World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:8, No:11, 2014

The Relative Efficiency of Parameter Estimation in Linear Weighted Regression

Baoguang Tian, Nan Chen

***Abstract*—**A new relative efficiency in linear model in reference is instructed into the linear weighted regression, and its upper and lower bound are proposed. In the linear weighted regression model, for the best linear unbiased estimation of mean matrix respect to the least-squares estimation, two new relative efficiencies are given, and their upper and lower bounds are also studied.

***Keywords*—**Linear weighted regression, Relative efficiency, Mean matrix, Trace.

I. Introduction

CONSIDERING the linear weighted regression model:

$$\begin{cases} WY = WX\beta + \varepsilon \\ E(\varepsilon)=0 \\ Cov(\varepsilon)=\Sigma \end{cases} \quad (1)$$

where Y is the $n\times 1$ observation vector, X is the $n\times p$ full column rank design matrix which we are known, β is the $p\times 1$ unknown parameter vector, ε is the $n\times 1$ observation vector, Σ is the $n\times n$ positive definite covariance matrix, $W = diag(w_1, w_2, ...w_n)$, $w_i \neq 0, i = 1, 2...n$ are constants.

There are two kinds of estimate class commonly used of parameter β:

One is the best linear unbiased estimation (BLUE), that is: when Σ is known, the best linear unbiased estimation (BLUE) of β in the model (1) is: $\beta^*=(X'W\Sigma^{-1}WX)^{-1}X'W\Sigma^{-1}WY$ and $\mu^{\bullet} = WX(X'W\Sigma^{-1}WX)^{-1}X'W\Sigma^{-1}WYX'W$, the covariance matrix is $Cov(\beta^*)=\sigma^2(X'W\Sigma^{-1}WX)^{-1}$;

Another is the least-squares estimation (LSE), that is: $\hat{\beta} = (X'W^2X)^{-1}X'W^2Y$ and $\hat{\mu} = WX(X'W^2X)^{-1}X'W^2YX'W$, the covariance matrix is $Cov(\hat{\beta})=\sigma^2(X'W^2X)^{-1}X'W\Sigma W$ $X(X'W^2X)^{-1}$. Where μ is the sequence characteristics root of W^2X.

When n is very large, the calculation of Σ^{-1} is very complicated, or people tend to use the LSE $\hat{\beta}$ instead of the BLUE β^* of β when Σ is unknown. By the theorem of Gauss-Markov: $Cov(\beta^*) < Cov(\hat{\beta})$. That is $\sigma^2(X'W\Sigma^{-1}WX)^{-1}$ $< \sigma^2(X'W^2X)^{-1}X'W\Sigma WX(X'W^2X)^{-1}$ .It will bring some losses to the estimation when uses the LSE $\hat{\beta}$ instead of the BLUE β^* of β. Because of that, the relative efficiency is cited. Following are commonly used [1]–[3]: $e_1(\hat{\beta}) = \frac{|Cov(\beta^*)|}{|Cov(\hat{\beta})|}$, $e_2(\hat{\beta}) = \frac{tr(Cov\beta^*)}{tr(Cov\hat{\beta})}$, $e_3(\hat{\beta}) = \frac{\|Cov\beta^*\|}{\|Cov\hat{\beta}\|}$ where $|A|$ means the determinant of A, trA means the trace of A, $\|A\|$ means the Euclidean mode of A.

However, the above three kinds of relative efficiency all have their drawbacks. The degree that $e_1(\hat{\beta})$ depends on the matrix X is too low, $e_2(\hat{\beta})$ does not consider the resulting effect of each component covariance, though the degree that $e_2(\hat{\beta})$ depends on the matrix X is improved. The sensitivity of $e_3(\hat{\beta})$ is no better than $e_1(\hat{\beta})$, though it measures the size of deviation arising from covariance and variance between the various components of LSE and BLUE.

So that, H.S. Liu et al. had introduced a new relative efficiency [4]: $e_4(\hat{\beta}) = [\frac{tr(Cov\beta^*)^q}{tr(Cov\hat{\beta})^q}]^{\frac{1}{q}}$, and also discussed the relationship between this new relative efficiency and other three existing relative efficiency. This means that, the dependence $e_4(\hat{\beta})$ on the matrix X is still too low despite the increase of the sensitivity. X.M. Liu et al. have defined another new efficiency in the linear model [8]: $e_5(\hat{\beta}) = \min_{1\le i\le p}\frac{\lambda_i(Cov\hat{\beta})}{\lambda_i(Cov\beta)}$. This paper could push it into the linear weighted regression model and study its lower bound.

In addition, this paper has defined two new relative efficiencies of the mean value matrix μ from another perspective: $e_6(\hat{\mu}/\mu^*) = tr(Cov\hat{\mu} - Cov\mu^*)$, $e_7(\hat{\mu}/\mu^*) = [tr(Cov\hat{\mu} - Cov\mu^*)^q]^{\frac{1}{q}}$, and their upper and lower bounds have given. In the last, it has discussed the relationship between the several kinds of relative efficiencies.

II. The Upper and Lower Bounds of $e_5(\beta)$

Lemma 1: Assume A, B is n order real symmetric matrix and $B \ge 0$, If $A > B$, then $\lambda_i(A) > \lambda_i(B)$, $i = 1, 2, \cdots, n$. [5]

The research is supported by the Natural Science Foundation of Shandong Province, China (No.ZR2010AL018)

Nan Chen and Baoguang Tian are with Mathematics and Physics College of Qingdao University of Science and Technology, China (e-mail: 601933395@qq.com, tianbaoguangqd@163.com, respectively).

Lemma 2: Assume A, B is n order symmetric matrix, there is $\lambda_n(B)\lambda_i(A^2) \le \lambda_i(ABA) \le \lambda_1(B)\lambda_i(A^2)$, where $\lambda_i(A)$ means the i characteristic root of the matrix A. [5]

Lemma 3: Assume A is the $n\times n$ Hermit matrix, U is the $n\times k$ column orthogonal array, that is $U'U = I_k$, then $\lambda_{n-k+i}(A) \le \lambda_i(U'AU) \le \lambda_i(A)$, $i = 1, 2, \cdots, k$. [5]

Lemma 4: In the model of (1), for any two unbiased estimation $\tilde{\beta}_1$, $\tilde{\beta}_2$ of β, if $Cov(\tilde{\beta}_1) > Cov(\tilde{\beta}_2) \ge 0$, then $e_5(\tilde{\beta}_1) < e_5(\tilde{\beta}_2)$. [7]

Theorem 1: In the model of (1), β_1^*, β_2^* is any two unbiased estimation of β, if $Cov(\beta_1^*) > Cov(\beta_2^*) \ge 0$, then there is $e_5(\beta_1^*) < e_5(\beta_2^*)$.

Proof: When $Cov(\beta_1^*) > Cov(\beta_2^*) \ge 0$, By the theorem 1: $\lambda_i(Cov(\beta_1^*)) > \lambda_i(Cov(\beta_2^*)) > 0$. Then it is believed that $\frac{\lambda_i(Cov(\beta^*))}{\lambda_i(Cov(\beta_1^*))} < \frac{\lambda_i(Cov(\beta^*))}{\lambda_i(Cov(\beta_2^*))}$, it can obtained that $\min_{1\le i\le p}\frac{\lambda_i(Cov(\beta^*))}{\lambda_i(Cov(\beta_1^*))} < \min_{1\le i\le p}\frac{\lambda_i(Cov(\beta^*))}{\lambda_i(Cov(\beta_2^*))}$, that is $e_5(\beta_1^*) < e_5(\beta_2^*)$.

The theorem 1 shows that, the increaser of $Cov(\hat{\beta})$, the more decrease of $e_5(\hat{\beta})$, Which shows that the greater loss of $\hat{\beta}$ takes the place of β^*; Otherwise, the greater of $Cov(\hat{\beta})$, the smaller of the loss. According to the theorem Gauss-Markov, for any linear unbiased estimation of β, there is $Cov(\beta^*) < Cov(\hat{\beta})$, then the upper bound of $e_5(\beta)$ is 1.

Theorem 2: In the model of (1), $\frac{\lambda_n(\Sigma)}{\lambda_1(\Sigma)} \le e_5(\hat{\beta}) \le 1$. Where $\lambda_i = \lambda_i(\Sigma), \mu_i = \mu_i(\Lambda)$.

Proof: In the model of (1), Assume $WX = P\Lambda Q$, that is the singular value decomposition of WX. Where $P'P = I_p Q'Q = QQ' = I_p \Lambda = diag\left(\sqrt{\mu_1}, \cdots, \sqrt{\mu_p}\right)$,

$$\begin{aligned}&\frac{\lambda_i\left(X'W\Sigma^{-1}WX\right)^{-1}}{\sigma^2}\\ &= \lambda_i\left(Q'\Lambda P'\Sigma^{-1}P\Lambda Q\right)^{-1},\\ &= \lambda_i\left(\Lambda P'\Sigma^{-1}P\Lambda\right)^{-1}\\ &= \lambda_{p-i+1}^{-1}\left(\Lambda P'\Sigma^{-1}P\Lambda\right)\end{aligned}$$

by the lemma 2 and the lemma 3:

$$\lambda_i\left(X'W\Sigma^{-1}WX\right)^{-1} \ge \lambda_n\mu_{p-i+1}^{-1} \tag{2}$$

and because

$$\begin{aligned}&\frac{\lambda_i\left(Cov\left(\hat{\beta}\right)\right)}{\sigma^2}\\ &= \lambda_i\left(X'W^2X\right)^{-1}X'W\Sigma WX\left(X'W^2X\right)^{-1}\\ &= \lambda_i\left[\left(Q'\Lambda P'P\Lambda Q\right)^{-1}Q'\Lambda P'\Sigma P\Lambda Q\left(Q'\Lambda P'P\Lambda Q\right)^{-1}\right]\\ &= \lambda_i\left(\Lambda^{-1}P'\Sigma P\Lambda^{-1}\right)\\ &\le \lambda_i\left(\Lambda^{-2}\right)\lambda_1\left(P'\Sigma P\right)\\ &\le \lambda_i\left(\Lambda^{-2}\right)\lambda_1(\Sigma)\\ &= \mu_{p-i+1}^{-1}\lambda_1(\Sigma)\end{aligned} \tag{3}$$

It is obtained by (2) and (3): $\frac{\lambda_i(Cov(\beta^*))}{\lambda_i(Cov(\hat{\beta}))} \ge \frac{\lambda_n(\Sigma)}{\lambda_1(\Sigma)}$.

According to the theorem Gauss-Markov, it is believed: $Cov(\beta^*) < Cov(\hat{\beta})$. And by the lemma 1, it achieves that: $\lambda_i Cov(\beta^*) < \lambda_i Cov(\hat{\beta})$, then it can obtain easily that: $e_5(\hat{\beta}) \le 1$. The theorem 2 has been proved.

In addition, according to the theorem Gauss-Markov, it is believed that for any unbiased estimation $\tilde{\beta}$ of β, the greater the deviation, the smaller relative efficiency of the estimation.

Theorem 3: $e_1(\hat{\beta}) = 1 \Leftrightarrow e_2(\hat{\beta}) = 1 \Leftrightarrow e_3(\hat{\beta}) = 1 \Leftrightarrow e_5(\hat{\beta}) = 1$

Proof: Assume $Cov(\tilde{\beta}) = A$, $Cov(\hat{\beta}) = B$, then $B \ge A > 0$. Thus it is believed that: $\lambda_i(B) > \lambda_i(\mathrm{A})$, $i = 1, \cdots, p$. The following are obtained:

$$|A| = |B| \Leftrightarrow \prod_{i=1}^{p}\lambda_i(A) = \prod_{i=1}^{p}\lambda_i(\mathrm{B}) \Leftrightarrow \lambda_i(A) = \lambda_i(\mathrm{B}),$$

$$\|A\| = \|B\| \Leftrightarrow \sum_{i=1}^{p}\lambda_i^2(A) = \sum_{i=1}^{p}\lambda_i^2(\mathrm{B}) \Leftrightarrow \lambda_i(A) = \lambda_i(\mathrm{B}),$$

$$trA^q = trB^q \Leftrightarrow \sum_{i=1}^{p}\lambda_i^q(A) = \sum_{i=1}^{p}\lambda_i^q(\mathrm{B}) \Leftrightarrow \lambda_i(A) = \lambda_i(\mathrm{B}).$$

The theorem 3 has been proved.

III. The Upper and Lower Bounds of $e_6(\hat{\mu}/\mu^*)$

Lemma 5: Assume A, B is n order positive definite matrix, then $\lambda_n(B)tr(A) \le trAB \le \lambda_1(B)tr(A)$, where $\lambda_1(B)$, $\lambda_n(B)$ means the maximum and minimum characteristic root of the matrix B respectively. [5]

Lemma 6: Assume $V_i = \lambda_i[U'AU - (U'A^{-1}U)^{-1}]$, U is the $n\times k$ column orthogonal array, and $1 \le m \le k$, then $\sum_{i=1}^{m}V_i \le \sum_{i=1}^{\min(m,n-k)}(\sqrt{\theta_i} - \sqrt{\theta_{n-i+1}})^2$, where θ_i is the sequence characteristics root of A, and $i = 1, 2, \cdots, n$. [6]

Lemma 7: Assume A is the $n \times n$ Hermit matrix, $\lambda_1, \lambda_2, \cdots, \lambda_n$ is the characteristic value of A, and $\lambda_1 \geq ... \geq \lambda_n$, X is the $n \times p$ order matrix and meets $X'X = I_p$, then $\sum_{i=n-p+1}^{n} \lambda_i \leq trX'AX$. When $X = (\varphi_{n-p+1}, ...\varphi_n)$, the equal sign is established. [5]

Lemma 8: Assume A is the $n \times n$ positive definite Hermit matrix, $\lambda_1, \lambda_2, \cdots, \lambda_n$ is the characteristic value of A, and $\lambda_1 \geq ... \geq \lambda_n$, then for all of the matrix X which meets $X'X = I_p$, there is $tr(X'AX)^{-1} \leq \sum_{i=1}^{p} \lambda_{n-p+i}^{-1}$ When $X = (\varphi_{n-p+1}, ..., \varphi_n)$, the equal sign is established. [5]

Lemma 9: Assume A, B is n order symmetrical matrix, there is $\lambda_n(B)\lambda_i(A) \leq \lambda_i(AB) \leq \lambda_1(B)\lambda_i(A)$, where $\lambda_i(A)$ means the i characteristic root of the matrix A, where $i = 1, 2, ...n$. [5]

Theorem 4: In the model of (1), if $r(X) = p$ then

$$\begin{aligned} &\frac{1}{\omega_1^2}\left(\sum_{i=n-p+1}^{n} \mu_i - \sum_{i=1}^{p} \mu_{n-p+i}^{-1}\right) \leq e_6 \\ &\leq \frac{1}{\omega_n^2} \sum_{i=1}^{\min(m,n-p)} (\sqrt{\mu_i} - \sqrt{\mu_{n-i+1}})^2 \end{aligned} \tag{4}$$

where μ_i, $i = 1, 2, ..., n$ is the sequence characteristics root of W^2X, and there is $1 \leq m \leq p$.

Proof: By the following formula:

$Cov(\mu^*) = WX(X'W\Sigma^{-1}WX)^{-1}X'W$

$Cov(\hat{\mu}) = WX(X'W^2X)^{-1}X'W\Sigma WX(X'W^2X)^{-1}X'W$

It can obtain that:

$tr(Cov\hat{\mu} - Cov\mu^*)$

$= tr[W^{-1}WX(X'W^2X)^{-1}WX'W\Sigma WWX(X'W^2X)^{-1}$

$X'W - W^{-1}WX(X'WW^{-1}\Sigma^{-1}W^{-1}WX)^{-1}X'WW^{-1}]$

Make $WX = S$, $W\Sigma W = T$, $P_S = S(S'S)^{-1}S'$, $R = P_S T P_S - S(S'T^{-1}S)^{-1}S'$, then $tr(Cov\hat{\mu} - Cov\mu^*) = tr(W^{-1}RW^{-1})$. By the lemma 5:

$$\lambda_n(W^{-2})tr(R) \leq e_5(\hat{\mu}/\mu^*) \leq \lambda_1(W^{-2})tr(R) \tag{5}$$

Make $Q = P_S(P_S'P_S)^{-\frac{1}{2}}$, then $Q'Q = I_p$, and

$trR = tr[P_S T P_S - S(S'T^{-1}S)^{-1}S']$

$= tr[QQ'TQQ' - Q(Q'T^{-1}Q)^{-1}Q']$

$= tr[Q'TQ - (Q'T^{-1}Q)^{-1}]$

By the lemma 6:

$$trR \leq \sum_{i=1}^{\min(m,n-p)} (\sqrt{\mu_i} - \sqrt{\mu_{n-i+1}})^2 \tag{6}$$

Insert (6) into the right equation of (5):

$$e_6 \leq \frac{1}{\omega_n^2} \sum_{i=1}^{\min(m,n-p)} (\sqrt{\mu_i} - \sqrt{\mu_{n-i+1}})^2 \tag{7}$$

By the lemma 7 and the lemma 8:

$$trR \leq \sum_{i=n-p+1}^{n} \mu_i - \sum_{i=1}^{p} \mu_{n-p+i}^{-1} \tag{8}$$

Insert (8) into the left equation of (5):

$$\frac{1}{\omega_1^2}\left(\sum_{i=n-p+1}^{n} \mu_i - \sum_{i=1}^{p} \mu_{n-p+i}^{-1}\right) \leq e_5 \tag{9}$$

Combined (7) with (9), the theorem 1 has been proved.

Inference 1: In the model of (1), if $r(X) = p$, then

$$\begin{aligned} &\frac{1}{\omega_1^2}\left(\sum_{i=n-p+1}^{n} \omega_n^2 \lambda_i - \sum_{i=1}^{p} \omega_n^{-2} \lambda_{n-p+i}^{-1}\right) \leq e_6 \\ &\leq \frac{1}{\omega_n^2} \sum_{i=1}^{\min(m,n-p)} (\sqrt{\lambda_i}\omega_1 - \sqrt{\lambda_{n-i+1}}\omega_n)^2 \end{aligned}$$

where λ_i is the sequence characteristics root of X, and $1 \leq m \leq p$, $i = 1, 2, ...n$.

Proof: Because $\mu_i = \lambda_i(W\Sigma W)$, by the lemma 5:

$$\omega_n^2 \lambda_i \leq \lambda_i(W^2\Sigma) \leq \omega_1^2 \lambda_i \tag{10}$$

$$\omega_n^2 \lambda_{n-i+1} \leq \lambda_{n-i+1}(W^2\Sigma) \leq \omega_1^2 \lambda_{n-i+1} \tag{11}$$

Insert (10) into (4),

$$\frac{1}{\omega_1^2}\left(\sum_{i=n-p+1}^{n} \omega_n^2 \lambda_i - \sum_{i=1}^{p} \omega_n^{-2} \lambda_{n-p+i}^{-1}\right) \leq \frac{1}{\omega_1^2}\left(\sum_{i=n-p+1}^{n} \mu_i - \sum_{i=1}^{p} \mu_{n-p+i}^{-1}\right)$$

Insert (11) into (4),

$$\frac{1}{\omega_n^2} \sum_{i=1}^{\min(m,n-p)} (\sqrt{\mu_i} - \sqrt{\mu_{n-i+1}})^2 \leq \frac{1}{\omega_n^2} \sum_{i=1}^{\min(m,n-p)} (\sqrt{\lambda_i}\omega_1 - \sqrt{\lambda_{n-i+1}}\omega_n)^2$$

The inference 1 has been proved.

IV. The Upper and Lower Bounds of $e_7(\hat{\mu}/\mu^*)$

Lemma 10: Assume D is the n order positive definite matrix, then $n^{1-p}(trD)^p \leq trD^p \leq (trD)^p$, $p \geq 1$. [9]

Proof: When $p = 1$, the equal sign in the inequality is established. The following we can proof that when $p > 1$, the Lemma 10 is founded.

Assume $\lambda_1, \lambda_2, \cdots, \lambda_n$ is the characteristic value of A, and $\lambda_1 \geq ... \geq \lambda_n$, then because of the Hoder inequality [7], it means that: $trA = \sum_{i=1}^{n} \lambda_i \leq \left| \sum_{i=1}^{n} \lambda_i^n \right|^{1/p} \left| \sum_{i=1}^{n} 1^{p/(p-1)} \right|^{(p-1)/p} = (trA^p)^{1/p} p^{1-1/p}$.

That is $trA^p \geq n^{1-p}(trA)^p$, and because $(trA)^p = (\sum_{i=1}^{n} \lambda_i)^p \geq \sum_{i=1}^{n} \lambda_i{}^p = trA^p$. Then $n^{1-p}(trA)^p \leq trA^p \leq (trA)^p$, $p \geq 1$.

The lemma 10 has been proved.

Theorem 5: In the model of (1), if $r(X) = p$, then

$$\frac{n^{\frac{1}{q}-1}}{\omega_1^2}\left(\sum_{i=n-p+1}^{n} \mu_i - \sum_{i=1}^{p} \mu_{n-p+i}^{-1}\right) \leq e_7 \leq \frac{1}{\omega_n^2} \sum_{i=1}^{\min(m,n-p)} (\sqrt{\mu_i} - \sqrt{\mu_{n-i+1}})^2 \tag{12}$$

where μ_i is the sequence characteristics root of W^2X, and there $1 \leq m \leq p$, $i = 1, 2, ...n$.

Proof: By the lemma 10:

$$n^{\frac{1}{q}-1}[tr(Cov\hat{\mu} - Cov\mu^*) \leq e_7(\hat{\mu}) = [tr(Cov\hat{\mu} - Cov\mu^*)^q]^{\frac{1}{q}} \leq tr(Cov\hat{\mu} - Cov\mu^*) \tag{13}$$

Insert (4) into (13),

$$e_7 = tr(Cov\hat{\mu} - Cov\mu^*) \leq \frac{1}{\omega_n^2} \sum_{i=1}^{\min(m,n-p)} (\sqrt{\mu_i} - \sqrt{\mu_{n-i+1}})^2$$

The theorem 5 has been proved.

Inference 2: In the model of (1), if $r(X) = p$, then

$$\frac{n^{\frac{1}{q}-1}}{\omega_1^2}\left(\sum_{i=n-p+1}^{n} \omega_n^2 \lambda_i - \sum_{i=1}^{p} \omega_n^{-2} \lambda_{n-p+i}^{-1}\right) \leq e_7 \leq \frac{1}{\omega_n^2} \sum_{i=1}^{\min(m,n-p)} (\sqrt{\lambda_i}\omega_1 - \sqrt{\lambda_{n-i+1}}\omega_n)^2 \tag{14}$$

where λ_i is the sequence characteristics root of X, and there is $1 \leq m \leq p$, $i = 1, 2, ...n$.

Proof: Because $\mu_i = \lambda_i(W\Sigma W)$, By the lemma 9: $\omega_n^2 \lambda_i \leq \lambda_i(W^2\Sigma) \leq \omega_1^2 \lambda_i$, $\omega_n^2 \lambda_{n-i+1} \leq \lambda_{n-i+1}(W^2\Sigma) \leq \omega_1^2 \lambda_{n-i+1}$. Then on the basis of (10), it is believed that: $\frac{n^{\frac{1}{q}-1}}{\omega_1^2}(\sum_{i=n-p+1}^{n} \omega_n^2 \lambda_i - \sum_{i=1}^{p} \omega_n^{-2} \lambda_{n-p+i}^{-1}) \leq \frac{n^{\frac{1}{q}-1}}{\omega_1^2}(\sum_{i=n-p+1}^{n} \mu_i - \sum_{i=1}^{p} \mu_{n-p+i}^{-1})$, $\frac{1}{\omega_n^2} \sum_{i=1}^{\min(m,n-p)} (\sqrt{\mu_i} - \sqrt{\mu_{n-i+1}})^2 \leq \frac{1}{\omega_n^2} \sum_{i=1}^{\min(m,n-p)} (\sqrt{\lambda_i}\omega_1 - \sqrt{\lambda_{n-i+1}}\omega_n)^2$

The inference 2 has been proved.

References

[1] Q.T. Duan, Y.H. Zhang, "Relative Efficiency of LSE in Linear Weighted Regression," *Pure mathematics and applied learning*, 1998, 4, pp. 25-29.

[2] J. Wang, W.X. Wang, "The Relative Efficiency of the Parameter Estimate in Linear Weighted Regression," *Guangxi normal learning*, 2000, 15, pp. 37-40.

[3] X.X. Chen, "A New Relative Efficiency of the Parameter estimation in Linear Model," *Journal of Jiangxi normal university*, 1999, 23, pp. 313-315.

[4] H.S. Liu, R.Q. Li. "The Relative Efficiency of the Parameter Estimate in Linear Weighted Regression," *Journal of Taiyuan university of technology*, 2005, 36, pp. 612-614.

[5] S.G. Wang, M.X. Wu, and Z.Z. Jia, *Matrix Inequality*, Beijing: Science Press. 2006.

[6] J.L. Wang, D.D. Gao, "The Relative Efficiency of the Generalized Least Squares Estimate in Euclidean Mode Sense," *Application of probability and statistics*, 1992, 7, pp. 361-366.

[7] S.G. Wang, Z.Z. Jia. *Matrix Inequality*, Hefei: Anhui Education Press. 1994.

[8] X.M. Liu, J. Zhang. "A new efficiency in linear model," *Mathematical theory and applications*, 2000, 20, pp. 26-29.

[9] A.Y. Liu, S.G. Wang. "A new efficiency of least squares estimate in the linear model," *Application of probability and statistics,* 1989. 15, pp. 97-104.