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The Application of Hybrid Orthonomal Bernstein and Block-Pulse Functions in Finding Numerical Solution of Fredholm Fuzzy Integral Equations

Mahmoud Zarrini, Sanaz Torkaman

Abstract—In this paper, we have proposed a numerical method for solving fuzzy Fredholm integral equation of the second kind. In this method a combination of orthonormal Bernstein and Block-Pulse functions are used. In most cases, the proposed method leads to the exact solution. The advantages of this method are shown by an example and calculate the error analysis.

Keywords—Fuzzy Fredholm Integral Equation, Bernstein, Block-Pulse, Orthonormal.

I. Introduction

INTEGRAL equation is an equation which has integral sign and an unknown function. Naturally, in such an equation there can occur other terms as well. The topic of fuzzy integral equations (FIE) has been developed in recent years. In the first step often including applicable definitions of the fuzzy integrals was followed by introducing FIE and establishing sufficient conditions for the existence of unique solutions to these equations. Finally, numerical methods for calculation approximates to these solutions were designed. The concept of fuzzy sets which was originally introduced by Zadeh [1] led to the definition of the fuzzy number and implementation in fuzzy control [2] and approximate reasoning problems [1].

In recent years, many different basic functions have been used to estimate the solution of integral equations, such as Block-Pulse functions [3, 4], Triangular functions [5, 6], Haar functions [7], Hybrid Legendre and Block-Pulse functions [8, 9], Hybrid Chebyshev and Block-Pulse functions [10, 11], Hybrid Taylor, Block-Pulse functions [12], Hybrid Fourier and Block-Pulse functions [13]. In the first time, Block-Pulse functions were introduced to electrical engineering by Harmuth and several researchers discussed the Block-Pulse [14, 15]. Fuzzy integral equations arise in many applications such as physics, geographic, medical, biology, social sciences, etc. Many practical problems in science and engineering can be transformed into Fuzzy Fredholm integral equations of the second kind, thus their solution is one of the main goals in various areas of applied sciences and engineering.

In this paper, a hybrid of orthonormal Bernstein and Block-Pulse functions for numerical solution of fredholm integral equations are used. Leading to the exact solution is the advantage of proposed method.

M. Zarrini is with the Department of Applied Mathematics, University of Ayatollah Boroujerdi, Boroujerd, Iran (e-mail: dr_mzarrini@yahoo.com).

S. Torkaman is in Islamic Azad University, Hamedan Branch, Hamedan, Iran.

II. Preliminaries

In this section, we introduce Bernestein polynomials and their properties to get better approximation and we used orthonormal these polynomials and hybrid them with Block-pulse functions.

A. Definition of Bernestein Polynomials

The Bernstein basis polynomial of degree n are defined by [16]

$$B_{i,n}(x) = \binom{n}{r} x^i (1-x)^{n-i} \tag{1}$$

By using binomial expansion of $(1-x)^{n-i}$, we have

$$\binom{n}{i} x^i (1-x)^{n-i} = \sum_{k=0}^{n-i} (-1)^k \binom{n}{i} \binom{n-i}{k} x^{i+k} \tag{2}$$

Then, $\{B_{0,n}, B_{1,n}, \ldots, B_{n,n}\}$ in Hilbert space $L^2[0,1]$ is a complete basis. Therefore, any polynomial of degree n can be expanded in terms of linear combination of $B_{i,n}(x)$ for $i = 0, 1, 2, \ldots, n$. By using Gram-schmid algorithm, we obtain orthonormal polynomials to construct new basis, these new basis are $OB_{i,n}(x)$.

B. Definition of Block-Pulse Functions and Their Properties

An M-set of Block-Pulse function is defined over the interval $[0, T)$ as

$$b_i(x) = \begin{cases} 1 & \frac{iT}{M} \leq x < \frac{(i+1)T}{M} \\ 0 & otherwise \end{cases} \tag{3}$$

where, $i = 0, 1, \ldots, M-1$ with M as a positive integer. Also, $h = \frac{T}{M}$ and b_i is the ith BPF. In this paper, it is assumed that $T = 1$, so BPFs are defined over $[0, 1)$ and $h = \frac{1}{M}$. [17]

$$b_i(x) b_j(x) = \begin{cases} b_i(x) & i = j \\ 0 & i \neq j \end{cases} \tag{4}$$

where, other property is orthogonality. It is clear that [18]

$$\int_0^1 b_i(x) b_j(x) dx = h\delta_{i,j} \tag{5}$$

where, $\delta_{i,j}$ is Kronecker delta. The third property is completeness. For every $f \in L^2[0,1]$ when M approaches to infinity, Parsevals identity holds [17]

$$\int_0^1 f^2(x)dx = \sum_{i=0}^{\infty} f_i^2 ||b_i(x)||^2 \tag{6}$$

where,

$$f_i = \frac{1}{h}\int_0^1 f(x)b_i(x)dx \tag{7}$$

Definition 1: We define OHB on the interval [0,1] as follow:

$$OBH_{i,j}(x) = \begin{cases} B_{j,n}(Mx-i+1) & \frac{i-1}{M} \le x < \frac{i}{M} \\ 0 & otherwise \end{cases} \tag{8}$$

where, $i = 1, 2, \cdots, M$ and $j = 0, 1, \cdots, n$.

thus, our new basis is $\{OBH_{1,0}, OBH_{1,1}, \ldots, OBH_{M,n}\}$ and we can approximate function with the base.

C. Function Approximation by Using OHB Functions

A function $u(x)$, square integrable in $[0, 1]$, may be expressed in terms of the OBH basis as follow [18]

$$u(x) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} c_{i,j}.OBH_{i,j}(x) \tag{9}$$

If we truncate the infinite series in (8), then we have

$$u(x) \simeq \sum_{i=1}^{M}\sum_{j=0}^{n} c_{i,j}.OBH_{i,j}(x) = C^T OBH(x)\ , \tag{10}$$

where,

$$OBH(x) = [OBH_{1,0}, OBH_{1,1}, \ldots, OBH_{M,n}]^T \tag{11}$$

and

$$C = [c_{1,0}, c_{1,1}, \cdots, c_{M,n}]^T$$

Therefore, we have

$$C^T < OBH(x), OBH(x) >=< u(x), OBH(x)) >$$

then

$$C = D^{-1} < u(x), OBH(x) >$$

where,

$$\begin{aligned} D &= < OBH(x), OBH(x) > \\ &= \int_0^1 OBH(x).OBH^T(x)dx \\ &= \begin{pmatrix} D_1 & 0 & \ldots & 0 \\ 0 & D_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & D_M \end{pmatrix} \end{aligned} \tag{12}$$

then, by using (7), $D_i(i = 0, 1, 2, \ldots, M)$ is defined as follow:

$$(D_n)_{i+1,j+1} = \int_{\frac{i-1}{M}}^{\frac{i}{M}} B_{i,n}(Mx-i+1)B_{j,n}(Mx-j+1)dx$$

$$\begin{aligned} (D_n)_{i+1,j+1} &= \frac{1}{M}\int_0^1 B_{i,n}(x)B_{j,n}(x)dx \\ &= \frac{\binom{n}{i}\binom{n}{j}}{M(2n+1)\binom{2n}{i+j}} \end{aligned} \tag{13}$$

We can also approximate the function $k(x, t) \in L^2[0, 1]$ as follow:

$$k(x,t) \simeq OBH^T(x)KOBH(t) \tag{14}$$

where, K is an $M(n + 1)$-matrix that we can obtain as follows:

$$K = D^{-1} < OBH(x) < k(x,t), OBH(t) >> D^{-1} \tag{15}$$

III. Basic Definitions Fuzzy

Definition 2 [19]: A fuzzy number is a set $v : R^1 \to I = [0, 1]$ which satisfies:

- v is upper semi continuous,
- $v(x) = 0$ outside some interval $[c, d]$,
- There are real numbers $a, b : c \le a \le b \le d$ which
  - $v(x)$ is monotonic increasing on $[c, d]$,
  - $v(x)$ is monotonic decreasing on $[b, d]$,
  - $v(x) = 1$, $a \le x \le b$.

The set of all such fuzzy number is denoted by R_F.

Definition 3 [19]: Let V be a fuzzy set on R. V is called a fuzzy interval if:

- V is normal: there exists $x_0 \in R$ such that $V(x_0) = 1$.
- V is convex: for all $x, t \in R$ and $0 \le \lambda \le 1$, it holds that $V(\lambda x + (1-\lambda)t) \ge min\{V(x), V(t)\}$,
- V is upper semi-continuous: for any $x_0 \in R$, it holds that $V(x_0) \ge \lim_{x\to x_0^{\mp}} V(x)$,
- $[V]^0 = Cl\{x \in R|V(x) > 0\}$ is a compact subset of R. The α-cut of a fuzzy interval V, with $0 < \alpha \le 1$ is the crisp set, $[V]^\alpha = \{x \in R|V(x) > \alpha\}$.
  For a fuzzy interval V, its α-cuts are closed intervals in R. Let denote them by $[V]^\alpha = [\underline{V}(\alpha), \overline{V}(\alpha)]$.

An alternative definition or parametric form of a fuzzy number which yields the same E^1 is given by Kaleva [19] as follows:

Definition 4 [20]: An arbitrary fuzzy number u in the parametric form is represented by an ordered pair of functions $(\underline{u}(r), \overline{u}(r))$ which satisfy the following requirements:

- $\underline{u}(r)$ is a bounded left-continuous non-decreasing function over $[0, 1]$,
- $\overline{u}(r)$ is a bounded right-continuous non-increasing function over $[0, 1]$,
- $\underline{u}(r) \le \overline{u}(r)$, for all $0 \le r \le 1$.

For arbitrary fuzzy numbers $v = (\underline{v}(r), \overline{v}(r))$, $w = (\underline{w}(r), \overline{w}(r))$ and real number λ, one may define the addition and the scalar multiplication of the fuzzy numbers by using the extension principle as follows:

- $v = w$ if and only if $\underline{v}(r) = \underline{w}(r)$ and $\overline{v}(r) = \overline{w}(r)$
- $v \oplus w = (\underline{v}(r) + \underline{w}(r), \overline{v}(r) + \overline{w}(r))$
- $(\lambda \otimes v) = \begin{cases} \lambda\underline{v}(r), \lambda\overline{v}(r) & \lambda \geq 0 \\ \lambda\overline{v}(r), \lambda\underline{v}(r) & \lambda < 0 \end{cases}$

Definition 5 [21]: For arbitrary numbers $v = (\underline{v}(r), \overline{v}(r))$ and $w = (\underline{w}(r), \overline{w}(r))$

$$D(v, w) = \max\{\sup_{0 \leq r \leq 1} |\overline{v}(r) - \overline{w}(r)|, \sup_{0 \leq r \leq 1} |\underline{v}(r) - \underline{w}(r)|\}$$

in the distance between v and w. It is proved that (R_F, D) is a complete metric space with the properties [21]

- $D(u \oplus w, v \oplus w) = D(u, v) \quad ; \quad \forall u, v, w \in R_F,$
- $D(k \otimes u, k \otimes v) = |k| D(u, v) \quad ; \quad \forall u, v \in R_F \quad \forall k \in R,$
- $D(u \oplus v, w \oplus e) \leq D(u, w) + D(v, e) \quad ; \quad \forall u, v, w, e \in R_F.$

Definition 6: [22] Let $f, g : [a, b] \to R_F$, be fuzzy real number valued functions. The uniform distance between f, g is defined by:

$$D_U(f, g) = \sup\{D(f(x), g(x)) | x \in [a, b]\} \tag{16}$$

In [23], the authors proved that if the fuzzy function, $f(t)$, is continuous in the metric D, its definite integral exists and also,

$$\underline{\int_a^b f(t, r)dt} = \int_a^b \underline{f}(t, r)dt$$

$$\overline{\int_a^b f(t, r)dt} = \int_a^b \overline{f}(t, r)dt$$

Definition 7 [24]: A fuzzy real number valued function $f : [a, b] \to R_F$ is said to be continuous in $x_0 \in [a, b]$, if for each $\epsilon > 0$ there is $\delta > 0$ such that $D(f(x), f(x_0)) < \epsilon$, whenever $x \in [a, b]$ and $|x - x_0| < \delta$. We say that f is fuzzy continuous on $[a, b]$ if f is continuous at each $x_0 \in [a, b]$ and denote the space of all such functions by $C_F([a, b])$.

Definition 8 [22]: Let $f : [a, b] \to R_F$ be a bounded function, then function $\omega_{[a,b]}(f, .) : R_+ \bigcup \{0\} \to R_+$

$$\omega_{[a,b]}(f, \delta) = \sup\{D(f(x), f(y)) | x, y \in [a, b], \ |x - y| \leq \delta\} \tag{17}$$

where, R_+ is the set of positive real numbers, is called the modulus of continuity of f on $[a, b]$.

Definition 9 [22]: Let $f : [a, b] \to R_F$, f is fuzzy-Riemann integrable to $I \in R_F$ if for any $\epsilon > 0$, there exists $\delta > 0$ such that for any division $P = \{[u, v] \ ; \ \xi\}$ of $[a, b]$ with the norms $\Delta(p) < \delta$, we have:

$$D\left(\sum_p {}^*(v - u) \otimes f(\xi), I\right) < \epsilon \tag{18}$$

where, $\sum {}^*$ denotes the fuzzy summation. In this case it is denoted by $I = (FR) \int_a^b f(x)dx$.

Lemma 1 [22] If $f, g : [a, b] \subseteq R \to R_F$ are fuzzy continuous functions, then the function $F : [a, b] \to R_+$ by $F(x) = D(f(x), g(x))$ is continuous on $[a, b]$ and by

$$D\left((FR)\int_a^b f(x)dx, (FR)\int_a^b g(x)dx\right) \leq \int_a^b D(f(x), g(x))dx$$

IV. Solving Fuzzy Fredholm Integral Equation via OBH Functions

In this section, first the fuzzy integral equations of the second kind are introduced then we solving it's via OBH function. The Fredholm fuzzy integral equation of the second kind is [25]

$$\tilde{u}(x) = \tilde{f}(x) + \lambda \int_a^b k(x, t)\tilde{u}(t)dt \tag{19}$$

Where, $\lambda > 0$, $k(x, t)$ is an arbitrary kernel function over the square $a \leq x, t \leq b$ and $\tilde{u}(x)$, $\tilde{f}(x)$ are fuzzy functions such that $k(x, t), \tilde{f}(x) \in L^2[0, 1]$. If $\tilde{f}(x)$ is a crisp function then the solutions of (19) is crisp .However, if $\tilde{f}(x)$ is a fuzzy function then this equation may only possess fuzzy solutions [25]. Now, if we introduce (19) by definition 4 then we have,

$$\underline{u}(x, r) = \underline{f}(x, r) + \lambda \int_a^b V_1(k(x, t)\underline{u}(t, r))dt \tag{20}$$

$$\overline{u}(x, r) = \overline{f}(x, r) + \lambda \int_a^b V_2(k(x, t)\overline{u}(t, r))dt \tag{21}$$

Where,

$$V_1(k(x, t)\underline{u}(t, r)) = \begin{cases} k(x, t)\underline{u}(t, r) & k(x, t) \geq 0 \\ k(x, t)\overline{u}(t, r) & k(x, t) < 0 \end{cases} \tag{22}$$

and

$$V_2(k(x, t)\overline{u}(t, r)) = \begin{cases} k(x, t)\overline{u}(t, r) & k(x, t) \geq 0 \\ k(x, t)\underline{u}(t, r) & k(x, t) < 0 \end{cases} \tag{23}$$

Throughout this paper, we consider Fuzzy Fredholm integral equation (19), with $a = 0$, $b = 1$ and $\lambda = 1$, then we write (19), in the following form:

$$\underline{u}(x, r) = \underline{f}(x, r) + \lambda \int_0^1 k(x, t)\underline{u}(t, r)dt \tag{24}$$

$$\overline{u}(x, r) = \overline{f}(x, r) + \lambda \int_0^1 k(x, t)\overline{u}(t, r)dt \tag{25}$$

we know $(\underline{u}(x, r), \overline{u}(x, r))$ is an unknown function with can be expanded into OBH function. Likewise, $k(x, t)$, $\underline{u}(x, r)$ and $\overline{u}(x, r)$ are also expanded into the OBH functions, then we must Let approximate $\underline{u}(x,r)$, $\overline{u}(x,r)$, $\underline{f}(x,r)$, $\overline{f}(x,r)$ and $k(x,t)$ by (14) as following for:

$$\begin{aligned}
k(x,t) &= OBH^T(x)KOBH(t) && (26)\\
\underline{u}(x,r) &= OBH^T(x)U1OBH(r) && (27)\\
\overline{u}(x,r) &= OBH^T(x)U2OBH(t) && (28)\\
\underline{f}(x,r) &= OBH^T(x)F1OBH(t) && (29)\\
\overline{f}(x,r) &= OBH^T(x)F2OBH(t) && (30)
\end{aligned}$$

With substituting above equations into (24) and (25), we have:

$$\begin{gathered}
OBH^T(x)U1OBH(r) = OBH^T(x)F1OBH(t) + \\
\int_0^1 OBH^T(x)KOBH(t)OBH^T(t)U1OBH(r)dt \\
OBH^T(x)U2OBH(r) = OBH^T(x)F2OBH(t) + \\
\int_0^1 OBH^T(x)KOBH(t)OBH^T(t)U2OBH(r)dt
\end{gathered}$$

By substituting (12) into above equation we get

$$\begin{aligned}
OBH^T(x)U1OBH(r) = OBH^T(x)F1OBH(t) + \\
OBH^T(x)KDU1OBH(r) \\
OBH^T(x)U2OBH(r) = OBH^T(x)F2OBH(t) + \\
OBH^T(x)KDU2OBH(r)
\end{aligned}$$

Therefore,

$$U1 = F1 + KDU1 \quad (31)$$
$$U2 = F2 + KDU2 \quad (32)$$

where, the dimensional subscripts have been dropped to simplify the notation. Rewriting (31) and (32), we have

$$U1 = (I - KD)^{-1}F1 \quad (33)$$
$$U2 = (I - KD)^{-1}F2 \quad (34)$$

where, I is $nM \times nM$-identity matrix. The unknowns matrix $U1$ and $U2$ can be obtained by solving (33) and (34). Thus the solutions $\underline{u}(x,r)$ and $\overline{u}(x,r)$ can be calculated in the OBH function expansion by using $U1$, $U2$ and (27)-(28).

V. The Convergence of the Method

In this section, we obtain error estimate for the numerical method proposed in previous section.
The solution of Fredholm fuzzy integral equation (19), by using OBH converges if $M < 1$, where

$$M = \max_{0 \le x,t \le 1} |k(x,t)|$$

Proof. Assume $\tilde{u}(x)$ and $\tilde{u}_{NM}(x)$ show approximate and exact solution of (19) respectively. then

$$\begin{aligned}
D(\tilde{u}(x), \tilde{u}_{NM}(x)) &= D(\int_0^1 k(x,t)\tilde{u}(t)dt, \\
&\int_0^1 k(x,t) \sum_{i=1}^{N} {}^* \sum_{j=0}^{M} {}^* c_{ij} OBH_{ij}(t)dt) \\
&\le M \int_0^1 D\left(\tilde{u}(t), \sum_{i=1}^{N} {}^* \sum_{j=0}^{M} {}^* c_{ij} OBH_{ij}(t)\right) dt
\end{aligned}$$

therefore, we have

$$\begin{aligned}
D(\tilde{u}(x), \tilde{u}_{NM}(x)) &\le M \int_0^1 D(\tilde{u}(t), \tilde{u}_{NM}(t))dt \\
\sup_{x\in[0,1]} D(\tilde{u}(x), \tilde{u}_{NM}(x)) &\le M \sup_{x\in[0,1]} D(\tilde{u}(t), \tilde{u}_{NM}(t))
\end{aligned}$$

Therefore, if $M < 1$, we will have:

$$\lim_{NM\to\infty} \sup_{x\in[0,1]} D(\tilde{u}(x), \tilde{u}_{NM}(x)) = 0$$

VI. Numerical Example

Consider the following linear fuzzy fredholm integral equation with

$$\underline{f}(x,r) = rx - x^2[\frac{2}{3}rx^3 - \frac{4}{3}x^3 - \frac{1}{2}rx^2 + x^2 + \frac{1}{12}r - \frac{1}{12}]$$

$$\overline{f}(x,r) = (2-r)x + x^2[\frac{2}{3}rx^3 - \frac{1}{2}rx^2 + \frac{1}{12}r - \frac{1}{12}]$$

and

$$k\,(x,t) = x^2(1-2t), \quad 0 \le x,t \le 1 \ and \ \lambda = 1$$

The exact solution in this case is given by

$$\underline{u}(x,r) = rx$$

$$\overline{u}(x,r) = (2-r)x.$$

Results are shown in Table 1.

VII. Conclusion

In this paper, we presented a numerical method for solving the fuzzy Fredholm integral equation of second kind. We solved Fredholm integral equations by combination of orthonormal Bernstein and Block-Pulse functions. We have shown advantages of this numerical method by an example and calculate the error analysis.

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TABLE I
COMPARING OF EXACT AND OBH METHOD SOLUTIONS

r	Exact solution ($\underline{u}(x,r)$)	OBH method for $x=0.1$ and $M=2,\ N=1$	Exact solution $\overline{u}(x,r)$)	OBH method for $x=0.1$ and $M=2,\ N=1$
0.0	0.00000000	0.00000000	0.20000000	0.20000000
0.1	0.01000000	0.01000000	0.19000000	0.19000000
0.2	0.02000000	0.02000000	0.18000000	0.18000000
0.3	0.03000000	0.03000000	0.17000000	0.17000000
0.4	0.04000000	0.04000000	0.16000000	0.16000000
0.5	0.05000000	0.05000000	0.15000000	0.15000000
0.6	0.06000000	0.06000000	0.14000000	0.14000000
0.7	0.07000000	0.07000000	0.13000000	0.13000000
0.8	0.08000000	0.08000000	0.12000000	0.12000000
0.9	0.09000000	0.09000000	0.11000000	0.11000000

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Mahmoud Zarrini is an Assistant Professor in Department of Applied Mathematics, University of Ayatollah Boroujerdi, Boroujerd, Iran. He was born on August 11, 1970. He has received his Ph.D. degree (Applied Mathematics, Fluid dynamics) from Pune University, India and his M.Sc. degree (Applied Mathematics, Integral Equations) from K.N. Toosi University of Technology, Tehran, Iran. He has published more than 15 research articles in some journals and was Author of Four Books. He has published more than 30 papers in Proceedings of international conferences. He is interested in Numerical Analysis, Numerical Methods, Integral Equations, Heat/Mass transfer in Porous media, Fluid Dynamics, Cold and Explosive sciences.

Sanaz Torkaman has received the B.Sc degree in Education of mathematics from Islamic Azad University of Hamedan, Iran in 20009 and M.Sc degree in applied mathematics from Islamic Azad University of Hamedan, Iran in 2013. Her research interests are studying on numerical solution of integral equations, fuzzy Fredholm integral equations, fuzzy systems and differential equations. She has published some research articles in numerical solution of integral equations in some journals and international conferences.