On Chromaticity of Wheels

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Abstract**—**Let the vertices of a graph such that every two adjacent vertices have different color is a very common problem in the graph theory. This is known as proper coloring of graphs. The possible number of different proper colorings on a graph with a given number of colors can be represented by a function called the chromatic polynomial. Two graphs G and H are said to be chromatically equivalent, if they share the same chromatic polynomial. A Graph G is chromatically unique, if G is isomorphic to H for any graph H such that G is chromatically equivalent to H. The study of chromatically equivalent and chromatically unique problems is called chromaticity. This paper shows that a wheel W_{12} is chromatically unique.

Keywords**—**Chromatic Polynomial, Chromatically Equivalent, Chromatically Unique, Wheel.

I. Introduction

A graph *G* is planar if it can be drawn in the plane with no crossing edges. A λ-coloring of a graph *G* is a mapping *f : V(G) → {1,2,3,..., λ}* such that*: f(u) ≠ f(v)* for every edge *uv ε E(G).* A minimum number λ such that G has a proper coloring is called chromatic number, and *G* called *λ*-colorable. During their attempts to prove the four-color problem (Every planar graph is 4-colorable), Mathematicians found many useful tools for solving graph coloring problems. Birkhoff [1] proposed a way to attack the four-color problem by introducing a function P(M, λ), the number of ways of proper λ-colorings of a map M. P(M, λ) is a polynomial called chromatic polynomial of M. In 1968, Read [2] asked: What is a necessary and sufficient condition for two graphs to be chromatically equivalent; that is, to have the same chromatic polynomial?

Chao and Whitehead Jr. [3] defined a graph to be chromatically unique if no other graphs share its chromatic polynomial and another question appears: What is a necessary and sufficient condition for a graph to be chromatically unique?

Chromaticity, mean study of the above two questions of chromatically equivalent and chromatically unique.

During the period when the Four-Color Problem remained unsolved, which spanned more than a century, many approaches were introduced that would lead to a solution to this famous problem [4].

A wheel W_n is a graph of order n, where n ≥ 4, obtained from cycle C_{n-1} by adding a new vertex w adjacent to each vertex of the cycle. Each edge incident with w is a spoke of the wheel.

Xu and Li [5] proved that W_n, for odd n ≥ 5 is chromatically unique. They also showed that W_8 is not chromatically unique. Chao and Whitehead demonstrated that W_6 is not chromatically unique while Read [2] discovered that W_{10} is chromatically unique. Later on Li and Wgitehead Jr. [6] proved these results mathematically. This paper introduced mathematical proof of the chromatic uniqueness of W_{12}.

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II. Auxiliary Results

In this section, some known results are introduced some known results that help in proving the main result.

Theorem1. [7] Let G be a graph of order n and size m. Then $p(G,\lambda)$ is a polynomial of degree *n*. Moreover, if $(G,\lambda) = \sum_{i=0}^{n} a_i\,\lambda^{n-i}$, then

1- all coefficients a_i are integers and alternate in sign;

2- (i) $a_n = 0$

(ii) $a_0 = 1$

(iii) $a_1 = -m$

(iv) $a_2 = \binom{m}{2} - t_1(G)$

(v) $a_3 = -\binom{m}{3} + (m-2)t_1(G) + t_2(G) - 2t_3(G)$

Result (v) in the above theorem was obtained by Farrell [8] who also provided in [8] an expression for

$$a_4 = \binom{m}{4} - \binom{m-2}{2}t_1(G) - \binom{t_1(G)}{2} - (m-3)t_2(G) + (2m-9)t_3(G) - t_4(G) - 6t_5(G) + t_6(G) + 2t_7(G) + 3t_8(G) \quad (1)$$

Theorem 2. Let G be a graph of order n and size m. Then

$$p(G,\lambda) = \sum_{k=1}^{n}\left(\sum_{r=0}^{m}(-1)^r N(k,r)\right)\lambda^k \quad (2)$$

where N(k, r) denote the number of spanning subgraphs of G having exactly k components and r edges [7].

Theorem 3. [7] Let G be a K_r-gluing of graph G_1 and G_2. Then

$$p(G,\lambda) = \frac{p(G_1,\lambda)p(G_2,\lambda)}{p(K_r,\lambda)} \quad (3)$$

Theorem 4. [7] Let G and H be two chromatically equivalent graphs then we have:

1. $|V(G)|=|V(H)|$
2. $|E(G)|=|E(H)|$
3. $\chi(G)=\chi(H)$
4. $t_1(G) = t_1(H)$
5. $t_2(G) - 2t_3(G) = t_2(H) - 2t_3(H)$
6. $G$is connected if and only if H is connected
7. $G$is 2-connected if and only if H is 2-connected
8. g(G)=g(H)
9. $G$and H have the same number of shortest cycles .

III. Results

This section is devoted to prove the chromatic uniqueness of W_{12}.

Theorem 5. The wheel W_{12} is chromatically unique.

Proof:

$$p(W_{12},\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda^4 - 9\lambda^3 + 31\lambda^2 - 49\lambda + 31)(\lambda^4 - 7\lambda^3 + 19\lambda^2 - 23\lambda + 11) \quad (4)$$

Let G be a graph such that $p(G,\lambda)$=$p(\boldsymbol{W_{12}},\lambda)$.

From Theorem 4 we have the following conditions:

1. G has 12 vertices.
2. G has 22 edges.
3. G has 11 triangles.
4. $\chi(G) = 4$
5. G has no cut vertex since G is 2- connected by no.7 in Theorem 1.
6. Since G is connected then G has no vertex of degree 0.
7. Ghas no a vertex of degree 1, if G has a vertex of degree 1 then $(\lambda-1)^2$ divide $p(G,\lambda)$ but this is not the case.
8. G has no degree 2 vertex which is a triangle,if G has degree 2 then $(\lambda-2)^2$ divide $p(G,\lambda)$ but this is not the case.
9. G has no K_5 subgraph since $(\lambda-4)$ does not divide $p(G,\lambda)$.
10. In [7], Farrell derived formulas for the coefficients of $\lambda^{p-3} and \lambda^{p-4}$ in (H,λ), where H is a graph with p vertices. Specializing these formulas to $p(G,\lambda)$=$p(w12,\lambda)$.

The coefficients of $\lambda^{p-3}$is:

$$-\binom{m}{3} + (m-2)t_1(G) + t_2(G) - 2t_3(G) \quad (5)$$

where, m: edges , m = 22

$$t_1(G) = \binom{p}{3}$$
$$t_1(G) = \binom{12}{3}$$
$$t_1(G) = \frac{12!}{3!\,9!}$$
$$t_1(G) = 220$$
$$\binom{22}{3} = 1540$$

Now,

-(1540)+4400 $+t_2(G) - 2t_3(G)$

Derive the d formula for the coefficient of λ^{p-3}

$$t_2(G) = 2t_3(G)$$

The coefficient of λ^{p-4} is:

$$\binom{m}{4} - \binom{m-2}{2}t_1(G) - \binom{t_1(G)}{2} - (m-3)t_2(G) + (2m-9)t_3(G) - t_4(G) - 6t_5(G) + t_6(G) + 2t_7(G) + 3t_8(G)$$

where,

$t_4(G)$: the number of pure pentagons C_5 .

$t_5(G)$: the number of K_5 subgraph.

$t_6(G)$: the number of 2-3 complete bipartite graphs .

$t_7(G)$: the number of 5-vertex wheels with one spoke deleted X_4 .

$t_8(G)$: the number of wheel W_5 .

$$\binom{22}{4} - \binom{22-2}{2}t_1(G) - \binom{t_1(G)}{2} - (22-3)t_2(G) + (2(22)-9)t_3(G) - t_4(G) - 6t_5(G) + t_6(G) + 2t_7(G) + 3t_8(G)$$

$$\binom{22}{4} = 7315$$
$$t_5(G) = \binom{12}{5}$$
$$t_5(G) = 792$$
$$\binom{20}{2} = 190$$

$$(7315) - (190)(220) - \binom{220}{2} - (22-3)t_2(G) + (2(22)-9)t_3(G) - t_4(G) - 6(792) + t_6(G) + 2t_7(G) + 3t_8(G)$$

Derive the d formula for the coefficient of λ^{p-4}

$$-19t_2(G) + 35t_3(G) - t_4(G) + t_6(G) + 2t_7(G) + 3t_8(G) = 0 \quad (6)$$

11. G has no pure W_5 subgraph.

It is assumed that G contains a pure W_5 subgraph which implying that G contains a pure C_4 subgraph and a K_4 subgraph by (5). To consider various ways that the W_5 and K_4 subgraphs can overlap see Fig. 1.

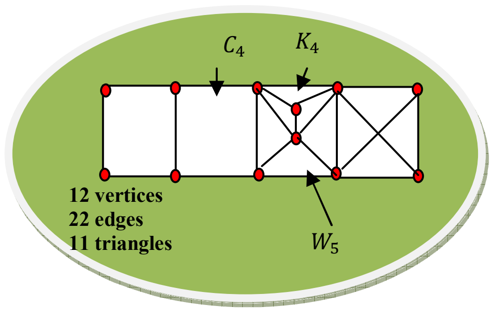

Fig. 1 The different ways of W_5 and K_4 subgraphs are overlapping

$$\boldsymbol{p(G,\lambda) = \lambda(\lambda-1)(\lambda-2)^2(\lambda-3)^2(\lambda^2-5\lambda+7)(\lambda^2-3\lambda+3)^2}$$

This is contradiction with equation (A) :

$t_2(G) = 2t_3(G)$and$p(G,\lambda)$ not equal to $p(w12,\lambda)$.

12. Ghas no K_4 subgraphs.

According to (6) this condition is equivalent to the statement G has no C_4 subgraphs.

Since G has no pure W_5 subgraph from (8) then $t_8(G) = 0$.

$$t_2(G) = 2t_3(G)$$
$$-19t_2(G) + 35t_3(G) - t_4(G) + t_6(G) + 2t_7(G) = 0 \quad (7)$$

Put (6) in (7):

$$-19(2)t_3(G) + 35t_3(G) - t_4(G) + t_6(G) + 2t_7(G) = 0$$
$$-38t_3(G) + 35t_3(G) - t_4(G) + t_6(G) + 2t_7(G) = 0$$
$$-3t_3(G) - t_4(G) + t_6(G) + 2t_7(G) = 0 \quad (8)$$

K_4 be overlap with X_4.We suppose that $t_3(G) = 1$ and $t_7(G) = 1$ then :

Case 1. $t_3(G) = 1$, $t_7(G) = 1$, $t_4(G) = 0$ and $t_6(G) = 1$. The vertices are equal to 9 not 12, which is a contradiction.

Case 2. $t_3(G) = 1$, $t_7(G) = 1$, $t_4(G) = 1$ and $t_6(G) = 2$. The triangles are equal to 9 not 11, which is a contradiction.

Case 3. $t_3(G) = 1$, $t_7(G) = 1$, $t_4(G) = 2$ and $t_6(G) = 3$

Case 4. The vertices must be greater than 12. Then the graph has no K_4 .

13. G has no separating edge(K_2-gluing). It is assumed that G consists of two subgraphs G_1 and G_2 which overlap in a separating edge and two cases are considered:

Case 1. G_1 andG_2 both contain odd cycles.

Case 2. Only G_1 or G_2 contain odd cycles.

Both cases shows contradiction.

14. Ghas no a pure C_5 subgraph.

Since G has no pure W_5 subgraph from (8) then $t_8(G) = 0$.

Since G has no K_4 subgraph from (9) then $t_3(G) = 0$ and G has no C_4 subgraph from (9) then $t_2(G) = 0$.

Then:

$$-t_4(G) + t_6(G) + 2t_7(G) = 0$$
$$t_4(G) = t_6(G) + 2t_7(G) \qquad (9)$$

All possible cases leads to contradiction.

15. G has no pure C_6 subgraph.

Since G has no W_5, K_4, C_4 and C_5 subgraphs then it is supposed that G has triangles with C_6. (see Fig. 2).

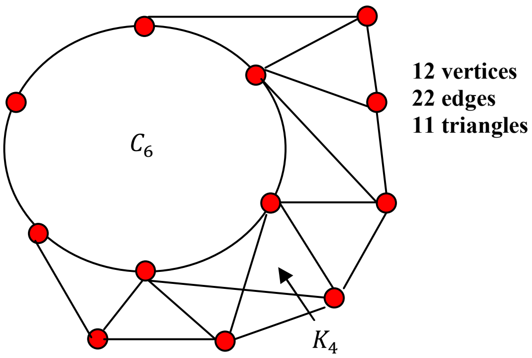

Fig. 2 G has no pure C_6

However, this graph must contain a K_4 subgraph. Therefore, G has no pure C_5 subgraph.

16. G has no pure C_7 subgraph.

Since G has no W_5, K_4, C_4, C_5 and C_6 subgraphs then it is supposed that G has triangles with C_7 see Fig. 3.

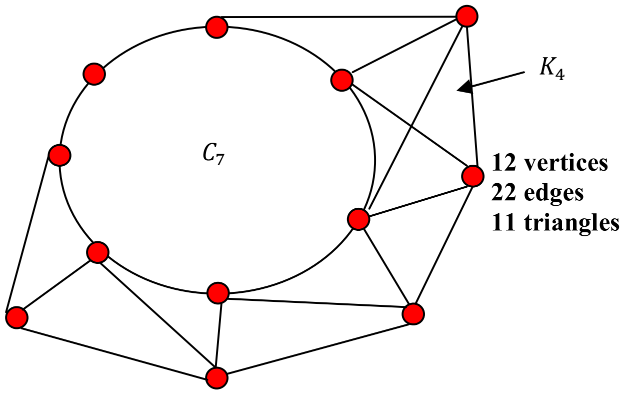

Fig. 3 G has no pure C_7

But this graph must contain K_4 subgraph. Therefore, G has no pure C_7 subgraph.

17. G has no a pure C_8 subgraph.

Since G has no W_5, K_4, C_4, C_5, C_6 and C_7 subgraphs then it is supposed that G has triangles with C_8 . See Fig. 4.

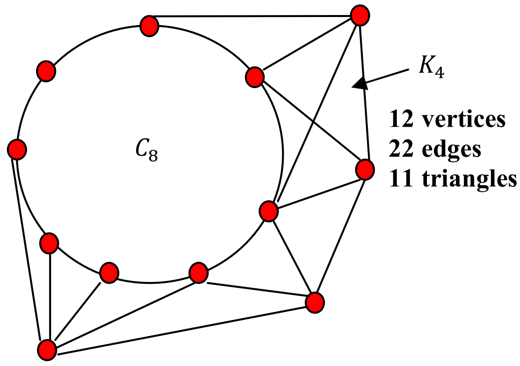

Fig. 4 G has no pure C_8

This graph must contain a K_4 subgraph. Therefore, G has no pure C_8 subgraph.

Since G is a 2-connected graph without separating edges and G satisfies the conditions then G is isomorphic to W_{12}.

IV. Conclusion

It is not easy to prove the chromatic uniqueness of a certain graph. In this paper, it is concluded that the graph W_{12} is chromatically unique.

Acknowledgment

The authors would like to thank the anonymous reviewers for their valuable comments and suggestions to improve the quality of the paper.

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