# Stable Time Reversed Integration of the Navier-Stokes Equation Using an Adjoint Gradient Method

**Authors :** Jurriaan Gillissen

**Abstract :** This work is concerned with stabilizing the numerical integration of the Navier-Stokes equation (NSE), backwards in time. Applications involve the detection of sources of, e.g., sound, heat, and pollutants. Stable reverse numerical integration of parabolic differential equations is also relevant for image de-blurring. While the literature addresses the reverse integration problem of the advection-diffusion equation, the problem of numerical reverse integration of the NSE has, to our knowledge, not yet been addressed. Owing to the presence of viscosity, the NSE is irreversible, i.e., when going backwards in time, the fluid behaves, as if it had a negative viscosity. As an effect, perturbations from the perfect solution, due to round off errors or discretization errors, grow exponentially in time, and reverse integration of the NSE is inherently unstable, regardless of using an implicit time integration scheme. Consequently, some sort of filtering is required, in order to achieve a stable, numerical, reversed integration. The challenge is to find a filter with a minimal adverse affect on the accuracy of the reversed integration. In the present work, we explore an adjoint gradient method (AGM) to achieve this goal, and we apply this technique to two-dimensional (2D), decaying turbulence. The AGM solves for the initial velocity field u0 at t = 0, that, when integrated forward in time, produces a final velocity field u1 at t = 1, that is as close as is feasibly possible to some specified target field v1. The initial field u0 defines a minimum of a cost-functional J, that measures the distance between u1 and v1. In the minimization procedure, the u0 is updated iteratively along the gradient of J w.r.t. u0, where the gradient is obtained by transporting J backwards in time from t = 1 to t = 0, using the adjoint NSE. The AGM thus effectively replaces the backward integration by multiple forward and backward adjoint integrations. Since the viscosity is negative in the adjoint NSE, each step of the AGM is numerically stable. Nevertheless, when applied to turbulence, the AGM develops instabilities, which limit the backward integration to small times. This is due to the exponential divergence of phase space trajectories in turbulent flow, which produces a multitude of local minima in J, when the integration time is large. As an effect, the AGM may select unphysical, noisy initial conditions. In order to improve this situation, we propose two remedies. First, we replace the integration by a sequence of smaller integrations, i.e., we divide the integration time into segments, where in each segment the target field v1 is taken as the initial field u0 from the previous segment. Second, we add an additional term (regularizer) to J, which is proportional to a high-order Laplacian of u0, and which dampens the gradients of u0. We show that suitable values for the segment size and for the regularizer, allow a stable reverse integration of 2D decaying turbulence, with accurate results for more then O(10) turbulent, integral time scales.

**Keywords :** time reversed integration, parabolic differential equations, adjoint gradient method, two dimensional turbulence
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