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Solutions to Probabilistic Constrained Optimal Control Problems Using Concentration Inequalities

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***Abstract*—**Recently, optimal control problems subject to probabilistic constraints have attracted much attention in many research field. Although probabilistic constraints are generally intractable in optimization problems, several methods haven been proposed to deal with probabilistic constraints. In most methods, probabilistic constraints are transformed to deterministic constraints that are tractable in optimization problems. This paper examines a method for transforming probabilistic constraints into deterministic constraints for a class of probabilistic constrained optimal control problems.

***Keywords*—**Optimal control, stochastic systems, discrete-time systems, probabilistic constraints.

I. Introduction

Model predictive control (MPC), also known as receding horizon control, is one of the most successful control methodologies [1]-[11] because it enables optimization of control performance while taking into account the constraints on state and control variables. Although classical MPC methods do not provide a systematic method to handle uncertain disturbances, recent MPC schemes guarantee constraint fulfillment under uncertain disturbances. The design methods of a robust MPC against uncertain disturbances are classified into deterministic and stochastic approaches.

In the deterministic setting, most studies are based on the min-max approach where the performance index is minimized over the worst possible disturbance scenario [12]-[16]. However, min-max approaches are often computationally demanding, and the control performance is highly conservative because the statistical properties of an occurring disturbance are not taken into account.

The other approach is the stochastic MPC (SMPC) where the expected values of the performance indices and probabilistic constraints are considered by exploiting the statistical information of a disturbance. In the deterministic MPC, the so-called hard constraints, which must hold with a probability of one, are taken into account for the optimization problem. In contrast, the SMPC handles the so-called soft constraints, which cannot be fulfilled with certainty but with a given probability. A small relaxation in the probability requirement is known to sometimes lead to a significant improvement in the achievable control performance. However, probabilistic constraints are generally intractable in an optimization problem. Recently, considerable attention has been devoted to the difficulty related to the SMPC problem. Thus, several tractable methods have been proposed to handle the probabilistic constraints.

In [17]-[18], a second-order cone approximation method was proposed based on the results from robust optimization to solve the stochastic linear-quadratic control problem. In [19]-[21], probabilistically constrained MPC problems were transformed into deterministically constrained MPC problems using a Gaussian assumption. In [22], a SMPC method was proposed, which considered the probabilistic polytopic sets instead of the deterministic bounds of uncertain disturbances. In addition, an alternate method for the convex approximation of probabilistic constraints with polytopic constraint functions was proposed in [23]. In [24], a sampling method that use scenario approximation was proposed for handling arbitrary probability distributions of uncertain disturbances.

Although the aforementioned studies [17]-[24] have achieved tremendous progress in handling the probabilistic constraints of SMPC, several restrictions are imposed on the probability distributions of stochastic disturbances, such as the normal (Gaussian) distribution, known distribution, finite support, and time invariance. On the other hand, the methods proposed in [25]-[29] enable us to address unknown arbitrary probability distributions, including non-Gaussian, infinitely supported, and time-variant distributions, only under the assumption of known expectation and variance in the disturbance. These studies in [25]-[29] aim to provide a SMPC method to successfully deal with probabilistic constraints with a lower computational load. For this purpose, concentration inequalities were applied to transform soft constraints on state variables into hard constraints on control inputs.

In [25]-[27], the Chebyshev's inequality was applied to transform probabilistic constraints on the state variables into deterministic constraints on the control inputs. Moreover, a sufficient condition for the stability of the closed-loop system with SMPC was provided in [26]. The results of computational simulations were provided in [27] to verify the effectiveness of the stability criteria obtained in [26]. In fact, there is a gap between the transformed deterministic constraints in case of known and unknown probability distribution. In [28], the conservativeness of probabilistic constrained optimization method for unknown probability distribution was examined. Therein, a quantitative assessment of the conservatism for tractable constraints in probabilistic constrained optimization with unknown probability distribution was provided. In [29], the Cantelli's inequality, which is a similar concentration inequality to the Chebyshev's inequality, was used to propose a solution method to the SMPC problem. It was shown

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that both the probabilistic component-wise and affine state constraints can be transformed into deterministic constraints on the control input variables using the Chebyshev's and Cantelli's inequalities, respectively.

In this paper, we consider bounded stochastic disturbances and probabilistic affine state constraints. The objective of this study is to provide an approach using the Hoeffding's inequality to solve the SMPC problem under bounded disturbances with unknown probability distributions.

This paper is organized as follows: In Section II, we introduce some notations. In Section III, the system model and stochastic MPC problems are formulated. In Section IV, we provide some preliminary results that are useful to construct the main results. The main results are provided in Section V. Finally, some concluding remarks are given in Section VI.

II. Notation

Let $\mathbb{R}$ and $\mathbb{N}$ denote the sets of real and natural numbers, respectively. Let $\mathbb{R}_+$ denote the set of non-negative real numbers. For matrix A, the transpose and trace of A are denoted by A' and $\mathrm{tr}A$, respectively. Let diag.$\{\cdots\}$ denote a diagonal block matrix. For matrices $A = \{a_{i,j}\}$ and $B = \{b_{i,j}\}$, let the inequalities between A and B, such as $A > B$ and $A \geq B$, indicate that they are component-wise satisfied, i.e., $a_{i,j} > b_{i,j}$ and $a_{i,j} \geq b_{i,j}$ is true for all i and j, respectively. Similarly, let each notation for absolute value $|A|$, square root $\sqrt{A}$, and multiplication $A \circ B$ indicate that it is true component-wise, i.e., $|A| = \{|a_{i,j}|\}$, $\sqrt{A} = \{\sqrt{a_{i,j}}\}$, and $A \circ B = \{a_{i,j} \times b_{i,j}\}$ for all i and j.

Let the triple $(\Omega, \mathcal{F}, \mathcal{P})$ denote a probability space where $\Omega \subseteq \mathbb{R}$ is the sampling space, $\mathcal{F}$ is the σ-algebra, and $\mathcal{P}$ is the probability measure [30]. Ω is non-empty and is not necessarily finite. $\mathcal{P}(E)$ denotes the probability that event E occurs. If $\mathcal{P}(E) = 1$, E almost surely occurs. For random variable $z : \Omega \to \mathbb{R}$ defined by $(\Omega, \mathcal{F}, \mathcal{P})$, let the expected value and variance of z be denoted by $\mathcal{E}(z)$ and $\mathcal{V}(z)$, respectively. For a random vector $z = [z_1, \cdots, z_n]'$, where each of its components is a random variable $z_i : \Omega \to \mathbb{R}$ $(i = 1, \cdots, n)$, which is defined on the same probability space $(\Omega, \mathcal{F}, \mathcal{P})$, we also adopt the same notations $\mathcal{E}(z)$ and $\mathcal{V}(z)$ to denote $c(z) = [\mathcal{E}(z_1), \cdots, \mathcal{E}(z_n)]'$ and $\mathcal{V}(z) = [\mathcal{V}(z_1), \cdots, \mathcal{V}(z_n)]'$ for notational simplicity. Furthermore, covariance matrix $\mathcal{C}_v(z)$ is defined by $\mathcal{C}_v(z) := \mathcal{E}[\{z - \mathcal{E}(z)\}\{z - \mathcal{E}(z)\}']$.

III. Problem Statement

Throughout this paper, we consider the following linear discrete-time system with stochastic disturbances:

$$x(t+1) = Ax(t) + Bu(t) + Cw(t), \tag{1}$$

where $t \in \mathbb{N}$ is the time step, $x(t) : \mathbb{N} \to \mathbb{R}^n$ is the state, $u(t) : \mathbb{N} \to \mathbb{R}^m$ is the control input, and $w(t) : \mathbb{N} \to \mathbb{R}^\ell$ is the unknown stochastic disturbance. More precisely, for each component $w_i : \mathbb{N} \times \Omega \to \mathbb{R}$ of w, the random sequence $\{w_i(t) : t \in \mathbb{N}\}$ is a collection of random variables in the same probability space $(\Omega, \mathcal{F}, \mathcal{P})$ equipped with a filtration $\{\mathcal{F}_t : t \in \mathbb{N}\}$ [30]. The system coefficients $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{n\times \ell}$ are all known as constant matrices. The pair (A, B) is assumed to be controllable. We also assume that the initial state $x(0)$ is given and that all components of state $x(t)$ are deterministically observable. Thus, we assume that $\mathcal{E}(x(t)) = x(t)$ and $\mathcal{V}(x(t)) = 0$ at present time t.

Next, we introduce some assumptions about the properties of the stochastic disturbances.

Assumption 1: $w_i(t)$ and $w_j(t)$ are independent of each other for all $i \neq j$ and $t \in \mathbb{N}$. Also, $w_i(t)$ and $w_j(k)$ are independent of each other for all $t \neq k$ and $j \in \{1, \cdots, \ell\}$.

In fact, most previous studies [17]-[29] typically assumed that random variables are mutually independent as well as Assumption 1. The case where random variables are mutually correlated requires more complicated analysis than the one provided here because $\mathcal{C}_v(w)$ cannot be neglected.

Assumption 2: $\mathcal{E}(w(t))$ and $\mathcal{V}(w(t))$ are assumed to be known for each time t.

Note that the probability distributions of random variables w_i are not necessarily assumed to be known. However, the probability distributions were assumed to be known in previous studies [17]-[24] to transform the soft constraints into hard constraints. In the present study, the assumption related to known probability distributions is relaxed to include arbitrary unknown probability distributions.

Hereafter, we formulate the stochastic optimal control problem of system (1). The control input at each time t is determined to minimize the performance index given by

$$J := \phi[x(t+N)] + \sum_{k=t}^{t+N-1} L[x(k), u(k)]. \tag{2a}$$

Here, $N \in \mathbb{N}$ denotes the length of the prediction horizon. ϕ and L are defined by

$$\phi := \mathcal{E}[x(t+N)'Px(t+N)], \tag{2b}$$

$$L := \mathcal{E}[x(k)'Qx(k)] + u(k)'Ru(k), \tag{2c}$$

where P, Q, and R are positive definite constant matrices. $\phi \in \mathbb{R}_+$ is the terminal cost function, and $L \in \mathbb{R}_+$ is the stage cost function over the prediction horizon.

Let $p(t) = [p_1(t), \cdots, p_n(t)]'$: $\mathbb{N} \to [0\ 1]^n$ denote the probability in vector form, which means that each component $p_i(t)$ belongs to $[0\ 1]$ for each time t.

For notational convenience, let $\mathbf{X} \in \mathbb{R}^{nN}$, $\mathbf{U} \in \mathbb{R}^{mN}$, $\mathbf{W} \in \mathbb{R}^{\ell N}$, $\mathbf{A} \in \mathbb{R}^{nN\times n}$, $\mathbf{B} \in \mathbb{R}^{nN\times mN}$, $\mathbf{C} \in \mathbb{R}^{nN\times \ell N}$, $\mathbf{Q} \in \mathbb{R}^{nN\times nN}$, $\mathbf{R} \in \mathbb{R}^{mN\times mN}$, and $\mathbf{p} \in \mathbb{R}^{nN}$, be defined by

$$\mathbf{X}(t) := \begin{bmatrix} x(t+1) \\ \vdots \\ x(t+N) \end{bmatrix}, \quad \mathbf{U}(t) := \begin{bmatrix} u(t) \\ \vdots \\ u(t+N-1) \end{bmatrix},$$

$$\mathbf{W}(t) := \begin{bmatrix} w(t) \\ \vdots \\ w(t+N-1) \end{bmatrix}, \quad \mathbf{A} := \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix},$$

$$\mathbf{B} := \begin{bmatrix} B & 0 & \cdots & 0 \\ AB & B & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix},$$

$$\mathbf{C} := \begin{bmatrix} C & 0 & \cdots & 0 \\ AC & C & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ A^{N-1}C & A^{N-2}C & \cdots & C \end{bmatrix},$$

$$\mathbf{Q} := \begin{bmatrix} Q & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & Q & 0 \\ 0 & \cdots & 0 & P \end{bmatrix}, \quad \mathbf{R} := \begin{bmatrix} R & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & R \end{bmatrix},$$

$$\mathbf{p}(t) = \begin{bmatrix} p(t+1) \\ \vdots \\ p(t+N) \end{bmatrix}.$$

Using the aforementioned notation, we rewrite the performance index in (2) as:

$$\begin{aligned} J[x(t), \mathbf{X}(t), \mathbf{U}(t)] &= \mathcal{E}[x(t)'Qx(t)] \\ &\quad + \mathcal{E}[\mathbf{X}(t)'\mathbf{Q}\mathbf{X}(t)] + \mathbf{U}(t)'\mathbf{R}\mathbf{U}(t), \end{aligned} \tag{3}$$

In addition, (1) over the prediction horizon can be rewritten as:

$$\mathbf{X}(t) = \mathbf{A}x(t) + \mathbf{B}\mathbf{U}(t) + \mathbf{C}\mathbf{W}(t). \tag{4}$$

Assumption 3: Each element of $x(t)$, $\mathbf{U}(t)$ and $\mathbf{W}(t)$ are assumed to be independent for each time t.

Here, we consider three types of stochastic optimal control problems.

A. SMPC Problem 1

We consider the probabilistic component-wise state constraints and unbounded stochastic disturbances with unknown probability distributions. Let $\underline{x}(t)$ and $\overline{x}(t) : \mathbb{N} \to \mathbb{R}^n$ denote the lower and upper bounds of $x(t)$, respectively. Here, we impose the following probabilistic constraint on the optimization problem: for $k = t+1, \cdots, t+N$ and $i = 1, \cdots, n$,

$$\mathcal{P}\left(\underline{x}_i(k) < x_i(k) < \overline{x}_i(k)\right) \geq p_i(k), \tag{5}$$

where $\underline{x}_i(k)$, $\overline{x}_i(k) \in \mathbb{R}$, and $p_i(k) \in [0\ 1]$ for $k = t + 1, \cdots, t+N$ are given constant sequences and their subscript indicates the ith element of the vector. Condition (5) indicates that state x_i over the prediction horizon must remain within the bound $[\underline{x}_i\ \ \overline{x}_i]$ at least with probability p_i.

Let $\underline{\mathbf{X}} \in \mathbb{R}^{nN}$ and $\overline{\mathbf{X}} \in \mathbb{R}^{nN}$ be defined by:

$$\underline{\mathbf{X}}(t) := \begin{bmatrix} \underline{x}(t+1) \\ \vdots \\ \underline{x}(t+N) \end{bmatrix}, \quad \overline{\mathbf{X}}(t) := \begin{bmatrix} \overline{x}(t+1) \\ \vdots \\ \overline{x}(t+N) \end{bmatrix}.$$

Using the above notation, probabilistic constraint (5) is rewritten in vector form as

$$\mathcal{P}\left(\underline{\mathbf{X}}(t) < \mathbf{X}(t) < \overline{\mathbf{X}}(t)\right) \geq \mathbf{p}(t). \tag{6}$$

More precisely, by using the components: $\underline{\mathbf{X}}_i, \mathbf{X}_i, \overline{\mathbf{X}}_i \in \mathbb{R}$, and $\mathbf{p}_i \in [0\ 1]$ of the vectors, condition (6) can be described as:

$$\bigwedge_{i=1}^{nN} \left\{\mathcal{P}\left(\underline{\mathbf{X}}_i(t) < \mathbf{X}_i(t) < \overline{\mathbf{X}}_i(t)\right) \geq \mathbf{p}_i(t)\right\}, \tag{7}$$

where notation $\wedge$ denotes the logical conjunction.

B. SMPC Problem 2

We consider the probabilistic affine state constraints and unbounded stochastic disturbances with unknown probability distributions. Thus, we impose the following probabilistic constraint on the optimization problem:

$$\mathcal{P}\left(\mathbf{D}\mathbf{X}(t) < \mathbf{h}\right) \geq \mathbf{p}. \tag{8}$$

where $\mathbf{D} \in \mathbb{R}^{\mathbf{s}\times\mathbf{nN}}$, $0 < \mathbf{h} \in \mathbb{R}^s_+$, $\mathbf{p} \in [0\ 1]^s$, and $s \in \mathbb{N}$ are given constant parameters.

C. SMPC Problem 3

We consider the probabilistic affine state constraints and bounded stochastic disturbances with unknown probability distributions. Here, we assume that $\mathbf{U}(t)$ and $\mathbf{W}(t)$ satisfy the following conditions for each time t:

$$\mathbf{F}\mathbf{U}(t) < \overline{\mathbf{U}}, \tag{9a}$$

$$\mathbf{G}\mathbf{W}(t) < \overline{\mathbf{W}}, \tag{9b}$$

where $\mathbf{F} \in \mathbb{R}^{\mathbf{f}\times\mathbf{mN}}$, $\mathbf{G} \in \mathbb{R}^{\mathbf{g}\times\ell\mathbf{N}}$, $\overline{\mathbf{U}} \in \mathbb{R}^f$, $\overline{\mathbf{W}} \in \mathbb{R}^g$, $f \in \mathbb{N}$, and $g \in \mathbb{N}$ are given constant parameters. Then, we impose probabilistic constraint (8) on the optimization problem.

IV. Preliminaries

In this section, we provide some preliminary results that are useful to derive the main results.

The inequality shown below is known as the Chebyshev's inequality.

Lemma 1 ([31]): For any random variable x and positive constant $\kappa \geq 1$, the following inequality holds:

$$\mathcal{P}\left(|x - \mathcal{E}(x)| \geq \kappa\sqrt{\mathcal{V}(x)}\right) \leq \frac{1}{\kappa^2}. \tag{10}$$

The inequality shown below is known as the Cantelli's inequality.

Lemma 2 ([31]): For any random variable x and positive constant $\kappa > 0$, the following inequality holds:

$$\mathcal{P}\left(x - \mathcal{E}(x) \geq \kappa\right) \leq \frac{\mathcal{V}(x)}{\mathcal{V}(x) + \kappa^2} \tag{11}$$

The inequality shown below is known as the Hoeffding's inequality.

Lemma 3 ([31]): Let $x_1, \cdots, x_n$ be independent random variables such that x_i takes its values in $[a_i, b_i]$ almost surely for all $i = 1, \cdots, n$. Then, for every positive constant $\kappa > 0$, the following inequality holds:

$$\mathcal{P}\left(\sum_{i=1}^{n}(x_i - E(x_i)) \geq \kappa\right) \leq \exp\left(-\frac{2\kappa^2}{\sum_{i=1}^{n}(b_i - a_i)^2}\right) \tag{12}$$

Hereafter, we provide the solutions to SMPC problems 1–2. First, we transform the minimization problem of (2) subject to (1) into a quadratic programming problem with respect to the sequence of control inputs over the prediction horizon.

From (4), $\mathcal{E}(\mathbf{X}(t))$ and $\mathcal{V}(\mathbf{X}(t))$ are given by:

$$\mathcal{E}(\mathbf{X}(t)) = \mathbf{A}x(t) + \mathbf{B}\mathbf{U}(t) + \mathbf{C}\mathcal{E}(\mathbf{W}(t)), \tag{13a}$$

$$\mathcal{V}(\mathbf{X}(t)) = (\mathbf{C} \circ \mathbf{C})\mathcal{V}(\mathbf{W}(t)). \tag{13b}$$

In (13a), we apply $\mathcal{E}(x(t)) = x(t)$ because the present state $x(t)$ is a deterministic vector. Moreover, (3) indicates that

$$\begin{aligned} J = {} & x(t)'Qx(t) + \mathbf{U}(t)'\mathbf{R}\mathbf{U}(t) \\ & \mathrm{tr}[Q\mathcal{C}_v(\mathbf{X}(t))] + \mathcal{E}(\mathbf{X}(t))'\mathbf{Q}\mathcal{E}(\mathbf{X}(t)). \end{aligned} \tag{14}$$

Note that covariance matrix $\mathcal{C}_v(\mathbf{X}(t))$ is independent of $\mathbf{U}(t)$:

$$\begin{aligned} \mathcal{C}_v(\mathbf{X}(t)) &= \mathcal{E}\left[\{\mathbf{X}(t) - \mathcal{E}(\mathbf{X}(t))\}\{\mathbf{X}(t) - \mathcal{E}(\mathbf{X}(t))\}'\right] \\ &= \mathcal{E}\left[\{\mathbf{C}\mathbf{W}(t) - \mathbf{C}\mathcal{E}(\mathbf{W}(t))\}\{\mathbf{C}\mathbf{W}(t) - \mathbf{C}\mathcal{E}(\mathbf{W}(t))\}'\right]. \end{aligned}$$

Substituting (13a) into (14) and neglecting the terms that do not contain $\mathbf{U}(t)$, we obtain

$$\begin{aligned} & \min_{\mathbf{U}(t)} J[x(t), \mathbf{X}(t), \mathbf{U}(t)] = \qquad (15) \\ & \min_{\mathbf{U}(t)} \left\{ \begin{array}{l} \mathbf{U}'(t)\left(\mathbf{B}'\mathbf{Q}\mathbf{B} + \mathbf{R}\right)\mathbf{U}(t) \\ \quad +2\left(\mathbf{A}x(t) + \mathbf{C}\mathcal{E}(\mathbf{W}(t))\right)'\mathbf{Q}\mathbf{B}\mathbf{U}(t) \end{array} \right\}. \end{aligned}$$

Note that the minimization problem of J in (2) subject to (1) has been reduced to a quadratic programming problem with respect to $\mathbf{U}$.

In general, solving the quadratic programming problem with probabilistic constraints is not straightforward. In [26] and [29], the methods for solving SMPC problems 1–2 were provided. The probabilistic constraints were converted into deterministic constraints using the concentration inequalities in Lemmas 1–2.

A. Solution to SMPC Problem 1

The following proposition has been proved in [26].

Proposition 1 ([26]): Suppose that the following condition holds:

$$\mathbf{U}_{\min}(t) \leq \mathbf{B}\mathbf{U}(t) \leq \mathbf{U}_{\max}(t), \tag{16}$$

where $\mathbf{U}_{\min}$ and $\mathbf{U}_{\max}$ are defined by:

$$\begin{aligned} \mathbf{U}_{\min}(t) := {} & \underline{\mathbf{X}}(t) + \boldsymbol{\kappa}(t) \circ \sqrt{(\mathbf{C} \circ \mathbf{C})\mathcal{V}(\mathbf{W}(t))} \\ & - \mathbf{A}x(t) - \mathbf{C}\mathcal{E}(\mathbf{W}(t)), \end{aligned} \tag{17a}$$

$$\begin{aligned} \mathbf{U}_{\max}(t) := {} & \overline{\mathbf{X}}(t) - \boldsymbol{\kappa}(t) \circ \sqrt{(\mathbf{C} \circ \mathbf{C})\mathcal{V}(\mathbf{W}(t))} \\ & - \mathbf{A}x(t) - \mathbf{C}\mathcal{E}(\mathbf{W}(t)). \end{aligned} \tag{17b}$$

$$\boldsymbol{\kappa}(t) := \left[\frac{1}{\sqrt{1 - \mathbf{p}_1(t)}}, \cdots, \frac{1}{\sqrt{1 - \mathbf{p}_{nN}(t)}}\right]'. \tag{17c}$$

Then, the probabilistic condition (6) is fulfilled.

Remark 1: From Proposition 1, the minimization problem of (15) with probabilistic constraint (6) is reduced to a quadratic programming problem with deterministic constraint (16), which can be solved using a conventional algorithm [32].

Remark 2: Suppose that we impose not only probabilistic state constraint (6) but also control input constraint (9a) on the optimization problem. Then, the optimization problem can be reduced to a quadratic programming problem (15) subject to the following constraint:

$$\begin{bmatrix} -\mathbf{B} \\ \mathbf{B} \\ \mathbf{F} \end{bmatrix} \mathbf{U}(t) \leq \begin{bmatrix} \mathbf{U}_{\min} \\ \mathbf{U}_{\max} \\ \overline{\mathbf{U}} \end{bmatrix}. \tag{18}$$

Solving quadratic programming problem (15) subject to constraint (18) is also straightforward using a conventional algorithm [32].

B. Solution to SMPC Problem 2

The following proposition has been proved in [29].

Proposition 2 ([29]): Suppose that the following condition holds:

$$\mathbf{D}\mathbf{B}\mathbf{U}(t) \leq \mathbf{h} - \mathbf{D}\left(\mathbf{A}x(t) + \mathbf{C}\mathcal{E}(\mathbf{W}(t))\right) - \mathbf{V}(t), \tag{19}$$

where the ith element of $\mathbf{V} \in \mathbb{R}^s$ is given by:

$$\mathbf{V}_i = \sqrt{\frac{\mathbf{p}_i}{1 - \mathbf{p}_i}\left((\mathbf{DC} \circ \mathbf{DC})\mathcal{V}(\mathbf{W}(t))\right)_i}. \tag{20}$$

Then, the probabilistic condition (8) is fulfilled.

Remark 3: From Proposition 2, the minimization problem of (15) with probabilistic constraint (8) is reduced to a quadratic programming problem with deterministic constraint (19), which can be solved using a conventional algorithm [32].

Remark 4: Consider an example of affine state constraint (8) as shown below:

$$\mathcal{P}\left(\sum_{i=1}^{nN}\mathbf{X}_i < h\right) \geq p. \tag{21}$$

In contrast, the following condition is an example of component-wise state constraint (6).

$$\mathcal{P}\left(\mathbf{X} < \overline{\mathbf{X}}\right) \geq \mathbf{p}. \tag{22}$$

Suppose that $\overline{\mathbf{X}}_i$ is given by: $\overline{\mathbf{X}}_i = h/nN$ for all $i = 1, \cdots, nN$. Then, we have the following condition:

$$\begin{aligned} & \mathcal{P}\left(\sum_{i=1}^{nN}\mathbf{X}_i < h\right) \geq \beta, \qquad (23) \\ & \beta := \prod_{i=1}^{nN}\mathbf{p}_i. \end{aligned}$$

Comparing (21) with (23), we can see that β in (23) takes an underestimated value compared with p in (21). This is the advantage of applying probabilistic constraint (8) rather than (6).

V. Main Results

In this section, we provide the main results that are useful to solve SMPC problem 3.

A. Solution to SMPC Problem 3

Let $\mathbf{U}^*$ be the solution of the minimization problem of (15) subject to (9a). Because $\mathbf{W}(t)$ is bounded to satisfy (9b), $\mathbf{X}(t)$ is also bounded. In fact, the lower and upper bounds ($\underline{\mathbf{X}}_i$ and $\overline{\mathbf{X}}_i$, respectively) of $\mathbf{X}_i$ are obtained by solving the linear programming problems as follows: For given $x(t)$ and $\mathbf{U}^*$,

$$\underline{\mathbf{X}}_i(t) = \min_{\mathbf{W}(t)} \left(\mathbf{A}x(t) + \mathbf{BU}^* + \mathbf{CW}(t)\right)_i, \quad (24)$$

$$\overline{\mathbf{X}}_i(t) = \max_{\mathbf{W}(t)} \left(\mathbf{A}x(t) + \mathbf{BU}^* + \mathbf{CW}(t)\right)_i, \quad (25)$$

subject to (9b).

Thus, $\mathbf{X}$ is bounded as $\underline{\mathbf{X}} \le \mathbf{X} \le \overline{\mathbf{X}}$. Using this property, we can state the following theorem:

Theorem 1: Suppose that the following condition holds:

$$\mathbf{DBU}(t) \le \mathbf{h} - \mathbf{D}\left(\mathbf{A}x(t) + \mathbf{C}\mathcal{E}(\mathbf{W}(t))\right) - \mathbf{Z}(t), \quad (26)$$

where the ith element of $\mathbf{Z} \in \mathbb{R}^s$ is given by:

$$\mathbf{Z}_i = \sqrt{\frac{-\log(1-\mathbf{p}_i)}{2}\left((\mathbf{D}\circ\mathbf{D})(\overline{\mathbf{X}} - \underline{\mathbf{X}}) \circ (\overline{\mathbf{X}} - \underline{\mathbf{X}})\right)_i}. \quad (27)$$

Then, the probabilistic condition (8) is fulfilled.

Proof: Using Lemma 3, we have the following inequality in the component-wise form:

$$\mathcal{P}\left((\mathbf{DX})_i - \mathcal{E}(\mathbf{DX})_i \ge \kappa\right) \le \exp\left(\frac{-2\kappa^2}{\left((\mathbf{D}\circ\mathbf{D})(\overline{\mathbf{X}} - \underline{\mathbf{X}}) \circ (\overline{\mathbf{X}} - \underline{\mathbf{X}})\right)_i}\right). \quad (28)$$

Accordingly, we have the following inequality:

$$\mathcal{P}\left((\mathbf{DX})_i - \mathcal{E}(\mathbf{DX})_i < \kappa\right) \le 1 - \exp\left(\frac{-2\kappa^2}{\left((\mathbf{D}\circ\mathbf{D})(\overline{\mathbf{X}} - \underline{\mathbf{X}}) \circ (\overline{\mathbf{X}} - \underline{\mathbf{X}})\right)_i}\right). \quad (29)$$

Suppose that the following equation holds:

$$\mathbf{p}_i = 1 - \exp\left(\frac{-2\kappa^2}{\left((\mathbf{D}\circ\mathbf{D})(\overline{\mathbf{X}} - \underline{\mathbf{X}}) \circ (\overline{\mathbf{X}} - \underline{\mathbf{X}})\right)_i}\right). \quad (30)$$

Then, it follows from (30) that

$$\kappa = \mathbf{Z}_i. \quad (31)$$

Consequently, we have the following inequality:

$$\mathcal{P}\left((\mathbf{DX})_i - \mathcal{E}(\mathbf{DX})_i < \mathbf{Z}_i\right) \ge \mathbf{p}_i. \quad (32)$$

For notational simplicity, we rewrite inequality (32) in a vector form, i.e.,

$$\mathcal{P}\left(\mathbf{DX}(t) < \mathbf{D}\mathcal{E}(\mathbf{X}(t)) + \mathbf{Z}(t)\right) \ge \mathbf{p}. \quad (33)$$

Note that if the following condition:

$$\mathbf{D}\mathcal{E}(\mathbf{X}(t)) + \mathbf{V}(t) \le \mathbf{h} \quad (34)$$

is satisfied, then the probabilistic condition (8) is fulfilled. Substituting (13a) into (34), we can see that the condition (34) is equivalent to condition (26). Therefore, we conclude that if deterministic constraint (26) on $\mathbf{U}(t)$ is satisfied, then the probabilistic constraint (8) on $\mathbf{X}(t)$ is also satisfied. This completes the proof. ■

Remark 5: Note that condition (26) can be solved even if variance $\mathcal{V}(\mathbf{W})$ is unknown. This is the advantage of condition (26).

Remark 6: We provide a quantitative assessment of the conservatism between conditions (19) and (26). Subtracting the right-hand side of (19) from that of (26) yields

$$\mathbf{V}_i - \mathbf{Z}_i = \sqrt{\frac{\mathbf{p}_i}{1-\mathbf{p}_i}\left((\mathbf{DC}\circ\mathbf{DC})\mathcal{V}(\mathbf{W}(t))\right)_i} - \sqrt{\frac{-\log(1-\mathbf{p}_i)}{2}\left((\mathbf{D}\circ\mathbf{D})(\overline{\mathbf{X}} - \underline{\mathbf{X}}) \circ (\overline{\mathbf{X}} - \underline{\mathbf{X}})\right)_i}. \quad (35)$$

Let γ and η be defined by:

$$\gamma := \sqrt{\frac{\mathbf{p}_i}{1-\mathbf{p}_i}}, \qquad \eta := \sqrt{\frac{-\log(1-\mathbf{p}_i)}{2}}.$$

Noting that

$$0 \le \frac{\eta}{\gamma} < 1 \quad \text{for} \quad 0 \le \mathbf{p}_i \le 1, \quad (36)$$

we have the following:

$$\mathbf{V}_i - \mathbf{Z}_i = \gamma\frac{\mathbf{V}_i}{\gamma} - \left(\frac{\eta}{\gamma}\right)\gamma\frac{\mathbf{Z}_i}{\eta}, > \gamma\left(\frac{\mathbf{V}_i}{\gamma} - \frac{\mathbf{Z}_i}{\eta}\right). \quad (37)$$

Here, we suppose that

$$|\mathbf{C}|\sqrt{\mathcal{V}(\mathbf{W})} \ge \overline{\mathbf{X}} - \underline{\mathbf{X}}. \quad (38)$$

Then, it follows from (37) that

$$\mathbf{V}_i - \mathbf{Z}_i > 0. \quad (39)$$

Consequently, we can see that condition (26) is less conservative than (19) in the case of (38).

VI. Conclusion

In this study, we have examined a MPC design method for linear discrete-time systems with additive stochastic disturbances under probabilistic constraints. The advantage of the proposed method is its applicability to stochastic disturbances with unknown probability distribution. Concentration inequalities were applied to successfully handle probabilistic constraints with a lower computational load. Thus, the SMPC problem with probabilistic constraints was reduced to a quadratic programming problem with deterministic constraints, which can be solved using a conventional algorithm. The feasibility and stability analyses based on the proposed method are possible future research areas.

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