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