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Optimal Maintenance Policy for a Partially Observable Two-Unit System

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Abstract—In this paper, we present a maintenance model of a two-unit series system with economic dependence. Unit#1 which is considered to be more expensive and more important, is subject to condition monitoring (CM) at equidistant, discrete time epochs and unit#2, which is not subject to CM has a general lifetime distribution. The multivariate observation vectors obtained through condition monitoring carry partial information about the hidden state of unit#1, which can be in a healthy or a warning state while operating. Only the failure state is assumed to be observable for both units. The objective is to find an optimal opportunistic maintenance policy minimizing the long-run expected average cost per unit time. The problem is formulated and solved in the partially observable semi-Markov decision process framework. An effective computational algorithm for finding the optimal policy and the minimum average cost is developed, illustrated by a numerical example.

Keywords—Condition-Based Maintenance, Semi-Markov Decision Process, Multivariate Bayesian Control Chart, Partially Observable System, Two-unit System.

I. Introduction

Since the 1970s, there has been a growing interest in the maintenance modeling and optimization of single unit as well as multiple unit systems. The maintenance optimization surveys by [1]–[6] present the growing body of literature on multi-unit systems.

The interaction between units in the multi-unit systems is an important factor in maintenance decisions. In the early maintenance literature, three types of interactions between units have been considered, namely: economic dependence, structural dependence and stochastic dependence [5].

Economic dependence implies that lower maintenance costs can be incurred when several units are jointly maintained, that is the economies of scale can be incorporated. Structural dependence means that the units form a particular structure and one failed unit causes to perform the maintenance of other units as well. Stochastic dependence also referred to as failure interaction or probabilistic dependence, means that the state of a unit influences the lifetime distribution of other units. Most maintenance models for multi-unit systems consider just one interaction dependence among units, namely economic dependence or structural dependence, since combining these interactions makes the models too complex to be analyzed analytically. In this paper we will consider the economic dependence between two units.

The failure is more costly in multi-unit systems compared to single unit systems, especially in the case of series systems since each failure can cause the breakdown of the whole system. In the case of system failure, non-failed units should not be replaced if they are in a good condition, because their remaining useful life would be wasted. The condition of each unit can specify the condition of a multi-unit system. When maintenance activities are performed jointly, then economies of scale can be obtained and the maintenance cost can be minimized. Therefore, there is a potential to save maintenance costs considerably by implementing preventive maintenance on several units if an opportunity occurs.

To avoid costly failure, different multi-unit system models have been developed based on different maintenance or replacement policies, such as group maintenance policy, opportunistic maintenance policy, multi-level control-limit replacement policy and condition-based maintenance (CBM) methods for multi-unit systems. Except CBM, other policies do not take into account the condition monitoring(CM) information, therefore CBM can predict the system failure more precisely.

Condition-based maintenance is a maintenance program that recommends maintenance actions (decisions) based on the information collected through condition monitoring process [7]. A well established and effective CBM program can eliminate unnecessary maintenance actions, decrease maintenance costs, reduce system downtime and minimize unexpected catastrophic failures. Most existing work reported in the literature focuses only on determining the optimal CBM policy for single unit systems. Replacement and other maintenance decisions are made independently for each component, based on the components' age, condition monitoring data, and the CBM policy. ([8] and [9])

Reference[10] presented an overview of CBM and time-based maintenance (TBM) and compared the challenges of implementing each technique from a practical point of view. They made conclusion that the application of CBM is more realistic and more worthwhile to apply. So far, most published research on condition-based maintenance deals with simple one-unit systems, very few papers deal with multi-unit systems. Reference[11] studied the combination of CBM and opportunistic maintenance. Reference[12] presented a condition based maintenance model for two-unit series systems with exponential failures and fixed inspection intervals. The objective was to minimize the long-run average cost. Reference[13] presented a model for continuously monitored deterioration systems where the unit was subjected

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to stochastic degradation. They applied Monte Carlo (MC) simulation to find the optimal degradation threshold that minimizes the expected total maintenance cost over a given mission time, then they extended the model to multi-unit repairable systems.

Reference[14] proposed a model to maximize both profit and availability simultaneously. They applied Monte Carlo simulation method for degradation process, and Genetic Algorithms (GA) to find the optimal degradation level. Reference[15] considered condition-based maintenance to model a two-unit series system. They supposed that there is an economic dependence between units. The condition of the units was modelled by a stochastic process and the inspection intervals were not constant. They suggested four thresholds for each unit, two thresholds for individual preventive and corrective maintenance and two thresholds for the joint preventive and corrective maintenance. Reference[16] proposed a maintenance model for n-unit systems. The deterioration levels of the units were observed at sampling epochs and the degradation process was modelled by a continuous-time jump diffusion model. They supposed that there is an interaction among the units of the system. The proper maintenance action on the system or the units can be performed if the deterioration of the system or the unit crosses a critical threshold level. They developed a simulation-based optimization heuristic to minimize the long-run expected maintenance cost in order to obtain the optimum critical threshold value. Reference[17] proposed a model for a series system composed of two units, which were subject to continuous deterioration and stochastic failure. The units' condition was monitored periodically and they presented a model to determine the inspection interval as well as preventive maintenance and preventive replacement thresholds for the whole system.

Reference[18] developed a model for a multi-unit system. They assumed that the degradation condition of each unit can be accurately assessed through inspection. The degradation follows a Markov model with continuous time and discrete states. Two types of maintenance actions were considered: minimal and major maintenance. They considered resource constraints in their model (one repairman for three machines). Reference[19] built a proportional hazards model (PHM) for a multi-unit system and they supposed that there is economic dependence among units. Reference[20] presented a CBM policy for multi-unit system with continuous stochastic deterioration. They considered a high set-up cost of maintenance to minimize the long-run average cost as an objective and a joint maintenance interval is the decision variable. Unlike many previous efforts in multi-unit systems which consider the CM or age information for each unit, in this paper we will model a series system with two units, where economic dependency exists among these units and the combination of CM and age information will be considered. The objective is to derive the optimal maintenance policy to minimize the long-run expected average cost. Unit#1 is more expensive than unit#2 so that it is subject to condition monitoring and the obtained information is used for maintenance decision-making while the other unit is cheaper and only age information of this unit is available.

The remainder of the paper is organized as follows. Section II summarizes the assumptions and the details of the proposed model and presents the problem formulation. In Section III, we present the Bayesian control policy which is used for maintenance decision-making under partial information. In Section IV, an effective computational algorithm in the SMDP framework based on the policy iteration algorithm is developed. The effectiveness of the proposed model is investigated in Section V by using a numerical example. In Section VI, we discuss possible extensions to our model, and provide concluding remarks.

II. Model Formulation

Consider a series system consisting of two operating units. We assume that failure of either of the units causes a breakdown of the system. For the maintenance cost, economies of scale are incorporated in case maintenance activities are combined, so these two units have economic dependency. One unit is the core part of the system and more expensive than the other one. The more expensive unit (unit#1) is subject to condition monitoring, while only the age information is available for unit#2.

The condition of unit#1 can be categorized into one of three states: a healthy or "as good as new" state (state 0), unhealthy or warning state (state 1) and a failure state (state 2), where only failure state is observable. Let $\xi_1 = inf\{t \in R^+ : X_t = 2\}$ be the observable failure time of unit#1. We model the state process of unit#1 as a continuous-time homogeneous Markov process $(X_t : t \in R^+)$, with state space $\Omega = \{0, 1, 2\}$. The unit#1 is assumed to start in a healthy state, i.e. $P(X_0 = 0) = 1$. It is assumed that the sojourn time in the state i for $i = 0, 1$ is exponentially distributed.

The age information of unit#2 is available and its lifetime distribution function is of a general type denoted by $f_2(t)$ and ξ_2 represents the failure time.

The system state at any time is determined by the states of both units. In other words, the system state will be presented in two dimensions. Let S be the state space of the whole system. Then $S = \{(i, n, h) \mid i \in \Omega, n, h \in N\}$, i is the state of unit#1 which can be in healthy, unhealthy or failure state, whereas $n\Delta$ and $h\Delta$ represent age of unit#1 and unit#2, respectively and the sampling interval is indicated by Δ. The failure time of the system is denoted by ξ where $\xi = min(\xi_1, \xi_2)$.

Unit#1 can make transition from state 0 to state 1 with probability P_{01} or from state 0 to state 2 with probability P_{02}, where $P_{01} + P_{02} = 1$. The instantaneous transition rate $\lambda_{ij}, i, j \in \Omega$, is defined by:

$$\begin{aligned} \lambda_{ij} &= \lim_{u \to 0^+} \frac{P(X_{t+u} = j \mid X_t = i)}{u} < +\infty, i \neq j \in \Omega \\ \lambda_{ii} &= -\sum_{i \neq j} \lambda_{ij}. \end{aligned} \tag{1}$$

To model monotonic system deterioration, we assume that the state process is non-decreasing with probability 1, i.e. $\lambda_{ij} = 0$ for all $j < i$. We also assume that unit#1 is more likely to fail in the unhealthy state $i = 1$ than in healthy state $i = 0$, it means that $\lambda_{02} < \lambda_{12}$ and the failure state is absorbing state. Upon unit#1 failure, corrective maintenance is carried out which brings the unit#1 to the healthy state, and simultaneously the opportunistic maintenance will be performed for unit#2. The transition probability matrix $\mathbf{P}(t)$=$(P_{ij}(t))_{i,j\in\Omega}$, is obtained by solving the Kolmogorov backward differential equations, and it is given by

$$P_{ij}(t)=\begin{bmatrix} e^{-\nu_0 t} & \frac{\lambda_{01}(e^{-\nu_1 t}-e^{-\nu_0 t})}{\nu_0-\nu_1} & 1-e^{-\nu_0 t}-\frac{\lambda_{01}(e^{-\nu_1 t}-e^{-\nu_0 t})}{\nu_0-\nu_1} \\ 0 & e^{-\nu_1 t} & 1-e^{-\nu_1 t} \\ 0 & 0 & 1 \end{bmatrix} \quad (2)$$

where $P_{ij}(t) = P(X_{s+t} = j \mid X_s = i)$.

The unit#1 condition is monitored at equidistant sampling times $\Delta, 2\Delta, ...$ and the data $Y_1, Y_2, ... \in R^d$ are collected at these times which represent partial information about the system state. In this paper, we assume that the observations have d-dimensional normal distribution $N_d(\mu_i, \Sigma_i)$ and are conditionally independent given the system state i.e.

$$g_{Y_n|X_{n\Delta}}(y\,|\,i)=\frac{1}{\sqrt{(2\pi)^d det(\Sigma_i)}}\exp(-\frac{1}{2}(y-\mu_i)^T{\Sigma_i}^{-1}(y-\mu_i)) \quad (3)$$

where μ_0, $\mu_1 \in R^d$, Σ_0, $\Sigma_1 \in R^{d\times d}$ are known observation process parameters. We note that these assumptions are satisfied for the residuals obtained using a reference model approach (see e.g.[9]).

Consider unit#1. The decision maker takes a sample at each sampling epoch at a cost C_s and the posterior probability i.e. the probability of unit#1 being in the warning state (see e.g.[9]) is updated using the Bayes' rule. When the posterior probability exceeds a critical limit $CL \in [0, 1]$ on the Bayesian control chart, full inspection is initiated. If the unit is found to be in the healthy state, it is left operational and unit#2 adjustment is performed. If it is found to be in the warning state (state1), unit#1 is preventively replaced at a cost C_{P1}, which takes T_P time units and unit#2 will be adjusted, simultaneously. If unit#1 fails before the chart signals, failure replacement is triggered with corresponding cost C_{F1} accompanied by unit#2 adjustment and unit#1 failure replacement takes T_F time units. When the age of unit#1 exceeds the pre-determined age M then preventive maintenance will be triggered and unit#2 is adjusted. If unit#2 failure occurs, the posterior probability of unit#1 is updated, and if it crosses the opportunistic level (OL) then inspection of unit#1 is performed, which can be indicated false alarm or true alarm. If it is true alarm, then the opportunistic maintenance will be performed on both units, otherwise just unit#2 will be replaced at a cost C_{F2} and unit#1 will be left operational. Adjustment cost of unit#2 is equal to C_{P2}. The inspection cost C_I will be considered when inspection is performed, which takes T_I unit time. Whenever the system is stopped, the fixed cost K and C_{LP} cost rate will be charged. A fixed cost K is incurred for any maintenance operation and C_{LP} will be considered as the cost rate of lost production.

After collecting an observation sample and processing information of unit#1 and age information of unit#2, the decision maker follows the policy described above and the objective is to determine the optimal policy that minimizes the long-run expected average cost per unit time.

III. The Bayesian Control Policy

The Bayesian control chart monitors the posterior probability that the system is in the warning state. This approach has received a lot of attention ([21] and [9]) and was proven to be the optimal tool for decision making by [22]. It is well-known from the theory of partially observable Markov decision process [23] that the posterior probability that the system is in the warning state is sufficient for optimal decision making. Thus, at each sampling epoch the new observations are collected and the posterior probability is updated using Bayes' theorem. The posterior probability that unit#1 is in the warning state is denoted by:

$$\Pi_n = P(X_{n\Delta} = 1 \mid \xi_1 > n\Delta, Y_1, ..., Y_n, \Pi_{n-1}, n\Delta < M). \quad (4)$$

The posterior probability can be expressed as:

$$\Pi_n = \frac{P(X_{n\Delta} = 1, \xi_1 > n\Delta, Y_1, ..., Y_n, \Pi_{n-1}, n\Delta < M)}{P(\xi_1 > n\Delta, Y_1, ..., Y_n, \Pi_{n-1}, n\Delta < M)}. \quad (5)$$

The objective is to find the optimal value of the control and opportunistic limits $(CL^*, OL^*) \in (0, 1)$ such that the long-run expected average cost per unit time is minimized. From renewal theory, the long-run expected average cost per unit time is calculated for any control policy as the expected system cost incurred in one cycle (CC) divided by the expected cycle length (CL). The optimization problem can then be formulated as:

$$\frac{E_{(CL^*,OL^*)}(CC)}{E_{(CL^*,OL^*)}(CL)} = inf_{(CL,OL)\in[0,1]}\Big(\frac{E_{(CL,OL)}(CC)}{E_{(CL,OL)}(CL)}\Big) \quad (6)$$

where a cycle is completed when the system is brought back to the healthy state. We now develop an efficient computational algorithm in the semi-Markov decision process (SMDP) framework to determine the optimal control limit $(CL^*, OL^*) \in (0, 1)$. Using Bayes' Theorem, Π_n can be expressed as:

$$\begin{aligned}\Pi_n = {} & g(y\,|\,1).[P_{01}(\Delta)(1-\Pi_{n-1})+P_{11}(\Delta)\Pi_{n-1}]/\Big[g(y\,|\,0).P_{00}(\Delta) \\ & (1-\Pi_{n-1})+g(y\,|\,1).[P_{01}(\Delta)(1-\Pi_{n-1})+P_{11}(\Delta)\Pi_{n-1}]\Big]. \quad (7)\end{aligned}$$

We further need to simplify the posterior probability under the assumption $\Sigma_0 \neq \Sigma_1$, which is common in maintenance applications. We have

$$\begin{aligned}\frac{g_{Y_n|X_{n\Delta}}(y\,|\,0)}{g_{Y_n|X_{n\Delta}}(y\,|\,1)} &= \frac{\frac{1}{\sqrt{(2\pi)^d\ det(\Sigma_0)}}\exp(-(1/2)(y-\mu_0)^T{\Sigma_0}^{-1}(y-\mu_0))}{\frac{1}{\sqrt{(2\pi)^d\ det(\Sigma_1)}}\exp(-(1/2)(y-\mu_1)^T{\Sigma_1}^{-1}(y-\mu_1))} \\ &= S\times\exp\frac{1}{2}((Y_n-B)^TA(Y_n-B)+C), \quad (8)\end{aligned}$$

where constants A,B,C and S are given by

$$\begin{aligned} A &= \Sigma_1^{-1} - \Sigma_0^{-1} \\ B &= (\Sigma_1^{-1} - \Sigma_0^{-1})^{-1}(\Sigma_1^{-1}\mu_1 - \Sigma_0^{-1}\mu_0) \\ C &= (\mu_1^T\Sigma_1^{-1}\mu_1 - \mu_0^T\Sigma_0^{-1}\mu_0) - B^T(\Sigma_1^{-1}\mu_1 - \Sigma_0^{-1}\mu_0) \\ S &= (det(\Sigma_1).(det(\Sigma_0))^{-1})^{1/2}. \quad (9)\end{aligned}$$

So, (7) can be simplified to (10) if we denote $V_n = (Y_n - B)^T A(Y_n - B)$.

$$\Pi_n = \frac{P_{01}(\Delta)(1-\Pi_{n-1}) + P_{11}(\Delta)\Pi_{n-1}}{S \times \exp(\frac{V_n+C}{2}) P_{00}(\Delta).(1-\Pi_{n-1}) + [P_{01}(\Delta)(1-\Pi_{n-1}) + P_{11}(\Delta)\Pi_{n-1}]}$$
$$= \frac{C^1_{\Pi_{n-1}}}{S \times \exp(\frac{V_n+C}{2}).C^0_{\Pi_{n-1}} + C^1_{\Pi_{n-1}}}, \tag{10}$$

where

$$\begin{aligned} C^1_{\Pi_{n-1}} &= P_{01}(\Delta)(1-\Pi_{n-1}) + P_{11}(\Delta)\Pi_{n-1}, \\ C^0_{\Pi_{n-1}} &= P_{00}(\Delta).(1-\Pi_{n-1}). \end{aligned} \tag{11}$$

In the next section, we develop the computational algorithm using the SMDP formulation which will be used to find the optimal CBM policy.

IV. Computational Algorithm in the SMDP Framework

Typically, computing the long-run expected average cost in the SMDP framework requires discretization of [0,1], which is the state space of the posterior probability process $(\Pi_t, t \geq 0)$. Suppose that at sampling time $n\Delta$ the unit#1 is operational, and we obtain the posterior probability Π_n considering new data. We partition the state space of the posterior probability into L subintervals. The SMDP is defined to be in state l, $1 \leq l \leq L$, if the current value of the posterior probability Π_n lies in the interval $[\frac{l-1}{L}, \frac{l}{L})$. Before calculating the transition probabilities, the exact definition of states is required.

- State (0,n,0): both units are in as good as new condition and age of unit#1 is $n\Delta$.
- State (m, n, h): the first component represents the state of unit#1, i.e. $\Pi_n = m$, the second and third components are the age of unit#1 and unit#2, respectively. i.e. $n\Delta$ and $h\Delta$.
- State PM : unit#1 is in PM state which means that PM action is performed.
- State F : unit#1 failure occurs.

Now, the SMDP is determined by the following quantities:

$P_{(m,n-1,h-1)(l,n,h)}$= the probability that at the next decision epoch the system will be in state (l, n, h) given the current state is $(m, n-1, h-1)$,

$\tau_{(m,n-1,h-1)}$= Expected sojourn time until the next decision epoch given the current state is $(m, n-1, h-1)$,

$C_{(m,n-1,h-1)}$= Expected cost incurred until the next decision epoch given the current state is $(m, n-1, h-1)$.

Once all of these quantities are defined, for a fixed control and opportunistic limits (CL, OL), the long-run expected average cost $g(CL, OL)$ can be obtained by solving the following linear equations:

$$\begin{aligned} \nu_{(m,n-1,h-1)} &= C_{(m,n-1,h-1)} - g(CL,OL)\tau_{(m,n-1,h-1)} \\ &\quad + \sum_{(l,n,h)\in S} P_{(m,n-1,h-1)(l,n,h)}\nu_{(l,n,h)}, \\ \nu_{(s,n,h)} &= 0 \quad , \quad for \quad some \quad (s,n,h) \in S. \end{aligned} \tag{12}$$

So, the optimal control and opportunistic limits $(CL^*, OL^*) \in [0, 1]$ and the corresponding minimum long-run expected average cost per unit time $g(CL^*, OL^*)$ can be found by $g(CL^*, OL^*) = inf_{(CL,OL)\in[0,1]}\{g(CL, OL)\}$. Next, we will derive the closed form expression for the SMDP quantities.

A. Transition Probabilities

The SMDP transition probability $P_{(m,n-1,h-1)(l,n,h)}$ from state $(m, n-1, h-1)$ to state (l, n, h) where $0 < m, l < CL$ and unit#2 is working properly can be calculated as follows:

$$P_{(m,n-1,h-1)(l,n,h)} = P(\frac{l-1}{L} \leq \Pi_n < \frac{l}{L}, \xi_2 > h\Delta, \xi_1 > n\Delta \,|\, \xi_1 > (n-1)\Delta, \Pi_{n-1}, \xi_2 > (h-1)\Delta). \tag{13}$$

It means that the probability that the current value of the posterior probability Π_n lies in the interval $[\frac{l-1}{L}, \frac{l}{L})$, given that we have all the information up to now and we know that both units did not fail. Then, this probability can be calculated as:

$$\begin{aligned} &P_{(m,n-1,h-1)(l,n,h)} = P(\frac{l-1}{L} \leq \Pi_n < \frac{l}{L}, \xi_2 > h\Delta, \xi_1 > n\Delta \,|\, \xi_1 > (n-1)\Delta, \Pi_{n-1}, \xi_2 > (h-1)\Delta, n\Delta < M) \\ &= P(\frac{l-1}{L} \leq \Pi_n < \frac{l}{L} \mid \xi_1 > n\Delta, \Pi_{n-1}, n\Delta < M) \\ &\times P(\xi_1 > n\Delta \,|\, \xi_1 > (n-1)\Delta, \Pi_{n-1}, n\Delta < M) \times \frac{R_2(h\Delta)}{R_2((h-1)\Delta)} \\ &= P(\frac{l-1}{L} \leq \Pi_n < \frac{l}{L} \mid \xi_1 > n\Delta, \Pi_{n-1}, n\Delta < M) \\ &\times R_1(\Delta \mid \Pi_{n-1} = m) \times \frac{R_2(h\Delta)}{R_2((h-1)\Delta)} \quad for \quad 0 \leq m, l < CL. \end{aligned} \tag{14}$$

The closed form of the first term on the right hand side of (14) can be computed by using (10) as

$$\begin{aligned} &P(\frac{l-1}{L} \leq \Pi_n < \frac{l}{L} \mid \xi_1 > n\Delta, \Pi_{n-1}, n\Delta < M) \\ &= P\Big[\frac{l-1}{L} \leq \frac{C^1_{\Pi_{n-1}}}{S.\exp(\frac{V_n+C}{2}).C^0_{\Pi_{n-1}} + C^1_{\Pi_{n-1}}} < \frac{l}{L} \mid X_{n\Delta} = 0\Big] \\ &\qquad \times (\frac{C^0_{\Pi_{n-1}}}{C^0_{\Pi_{n-1}} + C^1_{\Pi_{n-1}}}) \\ &+ P\Big[\frac{l-1}{L} \leq \frac{C^1_{\Pi_{n-1}}}{S.\exp(\frac{V_n+C}{2}).C^0_{\Pi_{n-1}} + C^1_{\Pi_{n-1}}} < \frac{l}{L} \mid X_{n\Delta} = 1\Big] \\ &\qquad \times (\frac{C^1_{\Pi_{n-1}}}{C^0_{\Pi_{n-1}} + C^1_{\Pi_{n-1}}}) \\ &= P\Big[2ln[\frac{(1-\frac{l}{L})C^1_{\Pi_{n-1}}}{\frac{l}{L}C^0_{\Pi_{n-1}}.S}] - C \leq V_n < 2ln[\frac{(1-\frac{l-1}{L})C^1_{\Pi_{n-1}}}{\frac{l-1}{L}C^0_{\Pi_{n-1}}.S} - C \,|\, X_{n\Delta} = 0\Big] \\ &\qquad \times (\frac{C^0_{\Pi_{n-1}}}{C^0_{\Pi_{n-1}} + C^1_{\Pi_{n-1}}}) \\ &+ P\Big[2ln[\frac{(1-\frac{l}{L})C^1_{\Pi_{n-1}}}{\frac{l}{L}C^0_{\Pi_{n-1}}.S}] - C \leq V_n < 2ln[\frac{(1-\frac{l-1}{L})C^1_{\Pi_{n-1}}}{\frac{l-1}{L}C^0_{\Pi_{n-1}}.S} - C \,|\, X_{n\Delta} = 1\Big] \\ &\qquad \times (\frac{C^1_{\Pi_{n-1}}}{C^0_{\Pi_{n-1}} + C^1_{\Pi_{n-1}}}). \end{aligned} \tag{15}$$

Reference [24] showed that any indefinite quadratic form in multivariate normal vectors $Q = G^T AG$, where $G \sim N_r(\mu, \Sigma)$, $r \in N$, can be expressed as the difference of two linear combinations of independent non-central chi-square variables. Using this property, we can derive the first part of (15), since $(Y_n - B) \mid X_{n\Delta} = i \sim N_2(\mu_i - b, \Sigma_i)$. Therefore, (15) can be simplified as:

$$P(\frac{l-1}{L} \le \Pi_n < \frac{l}{L} \mid \xi_1 > n\Delta, \Pi_{n-1})$$
$$= D_{q_0}(Q_0).[\frac{C^0_{\Pi_{n-1}}}{C^0_{\Pi_{n-1}}+C^1_{\Pi_{n-1}}}] + D_{q_1}(Q_1).[\frac{C^1_{\Pi_{n-1}}}{C^0_{\Pi_{n-1}}+C^1_{\Pi_{n-1}}}] \quad (16)$$

where $D_{q_i}(Q_i), i = \{0,1\}$ is the cumulative density of a quadratic form in normal vectors.

The conditional reliability of unit#1 is obtained by

$$R_1(t|\Pi_n) = P(\xi_1 > n\Delta+t|\xi_1 > n\Delta, \Pi_n, (n\Delta+t) < M)$$
$$= (1-\Pi_n)(1-P_{02}(t)) + \Pi_n(1-P_{12}(t)). \quad (17)$$

Equation (14) can be calculated by substituting (16), (17) and the reliability function of unit#2.

When the posterior probability of unit#1 crosses the control limit $CL \le l < 1$, which means that it made transition from state $(m, n-1, h-1)$ to state (l, n, h), then after the inspection it can be either in state (0,n,0) or PM, where PM is the preventive maintenance state. When the result of inspection reveals that false alarm has occurred, then the system will be in as good as new condition or in state (0,n,0). Otherwise, true alarm occurs and the next state will be PM. The posterior probability is approximated by the mid-point of the interval. Thus, these transition probabilities are given by:

$$P_{(l,n,h)(PM)} = \frac{l-0.5}{L}. \quad (18)$$

$$P_{(l,n,h)(0,n,0)} = 1 - (\frac{l-0.5}{L}). \quad (19)$$

When unit#1 is operating well and unit#2 failure happens at time t before the sampling point, then the posterior probability is updated as:

$$\Pi'_t(\pi) = P(X_t = 1 \mid \Pi'_0 = \pi, \xi_1 > t, t < M)$$
$$= \frac{P_{01}(t)(1-\pi) + P_{11}(t)\pi}{R_1(t \mid \pi)}. \quad (20)$$

If the updated posterior probability does not reach the opportunistic limit, then unit#2 is replaced and the system will be ready at the next sampling epoch, so the state will be $(l, n, 0)$ and the transition probability is given below, where $\xi'_2 = \xi_2 - h\Delta$, $\xi'_1 = \xi_1 - n\Delta$ and $0 \le m < OL$:

$$P_{(m,n,h)(l,n,0)} = P(\xi'_2 < \xi'_1, \xi'_2 < \Delta, \Pi'_{\xi'_2}(m) < OL, \Pi'_\Delta = l|\xi'_1 > 0,$$
$$\Pi'_0 = m, \xi'_2 > 0)$$
$$= \int_0^\Delta P(\xi_1 > t, \Pi'_t(m) < OL, \Pi'_\Delta = l \mid \xi_1 > 0, \Pi'_0 = m) \times f_2(t|h)dt$$
$$= \int_0^\Delta P(\Pi'_t(m) < OL, \Pi'_\Delta = l \mid \Pi'_0 = m) \times f_2(t \mid h) \times R_1(t \mid m)dt$$
$$= \int_0^\Delta P(\Pi'_\Delta = l|\Pi'_0 = m) \times P(\Pi'_t(m) < OL \mid \Pi'_0 = m) \times f_2(t|h)$$
$$\times R_1(t|m)dt, \quad (21)$$

where $M > n\Delta + t$ and

$$f_2(t \mid h) = \frac{d}{dt} P\Big(\xi_2 \le (h\Delta + t) \mid \xi_2 > h\Delta\Big). \quad (22)$$

The first part of (21) is derived by using (15) as follows:

$$P(\Pi'_\Delta = l|\Pi'_0 = m) = P(\frac{l-1}{L} \le \Pi'_\Delta < \frac{l}{L} \mid \Pi'_0 = m)$$
$$= P\Big[2ln[\frac{(1-\frac{l}{L})C^1_{\Pi_0}}{\frac{l}{L}C^0_{\Pi_0}.S}] - C \le V < 2ln[\frac{(1-\frac{l-1}{L})C^1_{\Pi_0}}{\frac{l-1}{L}C^0_{\Pi_0}.S} - C \mid X_\Delta = 0\Big]$$
$$\times (\frac{C^0_{\Pi_0}}{C^0_{\Pi_0}+C^1_{\Pi_0}})$$
$$+ P\Big[2ln[\frac{(1-\frac{l}{L})C^1_{\Pi_t(m)}}{\frac{l}{L}C^0_{\Pi_0}.S}] - C \le V < 2ln[\frac{(1-\frac{l-1}{L})C^1_{\Pi_0}}{\frac{l-1}{L}C^0_{\Pi_0}.S} - C \mid X_\Delta = 1\Big]$$
$$\times (\frac{C^1_{\Pi_0}}{C^0_{\Pi_0}+C^1_{\Pi_0}}). \quad (23)$$

The second part of (21) i.e. $P(\Pi'_t(m) < OL \mid \Pi_0 = m)$ can be zero or one which depends on the value of the updated posterior probability of unit#1 at time t. We solve the following inequality:

$$\frac{P_{01}(t)(1-m) + P_{11}(t)m}{R_1(t \mid m)} < OL$$
$$= \frac{P_{01}(t)(1-m) + P_{11}(t)m}{(1-m)(1-P_{02}(t)) + m(1-P_{12}(t))} < OL$$
$$= \frac{\frac{\lambda_{01}(e^{-\nu_1 t} - e^{-\nu_0 t})}{\nu_0 - \nu_1}(1-m) + me^{-\nu_1 t}}{(1-m)(e^{-\nu_0 t} + \frac{\lambda_{01}(e^{-\nu_1 t} - e^{-\nu_0 t})}{\nu_0 - \nu_1}) + m(e^{-\nu_1 t})} < OL$$
$$\frac{\lambda_{01}(e^{-\nu_1 t} - e^{-\nu_0 t})(1-m) + m(\nu_0 - \nu_1)e^{-\nu_1 t}}{(1-m)\Big((\nu_0 - \nu_1)e^{-\nu_0 t} + \lambda_{01}(e^{-\nu_1 t} - e^{-\nu_0 t})\Big) + m(\nu_0 - \nu_1)e^{-\nu_1 t}} < OL$$
$$\frac{\lambda_{01}(e^{(-\nu_1+\nu_0)t} - 1)(1-m) + m(\nu_0 - \nu_1)e^{(-\nu_1+\nu_0)t}}{(1-m)\Big((\nu_0 - \nu_1) + \lambda_{01}(e^{(-\nu_1+\nu_0)t} - 1)\Big) + m(\nu_0 - \nu_1)e^{(-\nu_1+\nu_0)t}} < OL$$
$$e^{(-\nu_1+\nu_0)t} < \frac{OL(1-m)\Big((\nu_0 - \nu_1) - \lambda_{01}\Big)}{(1-OL)(\lambda_{01}(1-m) + m(\nu_0 - \nu_1))}$$
$$t < ln(\frac{OL(1-m)\Big((\nu_0 - \nu_1) - \lambda_{01}\Big)}{(1-OL)(\lambda_{01}(1-m) + m(\nu_0 - \nu_1))}) \times \frac{1}{(\nu_0 - \nu_1)} \quad (24)$$

where the posterior (m) is approximated by the mid-point of the interval, i.e. $\frac{m-0.5}{L}$, so we substitute it in (24) as:

$$t < ln(\frac{OL(L-m+0.5)(\nu_0 - \nu_1 - \lambda_{01})}{(1-OL)(\lambda_{01}(L-m+0.5) + (m-0.5)(\nu_0 - \nu_1))}) \times \frac{1}{(\nu_0 - \nu_1)} \quad (25)$$

and

$$P(\Pi'_t(m) < OL|m) = \begin{cases} 1 & ; t < \frac{ln(\frac{OL(L-m+0.5)(\nu_0 - \nu_1 - \lambda_{01})}{(1-OL)(\lambda_{01}(L-m+0.5) + (m-0.5)(\nu_0 - \nu_1))})}{(\nu_0 - \nu_1)} \\ 0 & ; O/W. \end{cases} \quad (26)$$

If false alarm has occurred, then unit#2 is replaced and the system will be ready at the next sampling epoch after inspection. Then, the system state will be $(0, n, 0)$ and the transition probability is given by:

$$
\begin{aligned}
&P_{(m,n,h)(0,n,0)} = P(\xi_2^{'} < \xi_1^{'}, \xi_2^{'} < \Delta, \Pi_{\xi_2^{'}}^{'}(m) \geq OL, X_{\xi_2^{'}} = 0 \mid \xi_1^{'} > 0, \\
&\Pi_0^{'} = m, \xi_2^{'} > 0) \\
&= \int_0^{\Delta} P(\xi_1^{'} > t, \Pi_t^{'}(m) \geq OL, X_t = 0 \mid \xi_1^{'} > 0, \Pi_0^{'} = m) \times f_2(t \mid h) dt \\
&= \int_0^{\Delta} P(\Pi_t^{'}(m) \geq OL, X_t = 0 \mid \Pi_0^{'} = m) \times f_2(t \mid h) dt \\
&= \int_0^{\Delta} P(X_t = 0 \mid \Pi_t^{'}(m) \geq OL, \Pi_0^{'} = m) \times P(\Pi_t^{'}(m) \geq OL \mid \Pi_0^{'} = m) \times f_2(t \mid h) dt \\
&= \int_0^{\Delta} P_{00}(t)(1 - \Pi_0^{'}) \times P(\Pi_t^{'}(m) \geq OL \mid \Pi_0^{'} = m) \times f_2(t \mid h) dt \qquad (27)
\end{aligned}
$$

where $M > n\Delta + t$ and

$$
P(\Pi_t^{'}(m) \geq OL \mid m) = \begin{cases} 1 & ; t \geq \dfrac{ln\left(\frac{OL(L-m+0.5)(\nu_0-\nu_1-\lambda_{01})}{(1-OL)(\lambda_{01}(L-m+0.5)+(m-0.5)(\nu_0-\nu_1))}\right)}{(\nu_0-\nu_1)} \\ 0 & ; O/W. \end{cases} \qquad (28)
$$

Otherwise, true alarm occurs and the next state will be PM.

$$
\begin{aligned}
&P_{(m,n,h)(PM)} = P(\xi_2^{'} < \xi_1^{'}, \xi_2^{'} < \Delta, \Pi_{\xi_2^{'}}^{'}(m) \geq OL, X_{\xi_2^{'}} = 1 \mid \xi_1^{'} > 0, \\
&\Pi_0^{'} = m, \xi_2^{'} > 0) \\
&= \int_0^{\Delta} P(\xi_1^{'} > t, \Pi_t^{'}(m) \geq OL, X_t = 1 \mid \xi_1 > 0, \Pi_0^{'} = m) \times f_2(t \mid h) dt \\
&= \int_0^{\Delta} P(\Pi_t^{'}(m) \geq OL, X_t = 1 \mid \Pi_0^{'} = m) \times f_2(t \mid h) dt \\
&= \int_0^{\Delta} P(X_t = 1 \mid \Pi_t^{'}(m) \geq OL, \Pi_0^{'} = m) \times P(\Pi_t^{'}(m) \geq OL \mid \Pi_0^{'} = m) \times f_2(t \mid h) dt \\
&= \int_0^{\Delta} (P_{01}(t)(1 - \Pi_0^{'}) + P_{11}(t)\Pi_0^{'}) \times P(\Pi_t^{'}(m) \geq OL \mid \Pi_0^{'} = m) \times f_2(t \mid h) dt. \qquad (29)
\end{aligned}
$$

When the age of unit#1 exceeds the pre-determined age M, then preventive maintenance will be triggered and unit#2 is adjusted simultaneously, so the transition probability is given by:

$$
P_{(m,n,h)(PM)} = 1; \quad 0 < m < CL, \quad n\Delta \geq M. \qquad (30)
$$

When unit#1 is in the PM state, then mandatory replacement of unit#1 and adjustment of unit#2 are performed and the system goes back to state (0,0,0). We have:

$$
P_{(PM)(0,0,0)} = 1. \qquad (31)
$$

When unit#1 failure occurs, then the next state will be (F) which is the failure state and the transition probability $P_{(m,n-1,h-1)(F)}$ from state $(m, n-1, h-1)$ to state (F), where $0 \leq m < CL$ can be calculated as follows:

$$
\begin{aligned}
&P_{(m,n-1,h-1)(F)} = P(\xi_1^{'} \leq \Delta, \xi_1^{'} < \xi_2^{'} \mid \xi_1^{'} > 0, \Pi_0^{'} = m, n\Delta < M) \\
&= \int_0^{\Delta} P(\xi_1^{'} \leq \Delta, \xi_1^{'} < \xi_2^{'} \mid \xi_1^{'} > 0, \Pi_0^{'} = m, \xi_2^{'} = u) \times f_2(u \mid h) du \\
&= \int_0^{\Delta} (1 - R_1(u \mid \Pi_0^{'} = m)) \times f_2(u \mid h) du. \qquad (32)
\end{aligned}
$$

where $f_2(u \mid h)$ can be obtained from (22).

B. Expected Sojourn Times

The expected sojourn time given the state is (m, n, h) where $0 \leq m < CL$ can be derived as follows:

$$
\begin{aligned}
&\tau_{(m,n,h)} = E(Sojourn \;\; time \mid \Pi_0^{'} = m, \xi_1^{'} > 0, \xi_2^{'} > 0) \\
&= \sum_{l=m}^{L} E(Sojourn \;\; time \mid \Pi_0^{'} = m, \xi_1^{'} > 0, \xi_2^{'} > 0, \Pi_{\Delta}^{'} = l, \xi_1^{'} > \Delta, \\
&\xi_2^{'} > \Delta) \times P_{(m,n,h)(l,n+1,h+1)} + \sum_{l=m}^{L} E(Sojourn \;\; time \mid \Pi_0^{'} = m, \xi_1^{'} > 0, \\
&\xi_2^{'} > 0, \Pi_{\Delta}^{'} = l, \xi_2^{'} < \xi_1^{'}, \xi_2^{'} < \Delta, \Pi_{\xi_2^{'}}^{'}(m) < OL) \times P_{(m,n,h)(l,n+1,0)} + \\
&E(Sojourn \; time \mid \Pi_0^{'} = m, \xi_1^{'} > 0, \xi_2^{'} > 0, \xi_2^{'} < \xi_1^{'}, \xi_2^{'} < \Delta, \Pi_{\xi_2^{'}}^{'}(m) \geq OL, \\
&X_{\xi_2^{'}} = 0) \times P_{(m,n,h)(0,n+1,0)} + E(Sojourn \; time \mid \Pi_0^{'} = m, \xi_1^{'} > 0, \\
&\xi_2^{'} > 0, \xi_2^{'} < \xi_1^{'}, \xi_2^{'} < \Delta, \Pi_{\xi_2^{'}}^{'}(m) \geq OL, X_{\xi_2^{'}} = 1) \times P_{(m,n,h)(PM)} + \\
&E(Sojourn \; time \mid \Pi_0^{'} = m, \xi_1^{'} > 0, \xi_2^{'} > 0, \xi_1^{'} < \xi_2^{'}, \xi_1^{'} < \Delta) \times P_{(m,n,h)(F)} \\
&= \Delta \times \sum_{l=m}^{L} P_{(m,n,h)(l,n+1,h+1)} + \Delta \times \sum_{l=m}^{L} P_{(m,n,h)(l,n+1,0)} + \int_0^{\Delta} (t + T_I) \\
&P_{00}(t)(1 - \Pi_0^{'}) \times P(\Pi_t^{'}(m) \geq OL \mid \Pi_0^{'} = m) \times f_2(t \mid h) dt + \int_0^{\Delta} (t + T_I) \\
&(P_{01}(t)(1 - \Pi_0^{'}) + P_{11}(t)\Pi_0^{'}) \times P(\Pi_t^{'}(m) \geq OL \mid \Pi_0^{'} = m) \\
&\times f_2(t \mid h) dt + \int_0^{\Delta} t \times (1 - R_1(t \mid \Pi_0^{'} = m)) \times f_2(t \mid h) dt \\
&= \Delta \times \sum_{l=m}^{L} P_{(m,n,h)(l,n+1,h+1)} + \Delta \sum_{l=m}^{L} \times P_{(m,n,h)(l,n+1,0)} + \\
&\int_0^{\Delta} (t + T_I) \times R_1(t \mid \Pi_0^{'} = m) \times P(\Pi_t(m) \geq OL \mid \Pi_0^{'} = m) \times f_2(t \mid h) dt \\
&+ \int_0^{\Delta} t \times (1 - R_1(t \mid \Pi_0^{'} = m)) \times f_2(t \mid h) dt. \qquad (33)
\end{aligned}
$$

The mean sojourn time when the posterior probability crosses the control limit and still unit#2 is working properly, is given by:

$$
\tau_{(m,n,h)} = T_I, \quad CL \leq m < 1. \qquad (34)
$$

If the age of unit#1 exceeds the pre-determined age M or the result of inspection reveals that it was true alarm, then the sojourn time in PM state will be:

$$
\tau_{(PM)} = T_P. \qquad (35)
$$

The expected sojourn time when unit#1 failure occurs is given by:

$$
\tau_{(F)} = T_F. \qquad (36)
$$

C. Expected Cost

The average cost incurred until the next decision epoch for state (m, n, h) where $0 \leq m < CL$ is given by:

$$\begin{aligned}
&C_{(m,n,h)} = E(Cost \mid \Pi_0^{'}=m, \xi_1^{'}>0, \xi_2^{'}>0) \\
&= \sum_{l=m}^{L} E(Cost \mid \Pi_0^{'}=m, \xi_1^{'}>0, \xi_2^{'}>0, \Pi_\Delta^{'}=l, \xi_1^{'}>\Delta, \xi_2^{'}>\Delta) \times \\
&P_{(m,n,h)(l,n+1,h+1)} + \sum_{l=m}^{L} E(Cost \mid \Pi_0^{'}=m, \xi_1^{'}>0, \xi_2^{'}>0, \Pi_\Delta^{'}=l, \xi_2^{'}<\xi_1^{'}, \\
&\xi_2^{'}<\Delta, \Pi_{\xi_2^{'}}^{'}(m)<OL) \times P_{(m,n,h)(l,n+1,0)} + E(Cost \mid \Pi_0^{'}=m, \xi_1^{'}>0, \xi_2^{'}>0, \\
&\xi_2^{'}<\xi_1^{'}, \xi_2^{'}<\Delta, \Pi_{\xi_2^{'}}^{'}(m)\geq OL, X_{\xi_2^{'}}=0) \times P_{(m,n,h)(0,n+1,0)} + E(Cost \mid \Pi_0^{'}=m, \\
&\xi_1^{'}>0, \xi_2^{'}>0, \xi_2^{'}<\xi_1^{'}, \xi_2^{'}<\Delta, \Pi_{\xi_2^{'}}^{'}(m)\geq OL, X_{\xi_2^{'}}=1) \times P_{(m,n,h)(PM)} \\
&= C_s \times \sum_{l=m}^{L} P_{(m,n,h)(l,n+1,h+1)} + (K+C_{F2}+C_s) \times \sum_{l=m}^{L} P_{(m,n,h)(l,n+1,0)} \\
&+(K+C_{F2}+C_I+C_{LP}T_I) \int_0^\Delta P_{00}(t)(1-\Pi_0^{'}) \times P(\Pi_t^{'}(m)\geq OL \mid \Pi_0^{'}=m) \\
&\times f_2(t \mid h)dt + (K+C_{F2}+C_I+C_{LP}T_I) \int_0^\Delta (P_{01}(t)(1-\Pi_0^{'}) + P_{11}(t)\Pi_0^{'}) \\
&\times P(\Pi_t^{'}(m)\geq OL \mid \Pi_0^{'}=m) \times f_2(t \mid h)dt \\
&= C_s \times \sum_{l=m}^{L} P_{(m,n,h)(l,n+1,h+1)} + (K+C_{F2}+C_s) \times \sum_{l=m}^{L} P_{(m,n,h)(l,n+1,0)} \\
&+(K+C_{F2}+C_I+C_{LP}T_I) \int_0^\Delta R_1(t \mid \Pi_0^{'}=m) \times P(\Pi_t^{'}(m)\geq OL \mid \Pi_0^{'}=m) \\
&\times f_2(t \mid h)dt. \qquad (37)
\end{aligned}$$

The average cost incurred until the next decision epoch for state (m, n, h), where $CL \leq m < 1$ is as follows:

$$C_{(m,n,h)} = C_I + C_{LP}.T_I + C_{P2}. \qquad (38)$$

If the result of inspection reveals that it is true alarm, then the expected cost is given by:

$$C_{(PM)} = C_{P1} + C_{LP}.T_P. \qquad (39)$$

When unit#1 failure occurs then the expected cost is as follows:

$$C_{(F)} = K + C_{F1} + C_{P2} + C_{LP}.T_F. \qquad (40)$$

V. Experimental Results

We assume that the unit#1 deterioration follows a continuous-time homogeneous Markov chain $(X_t : t \in R^+)$, with state space $\Omega = \{0, 1, 2\}$. States 0 and 1 are unobservable, representing the healthy and unhealthy operational states respectively, and state 2 corresponds to the observable failure state. The sojourn time in healthy state has an exponential distribution with parameter $\nu_0 = \lambda_{01} + \lambda_{02}$ and the sojourn time in unhealthy state has an exponential distribution with parameter $\nu_1 = \lambda_{12}$. The transition rates of the state process are given by:

$$\lambda_{01} = 0.15, \quad \lambda_{02} = 0.02 \quad and \quad \lambda_{12} = 0.2$$

The parameters of the residual observation process $(Y_n : n \in N)$ are given by:

$$\mu_0 = \begin{pmatrix} 0.21 \\ -0.01 \end{pmatrix} \qquad \Sigma_0 = \begin{pmatrix} 1.5 & 0.61 \\ 0.61 & 1.9 \end{pmatrix}$$

$$\mu_1 = \begin{pmatrix} 0.75 \\ 0.54 \end{pmatrix} \qquad \Sigma_1 = \begin{pmatrix} 1.81 & 1.97 \\ 1.97 & 2.22 \end{pmatrix}.$$

The age information of unit#2 is available and its lifetime distribution follows Gamma distribution with parameters $k = 2$ and $\theta = 20$ where,

$$f_2(x) = \frac{1}{\Gamma(k)\theta^k} x^{k-1} e^{-x/\theta}; \qquad k > 0, \quad \theta > 0 \qquad (41)$$

The system inspection and replacement time parameters are given by:

$$T_I = 1, \quad T_P = 3, \quad T_F = 10$$

where T_I is the inspection time, T_P is the time to perform preventive maintenance, and T_F is the time to renew the system upon unit#1 failure. Times of preventive and corrective maintenance for unit#2 are negligible. When the age of unit#1 exceeds the pre-determined age 50 then preventive maintenance will be performed and unit#2 is adjusted.

The following costs will be considered in the experiment:

$$C_s = 1, C_I = 10, C_{LP} = 20, K = 50$$

$$C_{P1} = 500, C_{F1} = 1200, C_{P2} = 50, C_{F2} = 100$$

where C_s indicates the cost of sampling, C_I is the cost of inspection, C_{LP} is the cost rate of lost production during maintenance actions. C_{Pi} and C_{Fi} are the preventive and corrective maintenance costs of unit $i, i = \{1, 2\}$, respectively. We compute the optimal sampling interval, opportunistic limit and control limit to minimize the long-run expected average cost. We choose the partition parameter $L = 40$, and use (12) to obtain the results shown in table I. The algorithm takes 7.1832 seconds for each run on an Intel Core (TM) i5 CPU with 2.27 GHz.

The next stage of the analysis is to investigate the effect of the opportunistic maintenance limit on the optimal maintenance cost for two-unit series system. Table II shows the optimal control limit, sampling interval and the average cost when there is no opportunistic limit.

TABLE I
THE OPTIMAL MAINTENANCE POLICY FOR A TWO-UNIT SERIES SYSTEM WITH OPPORTUNISTIC MAINTENANCE LIMIT.

Optimal opportunistic limit	Optimal control limit	Optimal sampling interval	Average cost
0.2619	0.3810	2	35.4821

The results show that the policy with opportunistic limit is more economical than the optimal maintenance policy without opportunistic limit.

VI. Conclusions and Future Research

In this paper, we have developed a model and a computational algorithm that can be used to determine the

TABLE II
THE OPTIMAL MAINTENANCE POLICY FOR A TWO-UNIT SERIES SYSTEM WITHOUT OPPORTUNISTIC LIMIT.

Optimal control limit	Optimal sampling interval	Average cost
0.3333	2	40.1286

optimal maintenance policy for a two-unit series system where one unit is subject to condition monitoring while just the age information of unit#2 is available. Unit#1 deterioration is described by a hidden, 3-state homogeneous continuous-time Markov process. It is assumed that observations at regular sampling times are available which are related to the true state of the system. States 0 and 1 representing healthy and warning conditions are not observable and only failure state (state 2) is observable. We have developed a computational algorithm in the SMDP framework to find an optimal Bayesian control policy that minimizes the long-run expected average cost per unit time for the whole system. A numerical example has been provided to illustrate the new maintenance policy proposed for a two-unit series system. The optimal opportunistic maintenance policy has been compared with an optimal maintenance policy with no opportunistic limit. For this policy the optimal average cost increased by $\%13$ which indicates the cost effectiveness of the new opportunistic maintenance policy proposed in this paper.

We suggest a few possible directions for future research. First, a more general distribution such as an Erlang distribution of the state sojourn times can be considered. Another possible future research topic would be to test the effectiveness of our model using real data sets.

REFERENCES

[1] D. Cho, and M. Parlar, ”A survey of maintenance models for multi-unit systems,”*Operational Research*, vol. 51, pp.1–23, 1991.
[2] R. Dekker, and F. van der Duyn Schouten, and R. Wildeman, ”A review of multi-component maintenance models with economic dependence,” *Mathematical Methods of Operations Research*, vol.45, pp.411-435, 1997.
[3] R.P. Nicolai, and R. Dekker, ”Optimal maintenance of multi-component systems: a review,” *Econometric Institute Report*, pp.2006–29, 2006.
[4] T. Nowakowski, and S. Werbinka, ”On Problems of multi-component system maintenance modelling,” *International Journal of Automation and Computing*, vol. 4, pp.364–378, 2009.
[5] L.C. Thomas,”A survey of maintenance and replacement models for maintainability and reliability of multi-item systems,” *Reliability Engineering and system safety*, vol. 16, pp.297–309, 1986.
[6] H. Wang, ”A survey of maintenance policies of deteriorating systems,” *European journal of Operational Research*, vol. 139, pp.469–489, 2002.
[7] A. Jardine, D. Lin and D. Banjevic, ”A review on machinery diagnostics and prognostics implementing condition-based maintenance,” *Mechanical Systems and Signal Processing*, vol.20, pp.1483–1510, 2006.
[8] MJ. Kim and V. Makis,”Joint optimization of sampling and control of partially observable failing systems,” *Operations Research*, vol.61 pp.777–790, 2013.
[9] MJ. Kim, R. Jiang, V. Makis and CG. Lee, ”Optimal Bayesian fault prediction scheme for a partially observable system subject to random failure,” *European Journal of Operational Research*, vol. 214(2), pp.331–339, 2011.
[10] A. Rosmaini, and K. Shahrul, ”An overview of time-based and condition-based maintenance in industrial application,” *Journal of Computers and Industrial Engineering*, vol.63, pp.135–149, 2012.
[11] X. Zheng, N. Fard, ”A maintenance policy for repairable systems based on opportunistic failure rate tolerance,” *IEEE Transactions on Reliability* vol. 40, pp. 237–244, 1991.
[12] F. Barbera, H. Schneider and E.d. Watson, ”A condition based maintenance model for a two-unit series system,” *European Journal of Operational Research*, vol.116, pp.281–290, 1999.
[13] J. Barata, C. Guedes, M. Mareseguerra and E. Zio, ”Simulation modeling of repairable multi-component deteriorating systems for on condition maintenance optimization,” *Journal of Reliability Engineering and System Safety*, vol.76, pp.255–264, 2002.
[14] M. Marseguerra, E. Zio and L. Podofillini, ”Condition-based maintenance optimization by means of genetic algorithms and Monte Carlo simulation,” *Reliability Engineering and System Safety*, vol.77, pp.151–165, 2002.
[15] B. Castanier, A. Grall and C. Berenguer, ”A condition-based maintenance policy with non-periodic inspections for a two-unit series system,” *Reliability Engineering and System Safety* vol. 87, pp. 109–120, 2005.
[16] A. Gupta, C. Lawsirirat, ”Strategically optimum maintenance of monitoringenabled multi-component systems using continuous-time jump deterioration models,” *Journal of Quality in Maintenance Engineering*, vol.12, pp.306–329, 2006.
[17] M. Jalali Naini, B. Aryanezhad, A. Jabbarzadeh and H. Babaei, ”Condition Based Maintenance for Two-Component Systems with Reliability and Cost Considerations,” *International Journal of Industrial Engineering and Production Research*, vol.20, pp.107–116, 2009.
[18] A. Saunil, L. Lin and N. Jun, ”Condition-Based Maintenance Decision-Making for Multiple Machine Systems,” *Journal of Manufacturing Science and Engineering*, vol.131, pp.1–9, 2009.
[19] Z. Tian and H. Liao ”Condition based maintenance optimization for multi-component systems using proportional hazards model,” *Reliability Engineering and System Safety*, vol.96, pp.581–589,2011.
[20] Q. Zhu, H. Peng and G. Houtum, ”A Condition-Based Maintenance Policy for Multi-Component Systems with a High Maintenance Setup Cost,” *Beta working paper for operations management and logistics*, pp.1–20, 2012.
[21] G. Tagaras and Y. Nikoaidis,”Comparing the effectiveness of various Bayesian $\overline{X}$ control charts,” *Operations Research*, vol.50, pp.878–888, 2002.
[22] V. Makis, ”Multivariate Bayesian control chart,” *Operations Research*, vol. 56(2), pp.487–498, 2008.
[23] D.P. Bertsekas and S.E. Shreve, ”Stochastic optimal control: the discrete time case,” *Academic Press*, vol.23, pp.15–24, 1978.
[24] J.P. Imhof, ”Computing the distribution of quadratic forms in normal variables,” *Journal of Biometrika*, vol.48, pp.419–426, 1961.