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Generalized Chebyshev Collocation Method

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Abstract—In this paper, we introduce a generalized Chebyshev collocation method (GCCM) based on the generalized Chebyshev polynomials for solving stiff systems. For employing a technique of the embedded Runge-Kutta method used in explicit schemes, the property of the generalized Chebyshev polynomials is used, in which the nodes for the higher degree polynomial are overlapped with those for the lower degree polynomial. The constructed algorithm controls both the error and the time step size simultaneously and further the errors at each integration step are embedded in the algorithm itself, which provides the efficiency of the computational cost. For the assessment of the effectiveness, numerical results obtained by the proposed method and the Radau IIA are presented and compared.

Keywords—Generalized Chebyshev Collocation method, Generalized Chebyshev Polynomial, Initial value problem.

I. Introduction

THE embedded Runge-Kutta (ERK) method is a popular strategy for solving initial value problems described by

$$\frac{d\phi}{dt} = f(t,\phi(t)), \quad t \in [t_0, t_f]; \quad \phi(t_0) = \phi_0. \tag{1}$$

However, it is well known that ERK doesn't work well when the stiffness of system (1) is high (see [1]). The aim of this paper is to construct an algorithm of ERK type so called a generalized Chebyshev collocation method (GCCM) for solving the stiff system. The developed GCCM is based on the generalized Chebyshev polynomials introduced by [2], [3] and extensively used for approximating functions and integrals([7], [8]). The main theory of the generalized Chebyshev polynomial approximation is to use node points duplicately for Chebyshev polynomials of consequent degrees. The GCCM of ERK type is constructed from this idea together with the Chebyshev collocation method (CCM) developed by [5], [6], [9], [11]. In particular, it uses two different orders of CCM together with two Chebyshev polynomials of different degrees p and q ($q > p$) and hence we would like to denote GCCMp(q) simply. Practically, the solution at each integration scheme is calculated with the CCM of order 4 denoted by CCM4 using the Chebyshev interpolating polynomial on the Chebyshev-Gauss-Lobatto (CGL) nodes given by

$$s_j = cos(\frac{n-j}{n}\pi), \quad j = 0, 1, \cdots, 4.$$

To calculate the error, we develop the CCM of order 6 using the Chebyshev interpolating polynomial on the Clenshaw-Curtis (CC) nodes together with CGL nodes given by

$$\{\tau_k\} = \{s_j\} \cup \{\cos(\frac{3}{8}\pi), \cos(\frac{5}{8}\pi)\}, \quad k = 0, 1, \cdots, 6.$$

Assume that ϕ_m is known approximation for $\phi(t_m)$ and e_m is known approximation of $E_m = \phi(t_m) - \phi_t$ at time t_m. Our aim is to find approximations at time t_{m+1},ϕ_{m+1} and e_{m+1}. To find these values, we have to solve the IVP given by

$$\begin{cases} \phi'(t) = f(t,\phi(t)), \quad t \in [t_m, t_{m+1}], \\ \phi(t_m) = \phi_m + E_m. \end{cases} \tag{2}$$

First, to use Chebyshev Collocation Method, let $\bar{\phi}(s) := \phi(t(s))$ be a function on linear transformated domain from $[t_m, t_{m+1}]$ to $[-1, 1]$. Then, the IVP is changed to

$$\begin{cases} \bar{\phi}'(s) = \phi'(t(s))t'(s) = \frac{h}{2} f\big(t(s), \bar{\phi}(s)\big), \quad s \in [-1, 1], \\ \bar{\phi}(-1) = \phi(t_m) = \phi_m + E_m. \end{cases}$$

We now try to introduce generalized Chebyshev interpolating polynomials for approximating a given function g. From now on, we define

$$t_k = \begin{cases} s_k, & if \quad N = 4, \\ \tau_k, & if \quad N = 6, \end{cases}$$

where N means degree of polynomial. Let $p_N(s)$ be the Chebyshev interpolating polynomial of degree N satisfying

$$p_N(t_k) = g(t_k), \quad 0 \le k \le N.$$

Then, $p_N(s)$ can be expressed by the Lagrangian interpolation as follows:

$$p_N(s) := \sum_{k=0}^{N} g(t_k) l_k(s), \quad l_k(s) := \frac{q_N(s)}{(s-t_k)\dot{q}_N(t_k)},$$

where $\dot{q}_N(s) := \frac{dq_N(s)}{ds}$ and $q_N(s)$ is a polynomial of degree $N+1$ defined by

$$q_N(s) := \prod_{j=0}^{N} (s - t_j).$$

Using the points t_j and basis l_k, define a matrix $\mathcal{L}$ by $\mathcal{L} = (L_{jk})$, where $L_{jk} = \dot{l}_k(t_j)$. Then, for fourth-degree,

$$\mathcal{L} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \sqrt{2} & -\frac{1}{\sqrt{2}} & 1 - \frac{1}{\sqrt{2}} \\ -\sqrt{2} & 0 & \sqrt{2} & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & -\sqrt{2} & -\frac{1}{\sqrt{2}} & 1 + \frac{1}{\sqrt{2}} \\ 2(\sqrt{2}-2) & 2 & -2(2+\sqrt{2}) & \frac{11}{2} \end{pmatrix}$$

and for sixth-degree, one can get $\mathcal{L}$ similarly. Using the Chebyshev interpolating polynomial $p_N(s)$, let

$$\bar{\phi}(s) = p_N(s) + \rho_N(s) \tag{3}$$

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to approximate the solution $\bar{\phi}$, where ρ_N is the local truncation error. After differentiating both sides of (3) and collocating the resulted equation at the points t_j, we can obtain a discrete system of equations as follows :

$$\frac{h}{2}f\big(t_{m_j},\bar{\phi}(t_j)\big)-\sum_{k=0}^{N}\bar{\phi}(t_k)\dot{l}_k(t_j)=\dot{\rho}_N\big(t_j\big),\quad 1\le j\le N, \tag{4}$$

where $t_{m_j}:=t_m+\frac{h}{2}(1+t_j)$. By truncating the vector $\dot{\rho}_N(t_j)$ in (4) and perturbing the initial value of (2) with ϕ_m+e_m, the system (4) leads to

$$\sum_{k=1}^{N}\alpha_k\dot{l}_k(t_j)-\frac{h}{2}f\big(t_{m_j},\alpha_j\big)=-(\phi_m+e_m)\dot{l}_0(t_j) \tag{5}$$

for $1\le j\le N$, where α_j means an approximation for the exact solution $\bar{\phi}(t_j)$, $j=1,\cdots,N$. By introducing α defined by

$$\alpha:=\big[\alpha_1,\cdots,\alpha_N\big]^T,$$

the discrete system (5) can be simplified as follows.

$$\mathcal{L}\alpha-\frac{h}{2}f(t,\alpha)=-L_0(\phi_m+e_m). \tag{6}$$

Then, the nonlinear system (6) made from GCCM is solved with a Newton iteration method. By solving system (6), we can summarize the developed algorithm GCCM4(6) by

$$\begin{cases}\phi_{m+1}=F(\phi_m,e_m),\\ e_{m+1}=G(\phi_m,\phi_{m+1},e_m),\end{cases}$$

where the above two equations are obtained by the CCM4 and CCM6, respectively. In particular, the error formula of e_{m+1} is obtained with the observation that there are a Newton's formula between Chebyshev interpolating polynomial $p_6(s)$ and $p_4(s)$ given by

$$p_6(s):=p_4(s)+\sum_{k=1}^{2}c_k(T_{4-k}(s)-T_{4+k}(s)),$$

where coefficients $\{c_k\}$ are determined. Indeed, $e=p_6-p_4=\sum_{k=1}^{2}c_k(T_{4-k}(s)-T_{4+k}(s))$.

For the time step control, we use the formula for a given tolerance rtol and atol

$$h_{new}:=\alpha\times\frac{h_{old}}{E},$$

where α is the safety factor and $E:=atol+rtol\times\max(\phi_m,\phi_{m+1})$. The proposed algorithm controls both the errors and the time step size simultaneously and it must be noted that the errors at each integration step are embedded in the algorithm itself. This is a novelity of GCCM and reduce an accumulation of the estimated errors. In other words, by giving the usage of estimated error, we can improve the capability of the existing method, while existing method uses the estimated error only for the step-size selection.

II. Numerical Result

To assess the improvement and effectiveness of the proposed scheme, Lambert's problem [4] and Prothero-Robinson equation [10] are solved. For a comparison of the numerical result, a popular existing algorithm Radau5 [1] is used.

Example 1. Consider a problem from Lambert given by

$$\begin{aligned}y_1'(t)&=-2y_1(t)+y_2(t)+2\sin(t),\\ y_2'(t)&=998y_1(t)-999y_2(t)+999(\cos(t)-\sin(t)),\end{aligned} \tag{7}$$

with initial conditions $y_1(0)=2$, $y_2(0)=3$. The analytic solution is given by

$$y_1(t)=2e^{-t}+\sin(t),\qquad y_2(t)=2e^{-t}+\cos(t). \tag{8}$$

Note that the stiffness is high. We have numerically solved the problem on the interval [0,10] and use the numerical solutions as the sum of the approximate solution ϕ_m and the estimated error e_m to give more accurate results. As a measure of the effectiveness, we calculate both the required number of function evaluations (nfeval) and the computational time (cputime) to solve the problem. We calculate the L_∞ norm for the absolute error in log-scale at the final time for each problem. The problem is solved by varying the relative tolerance from 1.0e-5 to 1.0e-8 and absolute tolerance from 1.0e-7 to 1.0e-10. The numerical results are presented in Table I,II and displayed on Fig.1 (a) and (b), where y and x axes represent the absolute errors and either cputime or nfeval, respectively. All the marked points in figure from left to right are corresponding to the given tolerances from large to small, respectively. One can see that the proposed scheme is more efficient than the other scheme. For example, let us consider the point in the right corner. In Fig.1 (b), the point for Radau5 evaluate 2108 number of function with the error 7.0e-10. However, GCCM4(6) needs 505 nfeval only with the error 1.0e-12.

TABLE I
Comparison of errors spent cputime

Ex1	GCCM46		Radau5	
Rtol	Err	CPUtime	Err	CPUtime
10^{-5}	2.31×10^{-9}	0.0266	1.41×10^{-7}	0.0293
10^{-6}	1.77×10^{-10}	0.0378	2.59×10^{-8}	0.0446
10^{-7}	1.2×10^{-11}	0.0548	5.1×10^{-9}	0.0748
10^{-8}	1.0×10^{-12}	0.0850	7.0×10^{-10}	0.1401

TABLE II
Comparison of errors versus number of function-evaluations

Ex1	GCCM46		Radau5	
Rtol	Err	nfeval	Err	nfeval
10^{-5}	2.31×10^{-9}	162	1.41×10^{-7}	345
10^{-6}	1.77×10^{-10}	235	2.59×10^{-8}	555
10^{-7}	1.2×10^{-11}	335	5.1×10^{-9}	1045
10^{-8}	1.0×10^{-12}	505	7.0×10^{-10}	2108

Example 2. We test the Prothero-Robinson equation, which is a particular case of the family of scalar equation proposed by Prothero and Robinson and constitutes a stiff problem,

$$\phi'(t)=\nu(\phi(t)-g(t))+g'(t),\quad t\in(0,10];\quad \phi(0)=1,$$

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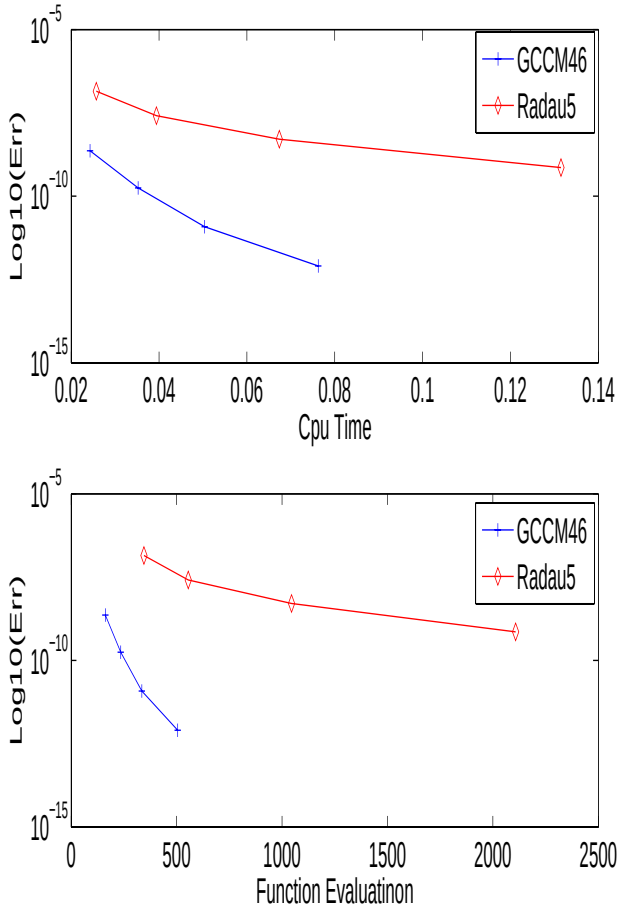


Fig. 1. Comparison of errors spent cputime (a) and versus number of function-evaluations (b)

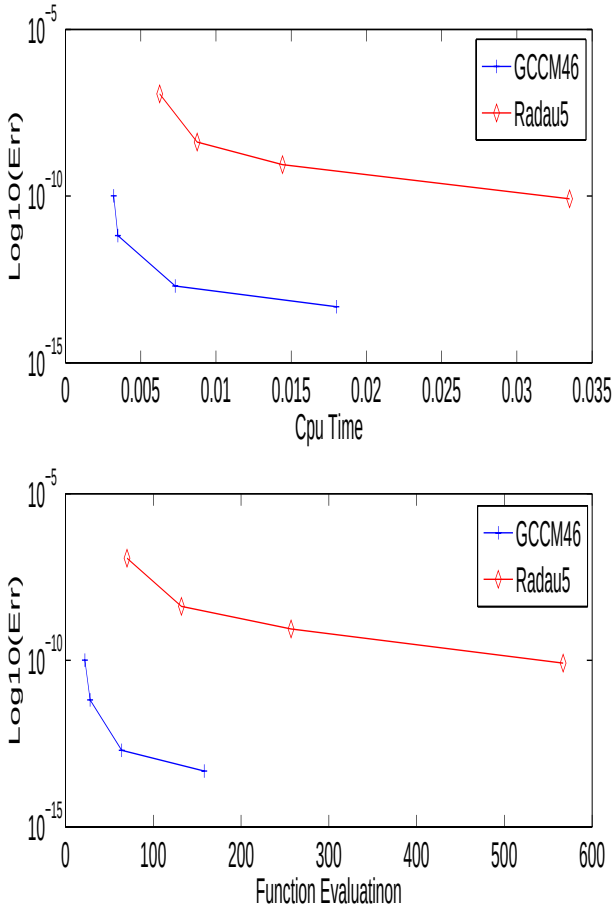


Fig. 2. Comparison of errors spent cputime (a) and versus number of function-evaluations (b)

where the eigenvalue ν is $\nu = -10^6$ and $g(t) = cos(t)$. The exact solution is given by $\phi(t) = sin(t)$. The problem is solved by varying the relative tolerance from 1.0e-7 to 1.0e-10 and absolute tolerance from 1.0e-9 to 1.0e-12. The numerical results are displayed in Fig. 2 and Table III,IV, from which one can see that the presented method are better than Radau5.

TABLE III
COMPARISON OF ERRORS SPENT CPUTIME

Ex2	GCCM46		Radau5	
Rtol	Err	CPUtime	Err	CPUtime
10^{-7}	3.59×10^{-11}	0.0031	1.25×10^{-7}	0.0067
10^{-8}	3.58×10^{-12}	0.0036	1.64×10^{-8}	0.0089
10^{-9}	2.1×10^{-13}	0.00083	9.0×10^{-10}	0.0148
10^{-10}	5.0×10^{-14}	0.0190	1.0×10^{-10}	0.0314

TABLE IV
COMPARISON OF ERRORS VERSUS NUMBER OF FUNCTION-EVALUATIONS

Ex2	GCCM46		Radau5	
Rtol	Err	nfeval	Err	nfeval
10^{-7}	3.59×10^{-11}	22	1.25×10^{-7}	74
10^{-8}	3.58×10^{-12}	28	1.64×10^{-8}	130
10^{-9}	2.1×10^{-13}	65	9.0×10^{-10}	264
10^{-10}	5.0×10^{-14}	158	1.0×10^{-10}	620

III. Conclusion

In summary, a generalized Chebyshev collocation method of ERK-type based on generalized Chebyshev approximations is newly introduced for solving stiff initial value problem. Using the CCM of two different convergence orders 4 and 6, the solution and the error are calculated efficiently and also we suggest a methodology that contains itself the estimated error at each integration step. One can see that the suggested algorithm is more efficient than Radau5 throughout several numerical tests.

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REFERENCES

[1] E. HAIRER, G. WANNER, *Solving Ordinary Differential Equations,II Stiff and Differential-Algebraic Problems*, Springer Series in Computational Mathematics, Springer, 1996.

[2] T. HASEGAWA, T. TORII, I. NINOMIYA, *Generalized Chebyshev interpolation and its application to automatic quadrature*, Math. Comp. 41(1983) pp. 537–543.

[3] T. HASEGAWA, T. TORII, H. SUGIURA, *An Algorithm based on The FFT for a Generalized Chebyshev Interpolation*, Mathematics of computation, Vol.54, Number 189(1990) pp. 195–210.

[4] L.G. IXARU, *Exponentially fitted variable two-step BDF algorithm for first order ODEs*, Comput. Appl. Math. 111(1999) pp. 93–111.

[5] P. S. KIM, X. PIAO, S. D. KIM, *An Error Corrected Euler Method for Solving Stiff Problems based on Chebyshev Collocation*, SIAM J. Numer. Anal., 48(2011) pp. 1759–1780.

[6] S. D. KIM, X. PIAO, D. H. KIM, P. S. KIM, *Convergence on Error Correction Methods for Solving Initial Value Problems*, J. Comp. and Applied Math., 236(2012) pp. 4448–4461.

[7] P. KIM, B.I. YUN, *On the convergence of interpolatory-type quadrature rules for evaluating Cauchy integrals*, J. Comp. and Applied Math., 149(2002) pp. 381–395.

[8] P. KIM, *A Trigonometric Quadrature Rule for Cauchy Integrals with Jacobi Weight*, J. Approx. 108(2001) pp. 18–35.

[9] S. KIM, J. KWON, X. PIAO, P. KIM, *A Chebyshev collocation method for stiff initial value problems and its stability* Kyungpook math. J. 51(2011) pp. 435–456.

[10] A. PROTHERO, A. ROBINSON, *On the stability and accuracy of one-step methods for solving stiff systems of ordinary differential equations*, Math. Comp., 28(1974) pp. 145–162.

[11] H. RAMOS, J. VIGO-AGUIAR, *A Fourth-Order Runge-Kutta Method based on BDF-type Chebyshev Approximations*, J. Comp. and Applied Math., 204(2007) pp. 124–136.

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