World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:8, No:10, 2014

Cubic Trigonometric B-spline Approach to Numerical Solution of Wave Equation

Shazalina Mat Zin, Ahmad Abd. Majid, Ahmad Izani Md. Ismail, Muhammad Abbas

***Abstract*—**The generalized wave equation models various problems in sciences and engineering. In this paper, a new three-time level implicit approach based on cubic trigonometric B-spline for the approximate solution of wave equation is developed. The usual finite difference approach is used to discretize the time derivative while cubic trigonometric B-spline is applied as an interpolating function in the space dimension. Von Neumann stability analysis is used to analyze the proposed method. Two problems are discussed to exhibit the feasibility and capability of the method. The absolute errors and maximum error are computed to assess the performance of the proposed method. The results were found to be in good agreement with known solutions and with existing schemes in literature.

***Keywords*—**Collocation method, Cubic trigonometric B-spline, Finite difference, Wave equation.

I. Introduction

CONSIDER a wave equation in the form of [1]

$$u_{tt} - u_{xx} = q(x,t) \tag{1}$$

with $a \le x \le b$ and $0 \le t \le T$ subject to the initial conditions

$$u(x,0) = \omega_1(x), \quad a \le x \le b \tag{2a}$$

$$u_t(x,0) = \omega_2(x), \quad a \le x \le b \tag{2b}$$

and the boundary conditions

$$u(a,t) = \phi_1(t), \quad 0 \le t \le T \tag{3a}$$

$$\int_a^b u(x,t) = \phi_2(t), \quad 0 \le t \le T \tag{3b}$$

where $q(x,t)$, $\omega_1(x)$, $\omega_2(x)$, $\phi_1(x)$ and $\phi_2(x)$ are known function.

In past few years, a number of papers have been focused on solving this hyperbolic partial differential equation numerically. Dehghan presented numerical techniques based on the three-level explicit finite difference schemes for solving this problem [2]. Ang solved the same problem using a scheme based on an integro-differential equation and local interpolating functions [3]. Then, B-spline functions were found to be an efficient method for solving wave equation. Dehghan et al [4], Khury et al. [5] and Goh et al. [6] proposed numerical methods based on cubic B-spline approach.

In this work, a new three-time level implicit approach based on B-spline will be presented for the approximate solution of wave equation. Central finite difference approach is used to discretize the time derivative and cubic trigonometric B-spline basis function are considered to interpolate the solution in space dimension. The stability of the proposed method is analyzed using von Neumann stability analysis. Two problems are solved to verify the proposed method.

Shazalina Mat Zin is PhD research student in School of Mathematical Sciences, Universiti Sains Malaysia, Malaysia and also a lecturer at Institute of Engineering Mathematics, Universiti Malaysia Perlis, Malaysia (e-mail: shazalina@unimap.edu.my).

Ahmad Abd Majid and Ahmad Izani Md Ismail are with School of Mathematical Sciences, Universiti Sains Malaysia, Malaysia.(e-mail: majid@cs.usm.my and izani@cs.usm.my).

Muhammad Abbas is with Department of Mathematics, University of Sarghoda, Pakistan (e-mail: m.abbas@uos.edu.pk).

II. Temporal discretization

Consider a grid points (x_j, t_k) to discretize the grid region $\Delta = [a,b] \times [0,T]$ with $x_j = a + jh$ and $t_k = k\Delta t$ where $j = 0,1,2,...,n$ and $k = 0,1,2,3,...,N$. h and Δt denote mesh space size and time step size, respectively. An approximation of the wave equation at t_{k+1}th time level is given as follows [2]

$$(u_{tt})_j^k - (1-\theta)(u_{xx})_j^k - \theta(u_{xx})_j^{k+1} = q(x_j, t_k) \tag{4}$$

The time derivative term in (4) is discretized by central difference approach. Thus,

$$\frac{u_j^{k+1} - 2u_j^k + u_j^{k-1}}{(\Delta t)^2} - (1-\theta)(u_{xx})_j^k - \theta(u_{xx})_j^{k+1} = q_j^k \tag{5}$$

In order for (5) to become half implicit and half explicit scheme, the value of θ is chosen to be 0.5. After simplification, the following scheme is produced

$$u_j^{k+1} - 0.5(\Delta t)^2 (u_{xx})_j^{k+1} = 2u_j^k + 0.5(\Delta t)^2 (u_{xx})_j^k + (\Delta t)^2 q_j^k - u_j^{k-1} \tag{6}$$

which is evaluated for $j = 0,1,\cdots,n$ at each time level k. Equation (6) is known as Crank-Nicolson scheme. The scheme is solved numerically by substituting cubic trigonometric B-spline function discussed in the following section.

III. Collocation Method

In this section, the approximate solution of wave equation is considered to be the following cubic trigonometric B-spline function

$$u(x,t)=\sum_{j=-3}^{n-1} C_j(t)T_{4,j}(x) \tag{7}$$

$C_j(t)$ is time dependent unknowns to be determined and $T_{4,j}(x)$ is cubic trigonometric B-spline basis function of order 4 given as

$$T_{4,j}(x)=\frac{1}{\kappa}\begin{cases}\rho^3(x_j) & x\in[x_j,x_{j+1}]\\ \rho^2(x_j)\sigma(x_{j+2}) +\rho(x_j)\sigma(x_{j+3})\rho(x_{j+1}) +\sigma(x_{j+4})\rho(x_{j+1}) & x\in[x_{j+1},x_{j+2}]\\ \rho(x_j)\sigma^2(x_{j+3}) +\sigma(x_{j+4})\rho(x_{j+1})\sigma(x_{j+3}) +\sigma^2(x_{j+4})\rho(x_{j+2}) & x\in[x_{j+2},x_{j+3}]\\ \sigma^3(x_{j+4}) & x\in[x_{j+3},x_{j+4}]\end{cases} \tag{8}$$

where $\rho(x_j)=\sin\left(\frac{x-x_j}{2}\right)$, $\sigma(x_j)=\sin\left(\frac{x_j-x}{2}\right)$ and $\kappa=\kappa_1\kappa_2\kappa_3$ with $\kappa_1=\sin\left(\frac{h}{2}\right)$, $\kappa_2=\sin(h)$, $\kappa_3=\sin\left(\frac{3h}{2}\right)$ and $\kappa_4=\sin(2h)$.

Due to local support properties of B-spline basis function, there are only three nonzero basis functions are included for evaluation at each x_j namely $T_{4,j-3}(x)$, $T_{4,j-2}(x)$ and $T_{4,j-1}(x)$. Thus, the approximate solution, $u(x_j,t_k)$ and the derivatives with respect to x can be obtained as follows

$$u_j^k=\eta_1 C_{j-3}^k+\eta_2 C_{j-2}^k+\eta_1 C_{j-1}^k \tag{9}$$

$$(u_x)_j^k=\eta_3 C_{j-3}^k-\eta_3 C_{j-1}^k \tag{10}$$

$$(u_{xx})_j^k=\eta_4 C_{j-3}^k+\eta_5 C_{j-2}^k+\eta_4 C_{j-1}^k \tag{11}$$

for $j=0,1,\ldots n$ where $\eta_1=\frac{\kappa_1^2}{\kappa_2\kappa_3}$, $\eta_2=\frac{2\kappa_1}{\kappa_3}$, $\eta_3=\frac{-3}{4\kappa_3}$, $\eta_4=\frac{6-9\kappa_1^2}{4\kappa_2\kappa_3}$ and $\eta_5=\frac{-3(\kappa_4+2\kappa_1^2\kappa_2)}{4\kappa_1\kappa_2\kappa_3}$.

Solution to (1) is obtained by substituting (9)–(11) into (6). Initially, time dependent unknowns $\mathbf{C}^0$ are calculated and shown in the next section. Then, the following initial condition is substituted into the last term of (6) for computing $\mathbf{C}^1$

$$u_j^{-1}=u_j^1-2\Delta t\omega_2(x) \tag{12}$$

Subsequently, the time dependent unknowns, $\mathbf{C}^k$ for $k\ge 1$ are calculated. The each system obtained consists $n+1$ linear equations with $n+3$ unknowns, namely $\mathbf{C^k}=(C_{-3}^k,C_{-2}^k,C_{-1}^k,\cdots,C_{n-1}^k)$ for $k\ge 1$. Hence, the following two additional equation from the boundary conditions given in (3a) and (3b) are needed for calculation.

i. $\eta_1 C_{-3}^{k+1}+\eta_2 C_{-2}^{k+1}+\eta_1 C_{-1}^{k+1}=\phi_1(t_{k+1})$

ii. $\eta_3 C_{n-3}^{k+1}-\eta_3 C_{n-1}^{k+1}-\eta_3 C_{-3}^{k+1}+\eta_3 C_{-1}^{k+1}=\phi_2''(t_{k+1})-\int_a^b q(x,t_{k+1})\,dx$

Thus, a $(n+3)\times(n+3)$ tridiagonal matrix system as below is obtained.

$$M\mathbf{C^{k+1}}=N\mathbf{C^k}-P\mathbf{C^{k-1}}+Q \tag{13}$$

System (13) are solved using the Thomas Algorithm repeatedly for $k=0,1,\cdots,N$.

IV. Initial States

Time dependent unknown $\mathbf{C}^0$ is calculated from the initial condition and boundary values of the derivatives of the initial condition as follows [7], [8]:

i. $(u_x)_j^0=\omega_1'(x_j)$ for $j=0$

$$\eta_3 C_{-3}^0-\eta_3 C_{-1}^0=\omega_1'(x_0)$$

ii. $u_j^0=\omega_1(x_j)$ for $j=0,1,2,...,n$

$$\eta_1 C_{j-3}^0+\eta_2 C_{j-2}^0+\eta_1 C_{j-1}^0=\omega_1(x_j)$$

iii. $(u_x)_j^0=\omega_1'(x_j)$ for $j=n$

$$\eta_3 C_{n-3}^0-\eta_3 C_{n-1}^0=\omega_1'(x_n)$$

This yields a $(n+3)\times(n+3)$ matrix system as

$$\mathbf{AC^0}=\mathbf{B} \tag{14}$$

where

$$\mathbf{A}=\begin{pmatrix}\eta_3 & 0 & -\eta_3 & 0 & \cdots & 0 & 0 & 0 & 0\\ \eta_1 & \eta_2 & \eta_1 & 0 & & & & & 0\\ 0 & \eta_1 & \eta_2 & \eta_1 & & & & & 0\\ \vdots & & & & \ddots & & & & \vdots\\ 0 & & & & & \eta_1 & \eta_2 & \eta_1 & 0\\ 0 & & & & & 0 & \eta_1 & \eta_2 & \eta_1\\ 0 & 0 & 0 & 0 & \cdots & 0 & \eta_3 & 0 & -\eta_3\end{pmatrix},$$

$$\mathbf{C}^0 = \begin{pmatrix} C_{-3}^0 \\ C_{-2}^0 \\ C_{-1}^0 \\ \vdots \\ C_{n-3}^0 \\ C_{n-2}^0 \\ C_{n-1}^0 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} \omega_1'(x_0) \\ \omega_1(x_0) \\ \vdots \\ \omega_1(x_n) \\ \omega_1'(x_n) \end{pmatrix}.$$

The solution of (14) can be obtained by using the Thomas Algorithm.

V. Stability Analysis

In this section, von Neumann stability analysis is applied for analyzing the stability of the proposed scheme. The growth of error in single Fourier mode is considered as

$$C_j^k = \delta^k e^{i\eta jh} \tag{15}$$

where $i=\sqrt{-1}$ and η is the mode number. It is known that this method can be used to analyze the stability of linear scheme. Thus, $q(x,t)$ in (1) is assumed to be 0 and the approximation is given by

$$u_j^{k+1} - \theta(\Delta t)^2 (u_{xx})_j^{k+1} = 2u_j^k + (1-\theta)(\Delta t)^2 (u_{xx})_j^k - u_j^{k-1} \tag{16}$$

Substituting (9)-(11) into (16) gives

$$\begin{aligned} &p_1 C_{j-3}^{k+1} + p_2 C_{j-2}^{k+1} + p_1 C_{j-1}^{k+1} \\ &= p_3 C_{j-3}^k + p_4 C_{j-2}^k + p_3 C_{j-1}^k - \eta_1 C_{j-3}^{k-1} - \eta_2 C_{j-2}^{k-1} - \eta_1 C_{j-1}^{k-1} \end{aligned} \tag{17}$$

where $p_1 = \eta_1 - \theta(\Delta t)^2 \eta_4$, $p_2 = \eta_2 - \theta(\Delta t)^2 \eta_5$, $p_3 = 2\eta_1 + (1-\theta)(\Delta t)^2 \eta_4$ and $p_4 = 2\eta_2 + (1-\theta)(\Delta t)^2 \eta_5$. In order to analyze the stability of the present scheme, (15) is inserted into (17). After simplification, it can be written as

$$A\delta^2 - B\delta + C = 0 \tag{18}$$

where $A = p_1[\cos(\eta h)] + p_2$, $B = p_3[\cos(\eta h)] + p_4$ and $C = \eta_1[\cos(\eta h)] + \eta_2$. Based on Routh-Hurwitz criterion, the transformation, $\delta = \frac{1+v}{1-v}$ is applied to (18) [9]. Then, the equation becomes

$$(A+B+C)v^2 + 2(A-C)v + (A-B+C) = 0 \tag{19}$$

The necessary and sufficient condition for $|\delta| \le 1$ are $A+B+C \ge 0$, $A-C \ge 0$ and $A-B+C \ge 0$. Thus, the following terms have been proved.

$$\eta_1 \cos(\eta h) + \eta_2 \ge 0 \tag{20}$$

$$\eta_4 \cos(\eta h) + \eta_5 \le 0 \tag{21}$$

Hence, the scheme is concluded to be is unconditionally stable.

VI. Numerical Experiments

A. Problem 1

The wave equation is considered as [6], [10]

$$u_{tt} - u_{xx} = 0, \quad 0 \le x \le 1, \quad 0 \le t \le T$$

subject to the initial and boundary conditions

$$u(x,0) = \cos(\pi x) \qquad u(0,t) = \cos(\pi t)$$
$$u_t(x,0) = 0 \qquad \int_0^1 u(x,t)\,dx = 0$$

The analytical solution is given by $\bar{u}(x,t) = \frac{1}{2}\{\cos[\pi(x+t)] + \cos[\pi(x-t)]\}$. The space-time plot for this analytical solution and approximate solution obtained with $h=0.02$ and $\Delta t = 0.1$ are shown in Figs. 1 and 2, respectively. The accuracy of the present method is tested by calculating the absolute error of the problem. Fig. 3 depicts the absolute error of Problem 1 at different time level with $h=0.02$ and $\Delta t = 0.01$. It can be seen that the errors decrease as time increases. Numerically, the absolute errors of this problem are listed in Table I. At $t=5$, the table shows that the present method gives smaller absolute error compare with [6].

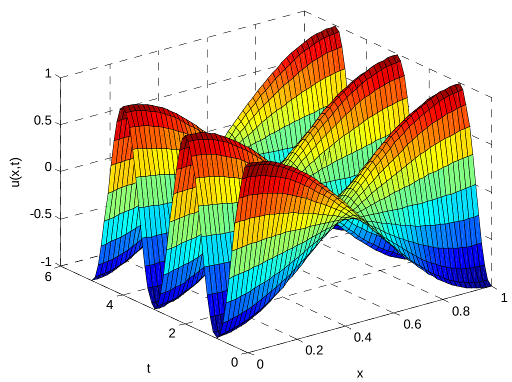

Fig. 1 Space-time graph of analytical solution of Problem 1 with $h=0.02$ and $\Delta t = 0.1$

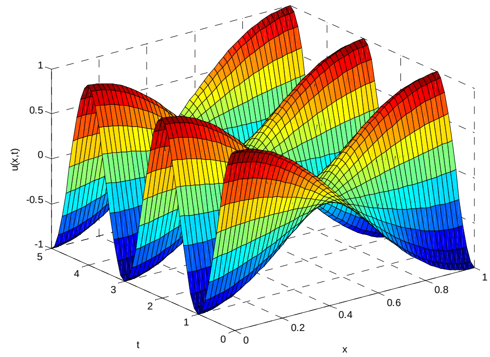

Fig. 2 Space-time graph of approximate solution for Problem 1 with $h = 0.02$ and $\Delta t = 0.1$

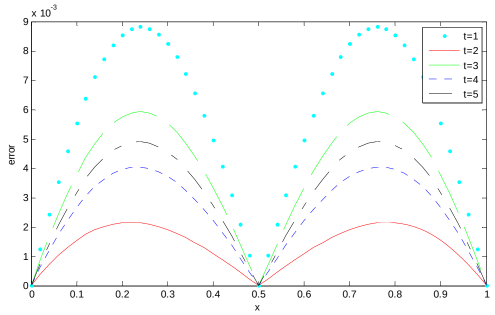

Fig. 3 Absolute error of Problem 1 with $h = 0.02$ and $\Delta t = 0.1$

B. Problem 2

The following one-dimensional wave equation is considered [3], [5], [10]

$$u_{tt} - u_{xx} = \left(\pi^2 + \frac{1}{4}\right) e^{-\frac{t}{2}} \sin(\pi x), \quad 0 \le x \le 1, \quad 0 \le t \le T$$

with the initial and boundary conditions

$$u(x,0) = \sin(\pi x) \qquad u(0,t) = \cos(\pi t)$$

$$u_t(x,0) = -\frac{1}{2}\sin(\pi x) \qquad \int_0^1 u(x,t)\,dx = \frac{2}{\pi} e^{-\frac{t}{2}}$$

The analytical solution of this problem is given as $\bar{u}(x,t) = e^{-t/2}\sin(\pi x)$. Figs. 4 and 5 show the space-time plot of the analytical solution and the approximate solution with $h = \Delta t = 0.02$, respectively. Fig. 6 depicts absolute error of Problem 2 at different time level with $h = \Delta t = 0.02$. The figure shows the errors increase when time increases. Table II lists the maximum error obtained from present method and Dehghan & Shokri method [10] at $t = 0.5$ and $t = 1$ with $h = 0.01$ and $\Delta t = 0.0001$. The comparison show the present method give better results.

TABLE I
ABSOLUTE ERROR OF PROBLEM 1 AT $t = 5$ WITH $h = 0.01$ AND $\Delta t = 0.1$

x	[6]	Present Method
0.2	1.21×10^{-4}	1.12×10^{-4}
0.3	1.15×10^{-4}	1.07×10^{-4}
0.4	6.88×10^{-5}	6.40×10^{-5}
0.5	2.03×10^{-13}	5.05×10^{-15}
0.6	6.88×10^{-5}	6.40×10^{-5}
0.7	1.15×10^{-4}	1.07×10^{-4}
0.8	1.21×10^{-4}	1.12×10^{-4}
0.9	7.97×10^{-5}	7.39×10^{-5}

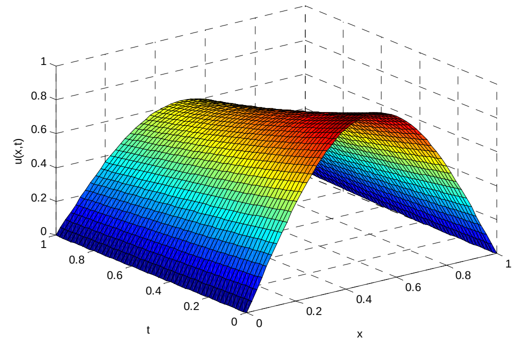

Fig. 4 Space-time graph of approximate solution for Problem 2

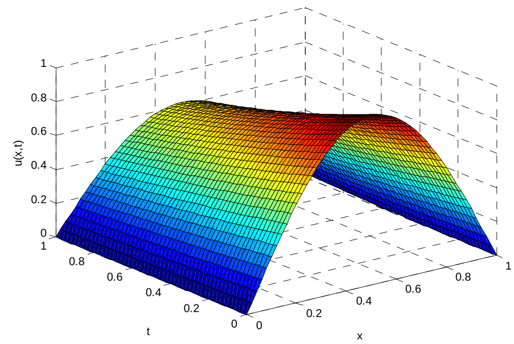

Fig. 5 Space-time graph of approximate solution for Problem 2

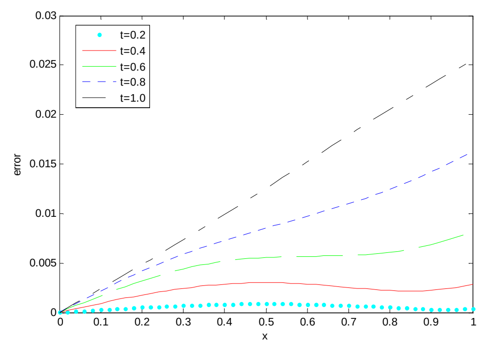

Fig. 6 Absolute error of Problem 2 using $h = \Delta t = 0.02$

TABLE II
MAXIMUM ERROR OF PROBLEM 2 AT $t = 0.5$ AND $t = 1$ WITH $h = 0.01$ AND $\Delta t = 0.0001$

t	MQ – RBF [10]	Present Method
0.5	1.3371×10^{-3}	3.9515×10^{-5}
1.0	2.3794×10^{-3}	2.008×10^{-4}

VII. CONCLUSION

In this work, a numerical method incorporating finite difference approach with cubic trigonometric B-spline had been developed to solve one-dimensional wave equation. B-spline function had been used to interpolate the solution in *x*-direction and finite difference approach had been applied to discretize the time derivative. Based on von Neumann stability analysis, this approach is proved to be unconditionally stable. Two problems were tested. It was found that the solutions are approximated very well. Tables I and II show the errors obtained from present method are less than the errors obtained from the method proposed in literature. Hence, we conclude that this present method approximates the solution very well.

ACKNOWLEDGMENT

The authors gratefully acknowledge the School of Mathematical Sciences, Universiti Sains Malaysia for the facilities used in this research. Thousand thanks is also given to Institute of Engineering Mathematics, Universiti Malaysia Perlis for the SLAB award to the first author.

REFERENCES

[1] F. Shakeri and M. Dehghan, "The method of lines for solution of the one-dimensional wave equation subject to an integral conservation condition," *Computer & Mathematics with Applications,* vol. 56, no. 9, pp. 2175-2188, 2008.

[2] M. Dehghan, "On the solution of an initial-boundary value problem that combines Neumann and integral condition for the wave equation," *Numerical Methods for Partial Differential Equations,* vol. 21, no. 1, pp. 24-40, 2005.

[3] W. T. Ang, "A numerical method for the wave equation subject to a non-local conservation condition," *Applied Numerical Mathematics,* vol. 56, pp. 1054-1060, 2006.

[4] M. Dehghan and M. Lakestani, "The Use of Cubic B-Spline Scaling Functions for Solving the One-dimensional Hyperbolic Equation with a Nonlocal Conservation Condition," *Numerical Methods for Partial Differential Equation,* vol. 23, pp. 1277-1289, 2007.

[5] S. A. Khuri and A. Sayfy, "A spline collocation approach for a generalized wave equation subject to non-local conservation condition," *Applied Mathematics and Computation,* vol. 217, no. 8, pp. 3993-4001, 2010.

[6] J. Goh, A. Abd. Majid and A. I. Md Ismail, "Numerical method using cubic B-spline for the heat and wave equation," *Computer & Mathematics with Application,* vol. 62, no. 12, pp. 4492-4498, 2011.

[7] I. Dag, D. Irk and B. Saka, "A numerical solution of the Burgers' equation using cubic B-splines,"*Applied Mathematics and Computation,* vol. 163, no. 1, pp. 199-211, 2005.

[8] H. Caglar, N. Caglar and K. Elfauturi, "B-spline interpolation compared with finite difference, finite element and finite volume methods which applied to two-point boundary value problems," *Applied Mathematics and Computation,* vol. 175, no. 1, pp. 72-79, 2006.

[9] S. S. Siddiqi and S. Arshed, "Quintic B-spline for the numerical solution of the good Boussinesq equation," *Journal of Egyption Mathematical Society*, to be published.

[10] M. Dehghan and A. Shokri, "A meshless method for numerical solution of the one-dimensional wave equation with an integral condition using radial basis functions," *Numerical Algorithms,* vol. 52, no. 3, pp. 461-477, 2009.