

# Quasi-Permutation Representations for the Group $SL(2, q)$ when Extended by a certain Group of order Two

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**Abstract**—A square matrix over the complex field with non-negative integral trace is called a quasi-permutation matrix. For a finite group  $G$  the minimal degree of a faithful representation of  $G$  by quasi-permutation matrices over the rationals and the complex numbers are denoted by  $q(G)$  and  $c(G)$  respectively. Finally  $r(G)$  denotes the minimal degree of a faithful rational valued complex character of  $G$ . The purpose of this paper is to calculate  $q(G)$ ,  $c(G)$  and  $r(G)$  for the group  $SL(2, q)$  when extended by a certain group of order two.

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In [10] Wong defined a quasi-permutation group of degree  $n$  to be a finite group  $G$  of automorphisms of an  $n$ -dimensional complex vector space such that every element of  $G$  has non-negative integral trace. Also Wong studied the extent to which some facts about permutation groups generalize to the quasi-permutation group situation. In [3] the authors investigated further the analogy between permutation groups and quasi-permutation groups. They also worked over the rational field and found some interesting results.

By a quasi-permutation matrix we mean a square matrix over the complex field  $C$  with non-negative integral trace. For a given finite group  $G$ , let  $q(G)$  denote the minimal degree of a faithful representation of  $G$  by quasi-permutation matrices over the rational field  $Q$  and let  $c(G)$  be the minimal degree of a faithful representation of  $G$  by complex quasi-permutation matrices and finally let  $r(G)$  denote the minimal degree of a faithful rational valued character of  $G$ . In this paper we will apply the algorithms in [1] to the group  $K_2^2(2^n)$ , where

$$K_2^2(q) = \langle SL(n, q), \theta | \theta^2 = 1, \theta^{-1}A\theta = (A^t)^{-1} \rangle.$$

We will prove

**Theorem 1: A)** Let  $G = K_2^2(2)$  then  $r(G) = 2, c(G) = q(G) = 4$

**B)** Let  $K_2^2(q)$ , where  $q = 2^n$ . Then

- 1) If  $q \equiv -1 \pmod{3}$  then  $r(G) = q - 1, c(G) = q(G) = 2(q - 1)$
- 2) Otherwise :  $r(G) = q, c(G) = q(G) = 2q$

Let  $SL(n, q)$  denote the special general linear group of a vector space of dimension  $n$  over a field with  $q$  elements. Let

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$\theta : SL(n, q) \rightarrow SL(n, q)$  be the automorphism of  $SL(n, q)$  given by  $\theta(A) = (A^t)^{-1}$ , where  $A^t$  denotes the transpose of the matrix  $A \in SL(n, q)$ . In this case one can define the split extension  $SL(n, q) \cdot \langle \theta \rangle$  that following the notations used in [6] is denoted by  $K_n^2(q)$ . Therefore we have  $K_n^2(q) = \langle SL(n, q), \theta | \theta^2 = 1, \theta^{-1}A\theta = (A^t)^{-1} \rangle$ , see [4].

Now let  $G$  denote the group  $SL(n, q)$  and let the split extension of  $G$  by the cyclic group  $\langle \theta \rangle$  of order 2 be denoted by  $G^+$ . Since  $[G^+ : G] = 2$ , we have  $G^+ = G \cup \theta G$ , and elements of  $G^+$  which lie in  $G$  are called positive and those outside  $G$  are called negative elements. A conjugacy class in  $G^+$  is called positive if it lies in  $G$  otherwise it is called negative. We may assume that using [7] one can obtain information about conjugacy classes and complex irreducible characters of  $G$ , therefore so far as conjugacy classes of  $G^+$  are concerned one must pay attention to negative conjugacy classes of  $G^+$ .

One can show that there is a one-to-one correspondence between the set of negative conjugacy classes of  $G^+$  and the set of equivalence classes of invertible matrices in  $G$ .

Now we begin with a summary of facts relevant to the irreducible complex characters of  $K_2^2(q)$ .

Complex irreducible characters of  $G^+$  are divided into two kinds. The group  $\langle \theta \rangle$  acts on the set of complex irreducible characters of  $G$  as follows. If  $\chi \in Irr(G)$ , then  $\chi^\theta(A) := \chi(\theta^{-1}A\theta)$ . If  $\chi^\theta = \chi$ , then we say that  $\chi$  is invariant under  $\langle \theta \rangle$  and in this case  $\chi$  forms an orbit of  $G^+$  acting on  $Irr(G)$ . Now by standard results that can be found in [8] there exists an irreducible character  $\varphi$  of  $G^+$  such that  $\varphi \downarrow_G = \chi$ . Since  $G^+/G \cong Z_2$  has two linear characters, therefore multiplication of  $\varphi$  with the non-trivial character of  $G^+/G$  gives another irreducible character  $\varphi'$  of  $G^+$  such that  $\varphi \downarrow_G = \chi$ . In this case we say that  $\chi$  extends to  $\varphi$  and  $\varphi'$  and it is enough to calculate one of them on the negative conjugacy classes of  $G^+$ .

As we mentioned earlier we have  $K_2^2(q) = SL(2, q) \cdot \langle \theta \rangle = \langle SL(2, q), \theta | \theta^2 = 1, \theta^{-1}A\theta = (A^t)^{-1}, \forall A \in SL(2, q) \rangle$ . In the following Lemma we give the structure of  $K_2^2(q)$ .

**Lemma 1:** Let  $G = K_2^2(q)$ . If  $q$  is even, then  $K_2^2(q) \cong SL(2, q) \times \langle \theta \rangle$  and if  $q$  is odd, then  $K_2^2(q) \cong SL(2, q) \circ 4$  a central product of  $SL(2, q)$  with the cyclic group of order 4.

**Proof.** The automorphism  $\theta : SL(2, q) \rightarrow SL(2, q)$  is

given by  $\theta(A) = (A^{-1})^t$  for all  $A \in SL(2, q)$ . If we set  $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ , then it is easy to verify that for  $A \in SL(2, q)$  we have  $J^{-1}AJ = (A^{-1})^t$  and therefore  $\theta$  is equal to an inner automorphism  $i_J$  of  $SL(2, q)$ . We have  $K_2^2(q) = \langle SL(2, q), \theta \rangle = \langle SL(2, q), \theta J \rangle = SL(2, q) \cdot \langle \theta J \rangle$  and since  $\theta J \in Z(K_2^2(q))$  hence  $K_2^2(q) \cong \frac{SL(2, q) \times \langle \theta J \rangle}{\{(I, I), (-I, -I)\}}$ . If the characteristic of  $GF(q)$  is even, then we get  $K_2^2(q) = SL(2, q) \times \langle \theta J \rangle \cong SL(2, q) \times \langle \theta \rangle$  and if the characteristic of  $GF(q)$  is odd we obtain  $K_2^2(q) \cong SL(2, q) \circ \langle \theta J \rangle$  the central product of  $SL(2, q)$  with a cyclic group of order 4.

By [5] we have two important lemmas as follows

**Lemma 2:** a) Let  $V_i (i = 1, 2)$  be  $KG$ -modules. Then the tensor product  $V_1 \otimes_K V_2$  over  $K$  obviously becomes a  $K[G_1 \times G_2]$  module by

$$(v_1 \otimes v_2)(g_1, g_2) = v_1 g_1 \otimes v_2 g_2$$

For  $v_i \in V_i, g_i \in G$ .

If  $\chi_i$  is the character of  $G_i$  on  $V_i$ , then the character  $\tau$  of  $G_1 \times G_2$  on  $V_1 \otimes V_2$  is given by

$$\tau((g_1, g_2)) = \chi_1(g_1)\chi_2(g_2)$$

For  $g_i \in G_i$ .

b) Let  $\chi_1, \dots, \chi_h$  be the irreducible characters of  $G_i$  over  $C$  and  $\psi_1, \dots, \psi_k$  the irreducible characters of  $G_2$  over  $C$ . Then the  $t_{ij}$  defined by  $t_{ij}((g_1, g_2)) = \chi_i(g_1)\psi_j(g_2)$  where  $i = 1, \dots, h$  and  $j = 1, \dots, k$  are all the irreducible characters of  $G_1 \times G_2$ .

**Lemma 3:** Let  $F$  be the finite field of  $q = 2^n$  elements, and let  $\nu$  be a generator of the cyclic group  $F^* = F - 0$ . Denote

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, c = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, a = \begin{pmatrix} \nu & 0 \\ 0 & \nu^{-1} \end{pmatrix}$$

in  $G = SL(2, F)$ .  $G$  contains an element  $b$  of order  $q + 1$ . For any  $x \in G$ , let  $(x)$  denote the conjugacy class of  $G$  containing  $x$ . Then  $G$  has exactly  $q + 1$  conjugacy classes  $(1), (c), (a), (a^2), \dots, (a^{(q-2)/2}), (b), \dots, (b^{q/2})$ , where

**Table (1)**  
Conjugacy Classes of  $SL(2, 2^n)$

$x$	1	$c$	$a^l$	$b^m$
$ (x) $	1	$q^2 - 1$	$q(q + 1)$	$q(q - 1)$

for  $1 \leq l \leq (q - 2)/2, 1 \leq m \leq q/2$ .

let  $\rho \in C$  be a primitive  $(q - 1)$ -th root of 1, table of  $G$  over  $C$  is

**Table (2)**  
Character Table of  $SL(2, 2^n)$

	1	$c$	$a^l$	$b^m$
$1_G$	1	1	1	1
$\psi$	$q$	0	$l$	-1
$\chi_i$	$q + 1$	1	$\rho^{il} + \rho^{-il}$	0
$\theta_j$	$q - 1$	-1	0	$-(\sigma^{jm} + \sigma^{-jm})$

for  $1 \leq i \leq (q - 2)/2, 1 \leq j \leq q/2, 1 \leq l \leq (q - 2)/2, 1 \leq m \leq q/2$ .

**Remark 1:** In the case of  $q$  even  $\langle \theta J \rangle$  has order 2 and its irreducible characters are denoted by  $\mu_0$  and  $\mu_1$  where  $\mu_0$  is the identity character. Regarding the structure of  $K_2^2(q)$  and Lemmas 2, 3 the irreducible characters of  $K_2^2(q)$  in the case of  $q$  even are  $\mu_k 1_G, \mu_k \psi, \mu_k \chi_i$  and  $\mu_k \theta_j$  where  $k = 0, 1$  and  $1 \leq i \leq \frac{q-2}{2}, 1 \leq j \leq \frac{q}{2}$ .

**Lemma 4:** Let  $G = SL(2, q)$ , if  $q$  is a power of 2 then the Schur index of any irreducible character of  $G$  over the rational numbers  $Q$  is 1.

*Proof.* See [9].

By [9] it is easy to see that:

**Lemma 5:** Let  $G = H \times K$  and  $\psi \in Irr(H)$  and  $\theta \in Irr(K)$ . Let  $\chi = \psi \times \theta$  and let  $F \subseteq C$ .

a)  $m_F(\chi)$  divides  $m_F(\psi)m_F(\theta)$ .

b) Equality occurs in (a) provided  $(m_F(\psi), \theta(1)|F(\theta) : F|) = 1$  and

$$(m_F(\theta), \psi(1)|F(\psi) : F|) = 1$$

**Lemma 6:** Let  $G$  be a finite group. If the Schur index of each non-principal irreducible character is equal to  $m$ , then  $q(G) = mc(G)$ .

*Proof.* See [1], Corollary 3.15.

We can see all the following statements in [1],[2].

**Definition 1:** Let  $\chi$  be a complex character of  $G$ , such that,  $\ker \chi = 1$ . Then define

- 1)  $d(\chi) = |\Gamma(\chi)|\chi(1)$
- 2)  $m(\chi) = \begin{cases} 0 & \chi = 1_G \\ |\min\{\sum_{\alpha \in \Gamma(\chi)} \chi^\alpha(g) : g \in G\}| & \text{otherwise} \end{cases}$
- 3)  $c(\chi) = \sum_{\alpha \in \Gamma(\chi)} \chi^\alpha + m(\chi)1_G$ .

Now according to Corollary 3.11 of [1] and above statements the following lemma is useful for calculation of  $r(G), c(G)$  and  $q(G)$ .

**Lemma 7:** Let  $G$  be a finite group with a unique minimal normal subgroup. Then

1)  $r(G) = \min\{d(\chi) : \chi \text{ is a faithful irreducible complex character of } G\}$

2)  $c(G) = \min\{c(\chi)(1) : \chi \text{ is a faithful irreducible complex character of } G\}$

**3)**  $q(G) = \min\{m_Q(\chi)c(\chi)(1) : \chi \text{ is a faithful irreducible complex character of } G\}$ .

By [2] we have the following lemmas .

**Lemma 8:** Let  $\varepsilon$  be a primitive  $n$ -th root of unity in  $C$ . Then  $\varepsilon + \varepsilon^{-1}$  is rational if and only if  $n = 1, 2, 3, 4, 6$ . The values which occur are as follows:

**Table (3)**

$n$	1	2	3	4	6
$+\varepsilon^{-1}$	2	-2	-1	0	1

**Lemma 9:** Let  $\varepsilon$  be a primitive  $n$ -th root of unity in  $C$  and  $m \in Z$ . If  $\varepsilon + \varepsilon^{-1}$  is rational, then so  $\varepsilon^m + \varepsilon^{-m}$ .

**Lemma 10:** Let  $\varepsilon$  be a primitive  $n$ -th root of unity. Then  $\varepsilon^j + \varepsilon^{-j}, 1 \leq j \leq n$  is rational if and only if  $n = j, 2j, 3j, 4j, 6j, \frac{3}{2}j, \frac{4}{3}j, \frac{6}{5}j$ .

**Lemma 11:** Let  $\chi \in Irr(G), \chi \neq 1_G$ . Then  $c(\chi)(1) \geq d(\chi) + 1 \geq \chi(1) + 1$ .

**Proof.** From Definition 1 it follows that  $c(\chi)(1)$  is a non-negative rational valued character of  $G$  so by [1], Lemma 3.2,  $m(\chi) \geq 1$ . Now the result follows from Definition 1.

**Lemma 12:** Let  $\chi \in Irr(G)$ . Then

(1)  $c(\chi)(1) \geq d(\chi) \geq \chi(1)$ ;

(2)  $c(\chi)(1) \leq 2d(\chi)$ . Equality occurs if and only if  $Z(\chi)/ker\chi$  is of even order.

**Proof.** (1) follows from the definition of  $c(\chi)(1)$  and  $d(\chi)$ . (2) follows from [1] Lemma 3.13.

**Lemma 13:** Let  $G = SL(2, q)$  where  $q = 2^n$  and  $n \geq 2$ . Then for each  $j, 1 \leq j \leq q/2$ ,

(1)  $\theta_j$  is rational if and only if  $q \equiv -1 \pmod 3$  and  $j = \frac{q+1}{3}$ ;

(2)  $d(\theta_j) \geq q - 1$  and equality holds if  $\theta_j$  is rational ;

(3)  $c(\theta_j) \geq q + 1$  and equality holds if  $\theta_j$  is rational .

**Proof .** As  $1 \leq j \leq \frac{q}{2} < \frac{q+1}{2}$  and as  $\sigma$  is a primitive  $(q + 1)$ -th root of unity, Lemmas 9 and 10 implies that  $\theta_j$  is rational if and only if  $j = \frac{q+1}{6}, \frac{q+1}{4}, \frac{q+1}{3}$ . Since  $q + 1$  is odd,  $\frac{q+1}{6}$  and  $\frac{q+1}{4}$  are not integers. Thus,  $\sigma^j + \sigma^{-j} \in Q$  if and only if  $3|q + 1$  and  $j = \frac{q+1}{3}$ . This proves (1). If  $\theta_j$  is not rational, then  $|\Gamma| \geq 2$  where  $\Gamma = \Gamma(Q(\theta_j) : Q)$  so that  $c(\theta_j)(1) \geq d(\theta_j) \geq 2(q - 1) > q + 1$  by Lemma 12. On the other hand if  $3|q + 1$ , then  $8 \leq q$ , so that  $3 \leq \frac{q}{2}$ ; but  $\theta_{\frac{q+1}{3}}(b_3) = -2 \leq \theta_{\frac{q+1}{3}}(g)$  for all  $g \in G$  so that  $m(\theta_{\frac{q+1}{3}}) = 2$ . Thus  $d(\theta_{\frac{q+1}{3}}) = q - 1$  and  $c(\theta_{\frac{q+1}{3}})(1) = q + 1$ . This completes the proofs of (2) and (3).

**Theorem 2:** Let  $G = K_2^2(2)$  then  $r(G) = 2, c(G) = q(G) = 4$

**Proof.** By Lemmas 4,5 Schur index of each irreducible characters is 1 and so by Lemma 6 we have  $c(G) = q(G)$ . Since the only faithful irreducible character of  $G$  is  $\mu_1\psi$  so by the character table of the group  $K_2^2(2)$  the result follows .

**Theorem 3:** Let  $G = K_2^2(q)$ , where  $q = 2^n$ . Then

1) If  $q \equiv -1 \pmod 3$  then  $r(G) = q - 1$ ,

$c(G) = q(G) = 2(q - 1)$

2) Otherwise  $r(G) = q, c(G) = q(G) = 2q$ .

**Proof.** By Lemmas 4,5 Schur index of each irreducible characters is 1 and so by Lemma 6 we have  $c(G) = q(G)$ .

By Lemma 3 we have the irreducible characters of  $SL(2, q)$  and by Remark 1 we have the irreducible characters of  $K_2^2(2^n)$ , now by definition of  $d(\chi), c(\chi)$  and character table of  $K_2^2(q)$  we obtain the Table (4) as follows

**Table (4)**

$\chi$ (faithful)	$d(\chi)$	$c(\chi)(1)$
$\mu_1\psi$	$q$	$2q$
$\mu_1\chi_i$	$\geq q + 1$	$\geq 2(q + 1)$
$\mu_1\theta_j$	$\geq (q - 1)$	$\geq 2(q - 1)$

Now by Lemma 7, Lemma 13 and Table (4) when  $q \equiv -1 \pmod 3$  we have

$\min \{d(\chi) : \chi \text{ is a faithful irreducible complex character of } G\} = q - 1$  and

$\min \{c(\chi)(1) : \chi \text{ is a faithful irreducible complex character of } G\} = 2(q - 1)$ .

Otherwise  $d(\mu_1\theta_j) > q - 1$  and so in this case  $\min d(\chi) = q$  and  $\min c(\chi) = 2q$ .

[1] H. Behraves, "Quasi-permutation representations of p-groups of class 2", *J.London Math. Soc.* (2)55(1997) 251-26 .  
 [2] H. Behraves, " The rational character table of special linear groups ", *J. Sci. I.R.I.* Vol. 9. No. 2(1998) 173-180.  
 [3] J.M. Burns, B. Goldsmith, B. Hartley and R. Sandling, "on quasi-permutation representations of finite groups ", *Glasgow Math. J.* 36(1994) 301-308.  
 [4] M. Darafsheh, F. Nowroozi Larki, "Equivalence Classes of Matrices in  $GL(2, q)$  and  $SL(2, q)$  and related topics ", *Korean J. Comput. Appl. Math.* 6(1999), no.2, 331 - 344 .  
 [5] L.Dornhoff, "Group Representation Theory", Part A, Marcel Dekker, New York, 1971 .  
 [6] W.Fiet, "Extension of Cuspidal Characters of  $GL(m, q)$ ", *Publications Mathematicae*, 34(1987), 273 - 297 .  
 [7] J.A.Green, "The Characters of the Finite General Linear Groups", *Trans. Amer. Math. Soc.* 80 (1955) 405-447 .  
 [8] I.M. Isaacs, "Character theory of finite groups", Academic Press, Newyork, 1976.  
 [9] M.A.Shahabi Shojaei, "Schur indices of irreducible characters of  $SL(2, q)$ ", *Arch.Math.* 221-131, 1983 .  
 [10] W.J. Wong, Linear groups analogous to permutation groups, *J. Austral. Math. Soc. (Sec. A)* 3, 180-184, 1963.