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The Bent and Hyper-Bent Properties of a Class of Boolean Functions

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Abstract—This paper considers the bent and hyper-bent properties of a class of Boolean functions. For one case, we present a detailed description for them to be hyper-bent functions, and give a necessary condition for them to be bent functions for another case.

Keywords—Boolean functions, bent functions, hyper-bent functions, character sums.

I. Introduction

BENT function is a class of Boolean functions with even variables and with the maximal distance to all affine functions. In fact, the distance of an n-variable bent function to any affine function equals $2^{n-1}-2^{\frac{n}{2}-1}$. Bent function was introduction by Rothaus [9] in 1976, later in 2001 Youssef et al [10] found a subclass of bent functions with even better cryptographic properties, which was named as hyper-bent functions. Thanks to their applications in cryptography, coding theory and combinatorial design, many interests have been put in bent and hyper-bent functions recently[2], [3], [4], [6], [7], [8].

In this paper, we consider a class of Boolean functions defined on $\mathbb{F}_{2^n}$ of the form:

$$f_{a,b}^{(r)}(x) := \mathrm{Tr}_1^n(ax^{r(2^m-1)}) + \mathrm{Tr}_1^4(bx^{\frac{2^n-1}{5}}), \qquad (1)$$

where $n = 2m$, $m \equiv 2k \pmod 4$, $k \in \{0,1\}$, $a \in \mathbb{F}_{2^n}$ and $b \in \mathbb{F}_{16}$. When $m = 2 \pmod 4$, with the help of the factorization of x^5+x+a^{-1} and Kloosterman sums, this paper characterizes the cases for $f_{a,b}^{(r)}$ to be hyper-bent. Further more , for $a \in \mathbb{F}_{2^{\frac{m}{2}}}$, we list all the hyper-bent functions of the form of $f_{a,b}^{(r)}$. When $m = 0 \pmod 4$, we give a necessary condition for $f_{a,b}^{(r)}$ to be bent.

The rest of paper is organized as follows. In Section II, we give some notations and recall some basic knowledge for this paper. Then we describe the hyper-bent properties of $f_{a,b}^{(r)}$ when $m \equiv 2 \pmod 4$ and study the bent properties of $f_{a,b}^{(r)}$ when $m \equiv 0 \pmod 4$ in Section III and Section IV respectively. Finally, we conclude our work in Section V.

II. Preliminaries

The ***sign*** function of Boolean function f is $\chi(f) := (-1)^f$.

Definition 1: A Boolean function $f : \mathbb{F}_{2^n} \to \mathbb{F}_2$ is called a bent function, if $\widehat{\chi}_f(w) = \sum_{x\in\mathbb{F}_{2^n}}(-1)^{f(x)+\mathrm{Tr}_1^n(wx)} = \pm 2^{\frac{n}{2}}$ $(\forall w \in \mathbb{F}_{2^n})$, where Tr_1^n is the absolute trace function defined as $\mathrm{Tr}_1^n(x) := x + x^2 + x^{2^2} + \cdots + x^{2^{n-1}}$.

Hyper-bent function is an important subclass of bent functions defined as

Definition 2: A bent function $f : \mathbb{F}_{2^n} \to \mathbb{F}_2$ is called a hyper-bent function, if, for any i satisfying $(i, 2^n-1) = 1$, $f(x^i)$ is also a bent function.

Charpin and Gong [4] gave the following property to determine a hyper-bent function.

Proposition 1: Let $n = 2m$, α be a primitive element of $\mathbb{F}_{2^n}$ and f be a Boolean function over $\mathbb{F}_{2^n}$ satisfying $f(\alpha^{2^{m+1}}x) = f(x)$ $(\forall x \in \mathbb{F}_{2^n})$ and $f(0) = 0$. Let ξ be a primitive 2^m+1-th root in $\mathbb{F}_{2^n}^*$. Then f is a hyper-bent function if and only if the cardinality of the set $\{i|f(\xi^i) = 1, 0 \le i \le 2^m\}$ is 2^{m-1}.

Kloosterman sum is a powerful tool to study the hyper-bent properties of some classes of boolean functions.

Kloosterman sums on $\mathbb{F}_{2^n}$ are defined as

$$K_m(a) := \sum_{x\in\mathbb{F}_{2^m}} \chi(\mathrm{Tr}_1^m(ax + \frac{1}{x})), \quad a \in \mathbb{F}_{2^m}.$$

Some properties of Kloosterman sums are given by the following proposition.

Proposition 2: ([5],Theorem 3.4]) Let $a \in \mathbb{F}_{2^m}$. Then $K_m(a) \in [1-2^{(m+2)/2}, 1+2^{(m+2)/2}]$ and $4 \mid K_m(a)$.

Quintic Weil sums on $\mathbb{F}_{2^m}$ are

$$Q_m(a) := \sum_{x\in\mathbb{F}_{2^m}} \chi(\mathrm{Tr}_1^m(a(x^5+x^3+x))), \quad a \in \mathbb{F}_{2^m}.$$

And the value of $Q_m(a)$ is related to the factorization of the polynomial $P(x) = x^5 + x + a^{-1}$ [1].

When $a \in \mathbb{F}_{2^{m_1}}^*$, $m = 2m_1$, $K_m(a)$ and $Q_m(a)$ have the following properties

Proposition 3: (Lemma 3 [1]) If $a \in \mathbb{F}_{2^{m_1}}^*$, $m = 2m_1$,

(1) $1 - K_m(a) = (1 - K_{m_1}(a))^2 - 2 \cdot 2^{m_1}$.

(1) if $m_1 \equiv 1 \pmod 2$, then $Q_m(a) \in \{0,\ 2 \cdot 2^{m/2}, -4 \cdot 2^{m/2}\}$.

Proposition 4: [11] The Ramanujan-Nagell equation $x^2 - D = 2^{n+2}$ has at most 4 solutions (x, n), which are

$$(x, n) := (2^k-3, 1),\ (2^k-1, k),\ (2^k+1, k+1),\ (3\cdot 2^k-1, 2k+1),$$

where $k \in \mathbb{N}$ and $D \in \mathbb{N}$ is odd.

With the help of the solutions of Ramanujan-Nagell equation,

Lemma 1: If $a \in \mathbb{F}_{2^{m_1}}$, $m = 2m_1$, $m_1 > 1$, then $K_m(a) \neq -4$.

Proof: By Propostion 3, if $K_m(a) = -4$,

$$(1 - K_{m_1}(a))^2 = 2 \cdot 2^{m_1} + 5. \qquad (2)$$

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It is easy to check that when $m_1 < 5$, $2\cdot 2^{m_1}+5$ is not a square. By Propostion 4, (2) has at most 4 solutions $(|(1-K_{m_1}(a))|, n)$, which are

$(|(1-K_{m_1}(a))|, m_1-1) =$
$(2^k-3, 1),\ (2^k-1, k),\ (2^k+1, k+1),\ (3\cdot 2^k-1, 2k+1),$

where $k \in \mathbb{N}$. We can check all the 4 solutions can not satisfy (2). For example, if $(|(1-K_{m_1}(a))|, m_1-1) = (3\cdot 2^k - 1, 2k+1)$, then

$$(3\cdot 2^k - 1)^2 = 2^{2k+1+2} + 5. \tag{3}$$

When $k = 1, 2$, $(3\cdot 2^k - 1)^2 \neq 2^{2k+1+2} + 5$. When $k \geq 3$, $(3\cdot 2^k - 1)^2 > 2^{2k+1+2} + 5$. Thus (3) has no integral solution, therefore (2) has no integral solution either, which concludes the proof. ■

III. The hyper-bent property of $f_{a,b}^{(r)}$ when $m = 2 \pmod 4$

In the this section, we consider the Boolean function $f_{a,b}^{(r)}$ defined by (1), where $n = 2m$, $m \equiv 2 \pmod 4$, $a \in \mathbb{F}_{2^n}$ and $b \in \mathbb{F}_{16}$. As the cyclotomic coset of 2 module $2^n - 1$ containing $\frac{2^n-1}{5}$ is

$$\{\frac{2^n-1}{5}, 2\cdot\frac{2^n-1}{5}, 2^2\cdot\frac{2^n-1}{5}, 2^3\cdot\frac{2^n-1}{5}\}.$$

Its size is 4, or $o(\frac{2^n-1}{5}) = 4$, which means $f_{a,b}^{(r)}$ is neither in the class considered by Charpin and Gong [4] nor in the class studied by Mesanager [6], [7].

Let α be a primitive element of $\mathbb{F}_{2^n}$, $\beta = \alpha^{\frac{2^n-1}{5}}$, $\xi = \alpha^{2^m-1}$, $U =< \xi >$, $V =< \xi^5 >$. Since $5|(2^m+1)$, V is the subgroup of U and $\#V = \frac{2^m+1}{5}$.

For any $i \in \mathbb{F}_{2^m}$, define

$$\begin{aligned} S_i &= \sum_{v\in V} \chi(\mathrm{Tr}_1^n(a\xi^{i(2^m-1)}v)) \\ &= \sum_{v\in V} \chi(\mathrm{Tr}_1^n(a\xi^{-2i}v)) = \sum_{v\in V} \chi(\mathrm{Tr}_1^n(a\xi^{-5i+3i}v)) \\ &= \sum_{v\in V} \chi(\mathrm{Tr}_1^n(a\xi^{3i}v)). \qquad (as\ \xi^{-5i} \in V) \end{aligned}$$

From the definition of S_i,

$$S_i = S_{i \pmod 5}. \tag{4}$$

To study the hyper-bent properties of $f_{a,b}^{(r)}$, we define the following character sum

$$\Lambda_r(a,b) := \sum_{u\in U} \chi(f_{a,b}^{(r)}(u)). \tag{5}$$

Similar to the proof of Proposition 9 in [1], the hyper-bent properties of $f_{a,b}^{(r)}$ can be described as

Proposition 5: $f_{a,b}^{(r)}$ is a hyper-bent function if and only if $\Lambda_r(a,b) = 1$.

Before our work on $f_{a,b}^{(r)}$, let us consider a *general* case of $f_{a,b}^{(r)}$ which is defined as

$$f_{a,b}^{(r,k)} := \mathrm{Tr}_1^n(ax^{r(2^m-1)}) + \mathrm{Tr}_1^4(bx^{k\frac{2^n-1}{5}}), \tag{6}$$

where a, b is defined as above and $k \in \mathbb{N}$.

When $k \equiv 0 \pmod 5$, $f_{a,b}^{(r,k)} = \mathrm{Tr}_1^n(ax^{r(2^m-1)}) + \mathrm{Tr}_1^4(b)$ is a special case studied by Charpin and Gong in [4]. In this paper we only consider the case of $k \not\equiv 0 \pmod 5$.

Proposition 6: The hyper-bent properties of $f_{a,b}^{(r,k)}$ can be represented by that of $f_{a,b}^{(r)}$ efficiently, where $a \in \mathbb{F}_{2^n}$, $b \in \mathbb{F}_{16}$, $k \not\equiv 0 \pmod 5$.

Proof: For $b \in \mathbb{F}_{16}^*$, b can be written as $b = \omega\beta^j$, where $\omega^3 = 1$, $0 \leq j \leq 4$. Thus

$$\mathrm{Tr}_1^4(bx^{k\frac{2^n-1}{5}}) = \mathrm{Tr}_1^4(\omega\beta^j x^{k\frac{2^n-1}{5}}) = \mathrm{Tr}_1^4(\omega(\beta^{\frac{j}{k}} x^{\frac{2^n-1}{5}})^k).$$

It is easy to check,

$$\begin{aligned} \mathrm{Tr}_1^4(\omega x^{\frac{2^n-1}{5}}) &= \mathrm{Tr}_1^4(\omega^2 x^{2\frac{2^n-1}{5}}) \\ &= \mathrm{Tr}_1^4(\omega x^{4\frac{2^n-1}{5}}) = \mathrm{Tr}_1^4(\omega^2 x^{3\frac{2^n-1}{5}}). \end{aligned}$$

Then $\mathrm{Tr}_1^4(bx^{k\frac{2^n-1}{5}}) = \mathrm{Tr}_1^4(b' x^{\frac{2^n-1}{5}})$, where $b' \in \mathbb{F}_{16}^*$.

Hence the result stands. ■

A step further, $f_{a,b}^{(r)}$ has following proposition.

Proposition 7: Let $f_{a,b}^{(r)}$ be defined as (1) and $(r,5) = 1$, then $f_{a,b}^{(r)}$ is a hyper-bent function if and only if $f_{a',b'}^{(r)}$ is a hyper-bent one, where $a = a'\xi^i \in \mathbb{F}_{2^n}$, $a' \in \mathbb{F}_{2^m}$, b, $b' = b\alpha^{-\frac{i}{r}\frac{2^n-1}{5}} \in \mathbb{F}_{16}$.

Proof: Notice that $\forall a \in \mathbb{F}_{2^n}$, $a = a'\xi^i$, where $a' \in \mathbb{F}_{2^m}$, $\xi = \alpha^{2^m-1}$ is a primitive 2^m+1-th root of unity in $\mathbb{F}_{2^n}$ and $0 \leq i \leq 2^m$. We have

$$\begin{aligned} f_{a,b}^{(r)}(x) &= \mathrm{Tr}_1^n(ax^{r(2^m-1)}) + \mathrm{Tr}_1^4(bx^{\frac{2^n-1}{5}}) \\ &= \mathrm{Tr}_1^n(a'(\alpha^{\frac{i}{r}}x)^{r(2^m-1)}) + \mathrm{Tr}_1^4(b\alpha^{-\frac{i}{r}\frac{2^n-1}{5}}(\alpha^{\frac{i}{r}}x)^{\frac{2^n-1}{5}}) \\ &= f_{a',b'}^{(r)}(\alpha^{-\frac{i}{r}}x), \end{aligned}$$

where $b' = b\alpha^{-\frac{i}{r}\frac{2^n-1}{5}} \in \mathbb{F}_{16}$.

Thus $f_{a,b}^{(r)}$ is linearly equivalent to $f_{a',b'}^{(r)}$, that is to say, $f_{a,b}^{(r)}$ is a hyper-bent function if and only if $f_{a',b'}^{(r)}$ is a hyper-bent one. ■

By Proposition 7, if $a = a'\xi^i$, and $\beta = \alpha^{\frac{2^n-1}{5}}$, we have the following results

- $f_{a,b}^{(1)}$ is linearly equivalent to $f_{a',b\beta^{4i}}^{(1)}$.
- $f_{a,b}^{(2)}$ is linearly equivalent to $f_{a',b\beta^{2i}}^{(2)}$.
- $f_{a,b}^{(3)}$ is linearly equivalent to $f_{a',b\beta^{3i}}^{(3)}$.
- $f_{a,b}^{(4)}$ is linearly equivalent to $f_{a',b\beta^{i}}^{(4)}$.

By Proposition 7 and Proposition 6, when $a \in \mathbb{F}_{2^n}, k \in \mathbb{N}$, $b \in \mathbb{F}_{16}$, the hyper-bent properties of $f_{a,b}^{(r,k)}$ can be fully represented by that of $f_{a,b}^{(r)}$, where $a \in \mathbb{F}_{2^m}$, $b \in \mathbb{F}_{16}$. Since the hyper-bent properties of $f_{a,b}^{(1)}$ had been studied elaborately in [1], in the following parts of this Section we only consider the rest cases of r.

A. The Case of $r = 5$

1) The hyper-bent properties of $f_{a,b}^{(5)}$, where $a \in \mathbb{F}_{2^m}$:

Proposition 8: Let $n = 2m$ and $m \equiv \pm 2, \pm 6 \pmod{20}$, If $b \in \{0\} \bigcup \{\beta^i | i = 0, 1, 2, 3, 4\}$, then the Boolean function

$f_{a,b}^{(5)}$ is not a hyper-bent function. Further, if $b \in \mathbb{F}_{16}^* \backslash \{\beta^i | 0 \leq i \leq 4\}$, $f_{a,b}^{(5)}$ is a hyper-bent function if and only if

$$\sum_{v \in V} \chi(\mathrm{Tr}_1^n(av)) = 1.$$

Proof: By (5),

$$\begin{aligned}\Lambda_5(a,b) &= \sum_{u \in U} \chi(f_{a,b}^{(5)}(u)) \\ &= \sum_{u \in U} \chi(\mathrm{Tr}_1^n(au^{5(2^m-1)}))\chi(\mathrm{Tr}_1^4(bu^{\frac{2^n-1}{5}})).\end{aligned}$$

Notice that $U =< \xi >$, $V =< \xi^5 >$ and $U = \xi^0 V \bigcup \xi^1 V \bigcup \xi^2 V \bigcup \xi^3 V \bigcup \xi^4 V$. Then,

$$\Lambda_5(a,b) = \tag{7}$$

$$\sum_{i=0}^{4} \sum_{v \in V} \chi(\mathrm{Tr}_1^4(b(\xi^i v)^{\frac{2^n-1}{5}}))\chi(\mathrm{Tr}_1^n(a(\xi^i v)^{5(2^m-1)}))$$

$$= \sum_{i=0}^{4} \sum_{v \in V} \chi(\mathrm{Tr}_1^4(b(\xi^i v)^{\frac{2^n-1}{5}}))\chi(\mathrm{Tr}_1^n(a(\xi^{5i})^{2^m-1} v^{5(2^m-1)})) \tag{8}$$

Since $(\xi^{5i})^{2^m-1} \in V$ and $m \equiv \pm 2, \pm 6 \pmod{20}$, $(5(2^m-1), \#V) = (5, \frac{2^m+1}{5}) = 1$. Then $v \longmapsto (\xi^{5i})^{2^m-1} v^{5(2^m-1)}$ is a permutation of V. Hence,

$$\begin{aligned}\Lambda_5(a,b) =& \sum_{i=0}^{4} \sum_{v \in V} \chi(\mathrm{Tr}_1^4(b(\xi^i v)^{\frac{2^n-1}{5}}))\chi(\mathrm{Tr}_1^n(av)) \\ =& (\sum_{i=0}^{4} \chi(\mathrm{Tr}_1^4(b\xi^{i\frac{2^n-1}{5}})))(\sum_{v \in V} \chi(\mathrm{Tr}_1^n(av))).\end{aligned}$$

As $\xi^{\frac{2^n-1}{5}} = (\alpha^{2^m-1})^{\frac{(2^m-1)(2^m+1)}{5}} = \beta^{2^m-1} = \beta^{2^m+1-2} = \beta^3$,

$$\begin{aligned}\Lambda_5(a,b) =& (\sum_{i=0}^{4} \chi(\mathrm{Tr}_1^4(b\beta^{3i}))(\sum_{v \in V} \chi(\mathrm{Tr}_1^n(av))) \\ =& (\sum_{i=0}^{4} \chi(\mathrm{Tr}_1^4(b\beta^i))(\sum_{v \in V} \chi(\mathrm{Tr}_1^n(av))).\end{aligned} \tag{9}$$

By (9), when $b = 0$, $\Lambda_5(a,0) = 5 \sum_{v \in V} \chi(\mathrm{Tr}_1^n(av))$, and thus $\Lambda_5(a,0) \neq 1$. By Proposition 5, $f_{a,0}^{(5)}$ is not a hyper-bent function.

When $b \neq 0$, b can be represented as $b = \omega\beta^j$, where $\omega^3 = 1$ and $0 \leq j \leq 4$. Then

$$\sum_{i=0}^{4} \chi(\mathrm{Tr}_1^4(b\beta^i)) = \sum_{i=0}^{4} \chi(\mathrm{Tr}_1^4(\omega\beta^{i+j})) = \sum_{i=0}^{4} \chi(\mathrm{Tr}_1^4(\omega\beta^i)). \tag{10}$$

Since $\omega^3 = 1$ and $\omega^4 = \omega$, we have

$$\mathrm{Tr}_1^4(\omega\beta^i) = \mathrm{Tr}_1^4(\omega^4\beta^{4i}) = \mathrm{Tr}_1^4(\omega\beta^{4i}).$$

If $\omega = 1$, $\sum_{i=0}^{4} \chi(\mathrm{Tr}_1^4(b\beta^i) = \sum_{i=0}^{4} \chi(\mathrm{Tr}_1^4(\beta^i))$. As β satisfies $\beta^4 + \beta^3 + \beta^2 + \beta + 1 = 0$, $\mathrm{Tr}_1^4(\beta^i) = 1, i \neq 0$. Then $\sum_{i=0}^{4} \chi(\mathrm{Tr}_1^4(b\beta^i)) = -3$. Therefore,

$$\Lambda_5(a,b) = -3 \sum_{v \in V} \chi(\mathrm{Tr}_1^n(av)), b = \beta^j, 0 \leq j \leq 4.$$

By Proposition 5, $f_{a,\beta^j}^{(5)}$ is not a hyper-bent function. When $\omega \neq 1$, we have

$$\begin{aligned}&\mathrm{Tr}_1^4(\omega\beta) + \mathrm{Tr}_1^4(\omega\beta^2) = \mathrm{Tr}_1^4(\omega(\beta + \beta^2)) \\ &= \omega(\beta + \beta^2 + \beta^3 + \beta^4) + \omega^2(\beta + \beta^2 + \beta^3 + \beta^4) \\ &= 1.\end{aligned}$$

Then $\chi(\mathrm{Tr}_1^4(\omega\beta)) + \chi(\mathrm{Tr}_1^4(\omega\beta^2)) = 0$. Similarly, $\chi(\mathrm{Tr}_1^4(\omega\beta^3)) + \chi(\mathrm{Tr}_1^4(\omega\beta^4)) = 0$. Therefore,

$$\Lambda_5(a,b) = \sum_{v \in V} \chi(\mathrm{Tr}_1^n(av)), b = \omega\beta^j, 0 \leq j \leq 4, \omega^3 = 1, \omega \neq 1.$$

By Proposition 5, the second part of this proposition follows. ■

In Proposition 8, we consider the hyper-bent properties of the Boolean function $f_{a,b}^{(5)}$ for $m \equiv \pm 2, \pm 6 \pmod{20}$. The proposition below discusses the hyper-bent properties of $f_{a,b}^{(5)}$ for $m \equiv 10 \pmod{20}$.

Proposition 9: Let $n = 2m$, $m \equiv 10 \pmod{20}$, $a \in \mathbb{F}_{2^m}$, $b \in \mathbb{F}_{16}$. then the Boolean function $f_{a,b}^{(5)}$ is not a hyper-bent function.

Proof: Notice that $\Lambda_5(a,b) = \sum_{i=0}^4 \sum_{v \in V} \chi(\mathrm{Tr}_1^4(b\xi^{i\frac{2^n-1}{5}}))\chi(\mathrm{Tr}_1^n(a(\xi^{5i})^{2^m-1} v^{5(2^m-1)}))$. Since $m \equiv 10 \pmod{20}$, $25 | (2^m + 1)$ and $(5(2^m-1), \frac{2^m+1}{5}) = 5$. Then $v \longmapsto v^{5(2^m-1)}$ is a 5 to 1 morphism from V to $V^5 := \{v^5 | v \in V\}$. Therefore,

$$\Lambda_5(a,b) = 5 \sum_{i=0}^{4} \sum_{v \in V^5} \chi(\mathrm{Tr}_1^4(b\xi^{i\frac{2^n-1}{5}}))\chi(\mathrm{Tr}_1^n(a(\xi^{5i})^{2^m-1} v)).$$

Hence, $5 | \Lambda_5(a,b)$ and $\Lambda_5(a,b)$ is not equal to 1, By Proposition 5, $f_{a,b}^{(5)}$ is not a hyper-bent function. ■

By Proposition 8,

$$\sum_{v \in V} \chi(\mathrm{Tr}_1^n(av)) = \sum_{v \in V} \chi(\mathrm{Tr}_1^n(av^{2^m-1})).$$

Notice that $\sum_{v \in V} \chi(\mathrm{Tr}_1^n(av)) = S_0$ in [1]. By Proposition 15 in [1],

$$\sum_{v \in V} \chi(\mathrm{Tr}_1^n(av)) = \frac{1}{5}[1 - K_m(a) + 2Q_m(a)]. \tag{11}$$

Further, By Proposition 16 and 18 in [1], we have the following results.

Proposition 10: Let $n = 2m$, $m \equiv \pm 2, \pm 6 \pmod{20}$, $m \geq 6$ and $b \in \mathbb{F}_{16}^* \backslash \{\beta^i | 0 \leq i \leq 4\}$, then $f_{a,b}^{(5)}$ is a hyper-bent function if and only if one of the assertions (1) and (2) holds.

(1) $Q_m(a) = 0$, $K_m(a) = -4$.

(2) $Q_m(a) = 2^{m_1}$, $K_m(a) = 2 \cdot 2^{m_1} - 4$.

2) The hyper-bent properties of $f_{a,b}^{(5)}$ where $a \in \mathbb{F}_{2^n}$: In this part, we always assume $n = 2m$, $m = 2m_1$, $m_1 \in \mathbb{N}$.

Lemma 2: Let $b \in \mathbb{F}_{16}^*$, $\gamma \in \{z \in \mathbb{F}_{2^n} : z^5 = 1, z \neq 1\} =< \alpha^{\frac{2^n-1}{5}} >$, then

$$\sum_{i=0}^{4} \chi(\mathrm{Tr}_1^4(b\gamma^i)) = \begin{cases} 1, & b^5 \neq 1 \\ -3, & b^5 = 1. \end{cases}$$

Proof: Firstly, if $b^5 = 1$,

$$\sum_{i=0}^{4} \chi(\mathrm{Tr}_1^4(b\gamma^i)) = \sum_{i=0}^{4} \chi(\mathrm{Tr}_1^4(\gamma^i)) = 1 + \sum_{i=0}^{3} \chi(\mathrm{Tr}_1^4(\gamma^{2^i})) = 1 + 4\chi(\mathrm{Tr}_1^4(\gamma)) = -3.$$

Secondly, if $b^5 \neq 1$,

$$\sum_{i=0}^{4} \chi(\mathrm{Tr}_1^4(b\gamma^i)) = \sum_{i=0}^{4} \chi(\mathrm{Tr}_1^4(b^2\gamma^{2i})) = \sum_{i=0}^{4} \chi(\mathrm{Tr}_1^4(b^2\gamma^i)).$$

Since $\forall b \in \mathbb{F}_{16}^*$, $b = \omega^j\gamma^i$, $0 \leq j \leq 2$, $0 \leq i \leq 4$, we have

$$\begin{aligned}\sum_{b\in\mathbb{F}_{16}} \chi(\mathrm{Tr}_1^4(b)) &= 1 + \sum_{b\in\mathbb{F}_{16}^*} \chi(\mathrm{Tr}_1^4(b)) \\ &= 1 + \sum_{j=0}^{2}\sum_{i=0}^{4} \chi(\mathrm{Tr}_1^4(\omega^j\gamma^i)) \\ &= 1 + \sum_{i=0}^{4} \chi(\mathrm{Tr}_1^4(\gamma^i)) + \sum_{i=0}^{4} \chi(\mathrm{Tr}_1^4(\omega\gamma^i)) + \sum_{i=0}^{4} \chi(\mathrm{Tr}_1^4(\omega^2\gamma^i)) \\ &= 1 + (-3) + 2\sum_{i=0}^{4} \chi(\mathrm{Tr}_1^4(\omega\gamma^i)).\end{aligned}$$

Notice that $\sum\limits_{b\in\mathbb{F}_{16}} \chi(\mathrm{Tr}_1^4(b)) = 0$, hence $\sum_{i=0}^4 \chi(\mathrm{Tr}_1^4(b\gamma^i)) = 1$, and the conclusion stands. ■

Theorem 1: If $a = a'\xi^i$, $a' \in \mathbb{F}_{2^m}$, the hyper-bent properties of $f_{a,b}^{(5)}$ can be described as follows:

(1) when $m \equiv 10 \pmod{20}$, $f_{a,b}^{(5)}$ is not hyper-bent.

(2) when $m \equiv \pm 2, \pm 6 \pmod{20}$, $f_{a,b}^{(5)}$ is hyper-bent if and only if $S_{2i} = 1$.

Proof: To the character sum of $f_{a,b}^{(5)}$:

$$\begin{aligned}\Lambda(a'\xi^i, b) &= \sum_{u\in U} \chi(f_{a'\xi^i,b}^{(5)}(u)) \\ &= \sum_{u\in U} \chi(\mathrm{Tr}_1^n(a'\xi^i u^{5(2^m-1)}))\chi(\mathrm{Tr}_1^4(bu^{\frac{2^n-1}{5}})) \\ &= \sum_{j=0}^{4}\sum_{v\in V} \chi(\mathrm{Tr}_1^n(a'\xi^i(\xi^j v)^{5(2^m-1)}))\chi(\mathrm{Tr}_1^4(b(\xi^j v)^{\frac{2^n-1}{5}})) \\ &= \sum_{j=0}^{4}\sum_{v\in V} \chi(\mathrm{Tr}_1^4(b\xi^{j\frac{2^n-1}{5}}))\chi(\mathrm{Tr}_1^n(a'\xi^i\xi^{5j(2^m-1)}v^{5(2^m-1)})).\end{aligned} \quad (12)$$

If $m \equiv 10 \pmod{20}$, then $(5, \#V) = 5$. By (12), $\Lambda(a'\xi^i, b) = 5\sum_{j=0}^4 \sum_{v'\in V^5} \chi(\mathrm{Tr}_1^4(b\xi^{j\frac{2^n-1}{5}}))\chi(\mathrm{Tr}_1^n(a'\xi^i\xi^{5j(2^m-1)}v'))$, where $V^5 = \{v^5 \mid v \in V\}$, $v \mapsto v^{5(2^m-1)}$ is a 5 *to* 1 morphism from V to V^5 . Thus $\Lambda(a'\xi^i, b) \neq 1$, and $f_{a,b}^{(5)}$ is not a hyper-bent function.

If $m \equiv \pm 2, \pm 6 \pmod{20}$, then $(5, \#V) = 1$. By (12) and (9),

$$\begin{aligned}\Lambda(a'\xi^i, b) &= \sum_{j=0}^{4}\sum_{v\in V} \chi(\mathrm{Tr}_1^4(b\beta^j))\chi(\mathrm{Tr}_1^n(a'\xi^i v)) \\ &= (\sum_{j=0}^{4} \chi(\mathrm{Tr}_1^4(b\beta^j)))(\sum_{v\in V} \chi(\mathrm{Tr}_1^n(a'(\xi^{\frac{i}{2^m-1}})^{2^m-1}v))),\end{aligned}$$

where $\beta = \alpha^{\frac{2^n-1}{5}}$, $\xi^{\frac{2^n-1}{5}} = \beta^3$. Since $\frac{1}{2^m-1} \equiv 2 \pmod 5$, then by (4),

$$\begin{aligned}\Lambda(a'\xi^i, b) &= (\sum_{j=0}^{4} \chi(\mathrm{Tr}_1^4(b\beta^j)))(\sum_{v\in V} \chi(\mathrm{Tr}_1^n(a'(\xi^{2i})^{2^m-1}v))) \\ &= (\sum_{j=0}^{4} \chi(\mathrm{Tr}_1^4(b\beta^j)))S_{2i}.\end{aligned}$$

By Lemma 2,

$$\Lambda(a'\xi^i, b) = \begin{cases} S_{2i}, & b^5 \neq 1 \\ -3S_{2i}, & b^5 = 1. \end{cases}$$

If $b^5 = 1$, $3 \mid \Lambda(a'\xi^i, b)$. Thus $f_{a'\xi^i,b}^{(5)}$ is not a hyper-bent function.

If $b^5 \neq 1$, then $f_{a'\xi^i,b}^{(5)}$ is a hyper-bent function if and only if $S_{2i} = 1$. ■

B. The Case of $r = 2$

When $b = 0$, the hyper-bent propriety of $f_{a,0}^{(2)}$ has been studied by Canteaut et al in [2]. We consider the case of $b \neq 0$.

Proposition 11: Let $a \in \mathbb{F}_{2^m}$, $b \in \mathbb{F}_{16}^*$, we have

(1) if $b = 1$, then $\Lambda_2(a, b) = S_0 - 2(S_1 + S_2) = 2S_0 - \Lambda_2(a, 0)$.

(2) if $b \in \{\beta+\beta^2, \beta+\beta^3, \beta^2+\beta^4, \beta^3+\beta^4\}$, then $\Lambda_2(a, b) = S_0$.

(3) if $b = \beta$ or β^4, then $\Lambda_2(a, b) = -S_0 - 2S_2$.

(4) if $b = \beta^2$ or β^3, then $\Lambda_2(a, b) = -S_0 - 2S_1$.

(5) if $b = 1 + \beta$ or $1 + \beta^4$, then $\Lambda_2(a, b) = -S_0 + 2S_2$.

(6) if $b = 1 + \beta^2$ or $1 + \beta^3$, then $\Lambda_2(a, b) = -S_0 + 2S_1$.

(7) if $b = \beta + \beta^4$, then $\Lambda_2(a, b) = S_0 + 2S_2 - 2S_1$.

(8) if $b = \beta^2 + \beta^3$, then $\Lambda_2(a, b) = S_0 - 2S_2 + 2S_1$.

Proof: Similar to proof of Proposition 13 in [1] the results hold. ■

Corollary 1: Let $a \in \mathbb{F}_{2^m}$, $b \in \mathbb{F}_{16}^*$, we have

(1) $f_{a,b}^{(2)}$ holds the same hyper-bent propertyies as $f_{a,b^2}^{(1)}$.

(2) if b satisfies $(b+1)(b^4+b+1) = 0$, then $f_{a,b}^{(2)}$ holds the same hyper-bent properties as $f_{a,b}^{(1)}$.

Proof: (1) By Proposition 11 and Proposition 13 in [1],

$$\Lambda_2(a, b) = \Lambda_1(a, b^2).$$

Hence $f_{a,b}^{(2)}$ is a hyper-bent function if and only if $f_{a,b^2}^{(1)}$ is.

(2) Similarly, if b satisfying $(b + 1)(b^4 + b + 1) = 0$, then,

$$\Lambda_2(a, b) = \Lambda_1(a, b).$$

Thus $f_{a,b}^{(2)}$ holds the same hyper-bent properties as $f_{a,b}^{(1)}$. ■

C. The General Case of r

Theorem 2: Let $n = 2m$, $m \equiv 2 \pmod 4$, $a \in \mathbb{F}_{2^m}$ and $b \in \mathbb{F}_{16}$. If $(r, \frac{2^m+1}{5}) > 1$, then $f_{a,b}^{(r)}$ is not a hyper-bent function. Further, if $(r, \frac{2^m+1}{5}) = 1$, then

(1) If $r \equiv 0 \pmod 5$, then $f_{a,b}^{(r)}$ and $f_{a,b}^{(5)}$ has the same hyper-bent properties.

(2) If $r \equiv \pm 1 \pmod 5$, then $f_{a,b}^{(r)}$ and $f_{a,b}^{(1)}$ has the same hyper-bent properties.

(3) If $r \equiv \pm 2 \pmod 5$, then $f_{a,b}^{(r)}$ and $f_{a,b}^{(2)}$ has the same hyper-bent properties.

Proof: Notice that

$$\Lambda_r(a,b) = \sum_{i=0}^{4}\sum_{v\in V}\chi(\mathrm{Tr}_1^4(b(\xi^i v)^{\frac{2^n-1}{5}}))\chi(\mathrm{Tr}_1^n(a(\xi^i v)^{r(2^m-1)}))$$

$$= \sum_{i=0}^{4}\sum_{v\in V}\chi(\mathrm{Tr}_1^4(b\xi^{i\frac{2^n-1}{5}}))\chi(\mathrm{Tr}_1^n(a\xi^{ri(2^m-1)}v^{r(2^m-1)})).$$

Let $d = (r(2^m-1), \#V) = (r, \frac{2^m+1}{5})$, then $\Lambda_r(a,b) = d\sum_{i=0}^{4}\chi(\mathrm{Tr}_1^4(b\xi^{i\frac{2^n-1}{5}}))\sum_{v\in V^d}\chi(\mathrm{Tr}_1^n(a\xi^{ri(2^m-1)}v^{r(2^m-1)}))$, where $V^d = \{v^d | v \in V\}$. If $d = (r, \frac{2^m+1}{5}) > 1$, $d|\Lambda_r(a,b)$ and $\Lambda_r(a,b) \neq 1$. Hence, $f_{a,b}^{(r)}$ is not a hyper-bent function.

When $d = (r, \frac{2^m+1}{5}) = 1$,

$$\Lambda_r(a,b) = \sum_{i=0}^{4}\chi(\mathrm{Tr}_1^4(b\xi^{i\frac{2^n-1}{5}}))\sum_{v\in V}\chi(\mathrm{Tr}_1^n(a\xi^{ri(2^m-1)}v)). \tag{13}$$

If $r \equiv 0 \pmod 5$, from $\xi^{\frac{2^n-1}{5}} = \beta^3$, we have

$$\Lambda_r(a,b) = \sum_{i=0}^{4}\chi(\mathrm{Tr}_1^4(b\beta^{3i}))\sum_{v\in V}\chi(\mathrm{Tr}_1^n(a\xi^{ri(2^m-1)}v))$$
$$= \sum_{i=0}^{4}\chi(\mathrm{Tr}_1^4(b\beta^{i}))\sum_{v\in V}\chi(\mathrm{Tr}_1^n(av)).$$

Then $\Lambda_r(a,b) = \Lambda_5(a,b)$. Therefore, $f_{a,b}^{(r)}$ and $f_{a,b}^{(5)}$ has the same hyper-bent properties.

If $r \equiv 1 \pmod 5$, then

$$\Lambda_r(a,b) = \sum_{i=0}^{4}\chi(\mathrm{Tr}_1^4(b\xi^{i\frac{2^n-1}{5}}))\sum_{v\in V}\chi(\mathrm{Tr}_1^n(a\xi^{i(2^m-1)}v)).$$

By Proposition 10 in [1], $\Lambda_r(a,b) = \Lambda_1(a,b)$. Hence, $f_{a,b}^{(r)}$ and $f_{a,b}^{(1)}$ has the same hyper-bent properties.

If $r \equiv 2 \pmod 5$, then

$$\Lambda_r(a,b) = \sum_{i=0}^{4}\chi(\mathrm{Tr}_1^4(b\xi^{i\frac{2^n-1}{5}}))\sum_{v\in V}\chi(\mathrm{Tr}_1^n(a\xi^{2i(2^m-1)}v))$$
$$= \sum_{i=0}^{4}\chi(\mathrm{Tr}_1^4(b\beta^{3i}))S_{2i}$$
$$= \sum_{i=0}^{4}\chi(\mathrm{Tr}_1^4(b\beta^{9i}))S_{6i} = \sum_{i=0}^{4}\chi(\mathrm{Tr}_1^4(b\beta^{4i}))S_i.$$

By Lemma 1 in [1],

$$\Lambda_r(a,b) = \chi(\mathrm{Tr}_1^4(b))S_0 + (\chi(\mathrm{Tr}_1^4(b\beta)) + \chi(\mathrm{Tr}_1^4(b\beta^4)))S_1 + (\chi(\mathrm{Tr}_1^4(b\beta^2)) + \chi(\mathrm{Tr}_1^4(b\beta^3)))S_2. \tag{14}$$

Hence, $\Lambda_r(a,b) = \Lambda_2(a,b)$. $f_{a,b}^{(r)}$ and $f_{a,b}^{(2)}$ has the same hyper-bent properties.

If $r \equiv 3 \pmod 5$,

$$\Lambda_r(a,b) = \sum_{i=0}^{4}\chi(\mathrm{Tr}_1^4(b\xi^{i\frac{2^n-1}{5}}))\sum_{v\in V}\chi(\mathrm{Tr}_1^n(a\xi^{3i(2^m-1)}v))$$
$$= \sum_{i=0}^{4}\chi(\mathrm{Tr}_1^4(b\beta^{3i}))S_{3i} = \sum_{i=0}^{4}\chi(\mathrm{Tr}_1^4(b\beta^{i}))S_i.$$

From Lemma 1 in [1],

$$\Lambda_r(a,b) = \chi(\mathrm{Tr}_1^4(b))S_0 + (\chi(\mathrm{Tr}_1^4(b\beta)) + \chi(\mathrm{Tr}_1^4(b\beta^4)))S_1 + (\chi(\mathrm{Tr}_1^4(b\beta^2)) + \chi(\mathrm{Tr}_1^4(b\beta^3)))S_2. \tag{15}$$

Hence, $\Lambda_r(a,b) = \Lambda_3(a,b)$. From (14) and (15), we have $\Lambda_2(a,b) = \Lambda_3(a,b)$. Thus, $f_{a,b}^{(r)}$ and $f_{a,b}^{(2)}$ have the same hyper-bent properties.

Similarly, if $r \equiv 4 \pmod 5$, then $\Lambda_r(a,b) = \Lambda_4(a,b) = \Lambda_1(a,b)$. Thus, $f_{a,b}^{(r)}$ and $f_{a,b}^{(1)}$ have the same hyper-bent properties.

Above all, the results stand. ■

From the above discussion, we have the following results on $f_{a,b}^{(r)}$.

Proposition 12: Let $a \in \mathbb{F}_{2^m}$ and $(r, \frac{2^m+1}{5}) = 1$, then

(1) If $\frac{1}{5}[1 - K_m(a) + 2Q_m(a)] = 1$, then the following Boolean functions

(a) $f_{a,b}^{(r)}$, $b \in \mathbb{F}_{16}^*\backslash\{\beta^i | i = 0,1,2,3,4\}$, $r \equiv 0 \pmod 5$.

(b) $f_{a,b}^{(r)}$, $r \not\equiv 0 \pmod 5$, $b^4 + b + 1 = 0$.

are hyper-bent functions.

(2) If $-\frac{1}{5}[3(1 - K_m(a)) - 4Q_m(a)] = 1$, then the Boolean function $f_{a,1}^{(r)}$ $(r \not\equiv 0 \pmod 5)$ is a hyper-bent function.

Proof: By Theorem 2, (11), Proposition 8 and Proposition 16 in [1], this proposition follows. ■

With Proposition 12, we can generalize Theorem 3 in [1] to the following theorem.

Theorem 3: Let $n = 2m$, $m = 2m_1$, $m_1 \equiv 1 \pmod 2$, $m_1 \geq 3$ and $(r, \frac{2^m+1}{5}) = 1$, If one of two assertions (1) and (2) holds,

(1) $p(x) = x^5 + x + a^{-1}$ over $\mathbb{F}_{2^m}$ is $(1)(2)^2$ and $K_m(a) = -4$.

(2) $p(x) = x^5 + x + a^{-1}$ is irreducible over $\mathbb{F}_{2^m}$. The quadratic form $\mathfrak{q}(x) = \mathrm{Tr}_1^m(x(ax^4 + ax^2 + a^2x))$ over $\mathbb{F}_{2^m}$ is even. $K_m(a) = 2 \cdot 2^{m_1} - 4$.

Then the Boolean functions

(a) $f_{a,b}^{(r)}$, $b \in \mathbb{F}_{16}^*\backslash\{\beta^i | i = 0,1,2,3,4\}$, $r \equiv 0 \pmod 5$.

(b) $f_{a,b}^{(r)}$, $r \not\equiv 0 \pmod 5$, $b^4 + b + 1 = 0$.

are hyper-bent functions.

Proof: By Proposition 16 and Theorem 3 in [1] and Proposition 12, this theorem follows. ■

By Proposition 16 , Proposition 12 and Theorem 2 in [1], we have the following results for the hyper-bent properties of $f_{a,b}^{(r)}$:

Theorem 4: Let $n = 2m$, $m = 2m_1$, $m_1 \equiv 1 \pmod 2$, $m_1 \geq 3$, $(r, \frac{2^m+1}{5}) = 1$ and $r \not\equiv 0 \pmod 5$, then $f_{a,1}^{(r)}$ is a hyper-bent function if and only if the following assertions holds.

(1) $p(x) = x^5 + x + a^{-1}$ is irreducible over $\mathbb{F}_{2^m}$.

(2) The quadratic form $\mathfrak{q}(x) = \mathrm{Tr}_1^m(x(ax^4 + ax^2 + a^2x))$ over $\mathbb{F}_{2^m}$ is even.

(3) $K_m(a) = \frac{4}{3}(2 - 2^{m_1})$.

If $a \in \mathbb{F}_{2^{\frac{m}{2}}}$, the hyper-bent properties of $f_{a,b}^{(r)}$ is

Theorem 5: Let $n = 2m$, $m = 2m_1$, $m_1 \equiv 1 \pmod 2$ and $m_1 \geq 3$. If $n \neq 12, 28$, any Boolean function in

$$\{f_{a,b}^{(r)} | a \in \mathbb{F}_{2^{\frac{m}{2}}}, b \in \mathbb{F}_{16}\} \quad (16)$$

is not a hyper-bent function. Further, if $n = 12$, all the hyper-bent functions in (16) are $\mathrm{Tr}_1^{12}(ax^{r(2^6-1)}) + \mathrm{Tr}_1^4(bx^{\frac{2^{12}-1}{5}})$, where $r \not\equiv 0 \pmod 5$, $(r, \frac{2^m+1}{5}) = 1$, $(a+1)(a^3 + a^2 + 1) = 0$ and $b = \beta^i, i = 1, 2, 3, 4$. If $n = 28$, all the hyper-bent functions in (16) are $\mathrm{Tr}_1^{28}(ax^{r(2^{14}-1)}) + \mathrm{Tr}_1^4(bx^{\frac{2^{28}-1}{5}})$, where $r \not\equiv 0 \pmod 5$, $(r, \frac{2^m+1}{5}) = 1$, $(a+1)(a^7+a^6+a^5+a^4+a^3+a^2+1) = 0$ and $b = \beta^i, i = 1, 2, 3, 4$.

Proof: Notice that $a \in \mathbb{F}_{2^{\frac{m}{2}}}$. By Theorem 2, if $f_{a,b}^{(r)}$ is a hyper-bent function, $(r, \frac{2^m+1}{5}) = 1$.

Suppose $(r, \frac{2^m+1}{5}) = 1$. we first prove that $f_{a,0}^{(r)}$ is not a hyper-bent function when $r \equiv 0 \pmod 5$. By Theorem 2, $f_{a,b}^{(r)}$ is a hyper-bent function if and only if $f_{a,b}^{(5)}$ is a hyper-bent function. If $b = 0$,

$$\Lambda_5(a, 0) = \sum_{u \in U} \chi(\mathrm{Tr}_1^n(au^{5(2^m-1)})) = 5 \sum_{v \in V} \chi(\mathrm{Tr}_1^n(av^{2^m-1})).$$

Hence, $5 | \Lambda_5(a, 0)$ and $\Lambda_5(a, 0) \neq 1$. Therefore, $f_{a,0}^{(5)}$ is not a hyper-bent function. Then $f_{a,0}^{(r)}$ is not a hyper-bent function.

When $b \neq 0$, by Theorem 3, $f_{a,b}^{(r)}$ is a hyper-bent function if and only if $f_{a,b'}^{(1)}$ $(b'^4 + b' + 1 = 0)$ is a hyper-bent function. By Theorem 5 in [1], $f_{a,b'}^{(1)}$ $(b'^4 + b' + 1 = 0)$ is not a hyper-bent function. Hence, $f_{a,b}^{(r)}$ is not a hyper-bent function when $r \equiv 0 \pmod 5$.

Now we discuss the case $r \equiv \pm 1 \pmod 5$ and $(r, \frac{2^m+1}{5}) = 1$. By Theorem 2, $f_{a,b}^{(r)}$ is a hyper-bent function if and only if $f_{a,b}^{(1)}$ is a hyper-bent function. By Theorem 5 in [1], there are only two cases. The first case is $n = 12$, where a and b satisfy

$$(a + 1)(a^3 + a^2 + 1) = 0, b = \beta^i, i = 1, 2, 3, 4.$$

The second case is $n = 28$, where a and b satisfy

$$(a+1)(a^7+a^6+a^5+a^4+a^3+a^2+1) = 0, b = \beta^i, i = 1, 2, 3, 4.$$

When $r \equiv \pm 2 \pmod 5$ and $(r, \frac{2^m+1}{5}) = 1$, we have similar results.

Above all, this theorem follows. ■

IV. The bent property of $f_{a,b}^{(r)}$ when $m = 0 \pmod 4$

In this section we consider the bent properties of $f_{a,b}^{(r)}$, where $m \equiv 0 \pmod 4$, $a \in \mathbb{F}_{2^n}$, $b \in \mathbb{F}_{16}$.

Proposition 13: Let $a = a'\xi^k \in \mathbb{F}_{2^n}^*$, $b \in \mathbb{F}_{16}^*$, $a' \in \mathbb{F}_{2^m}^*$, $0 \leq k \leq 2^m$, $m \equiv 0 \pmod 4$, $m = 2m_1$. One necessary condition for $f_{a,b}^{(r)}$ to be a bent function is: $(r, 2^m + 1) = 1$, $a' \in \mathbb{F}_{2^m} \setminus \mathbb{F}_{2^{m_1}}$, $b^5 \neq 1$, $\widehat{\chi}_{f_{a,b}^{(r)}}(0) = 2^m$ and $K_m(a') = -4$.

Proof: Notice that $\forall x \in \mathbb{F}_{2^n}^*$, $x = yu$, where $y \in \mathbb{F}_{2^m}^*$, $u \in U = < \alpha^{2^m-1} >$. Since $m \equiv 0 \pmod 4$, $5 \mid 2^m - 1$. Thus $u^{\frac{2^n-1}{5}} = (u^{2^m+1})^{\frac{2^m-1}{5}} = 1$. Now, consider the Walsh spectrum of $f_{a,b}^{(r)}$ at 0, which is

$$\begin{aligned}\widehat{\chi}_{f_{a,b}^{(r)}}(0) &= \sum_{x \in \mathbb{F}_{2^n}} \chi(f_{a,b}^{(r)}(x)) = 1 + \sum_{u \in U} \sum_{y \in \mathbb{F}_{2^m}^*} \chi(f_{a,b}^{(r)}(yu)) \\ &= 1 + \sum_{u \in U} \sum_{y \in \mathbb{F}_{2^m}^*} \chi(\mathrm{Tr}_1^n(a(yu)^{r(2^m-1)}))\chi(\mathrm{Tr}_1^4(b(yu)^{\frac{2^n-1}{5}})) \\ &= 1 + \sum_{u \in U} \chi(\mathrm{Tr}_1^n(au^{r(2^m-1)})) \sum_{y \in \mathbb{F}_{2^m}^*} \chi(\mathrm{Tr}_1^4(by^{\frac{2^n-1}{5}})) \quad (17)\end{aligned}$$

$\mathbb{F}_{2^m}^*$ can be written as $\mathbb{F}_{2^m}^* = \bigcup_{i=0}^{4} \beta^i V$, where $V = \{z^5 \mid z \in \mathbb{F}_{2^m}^*\}$, $\beta \in \mathbb{F}_{2^m}^* \setminus V$.

If $(r(2^m - 1), 2^m + 1) = 1$, by (17),

$$\begin{aligned}&\widehat{\chi}_{f_{a,b}^{(r)}}(0) = \\ &1 + \sum_{u \in U} \chi(\mathrm{Tr}_1^n(a'\xi^k u^{r(2^m-1)})) \sum_{i=0}^{4} \sum_{v \in V} \chi(\mathrm{Tr}_1^4(b(v\beta^i)^{\frac{2^n-1}{5}})) \\ &= 1 + \sum_{u \in U} \chi(\mathrm{Tr}_1^n(a'u)) \sum_{i=0}^{4} \sum_{v \in V} \chi(\mathrm{Tr}_1^4(b\beta^{i\frac{2^n-1}{5}})) \\ &= 1 + \sum_{u \in U} \chi(\mathrm{Tr}_1^n(a'u)) \sum_{v \in V} \sum_{i=0}^{4} \chi(\mathrm{Tr}_1^4(b\gamma^i)) \\ &= 1 + (1 - K_m(a'))\frac{2^m - 1}{5} \sum_{i=0}^{4} \chi(\mathrm{Tr}_1^4(b\gamma^i)), \quad (18)\end{aligned}$$

$(r(2^m - 1), 2^m + 1) = 1$, $u \mapsto \xi^k u^{r(2^m-1)}$ is a permutation in U, $\sum_{u \in U} \chi(\mathrm{Tr}_1^n(au^{2^m-1})) = 1 - K_m(a)$. $\gamma = \beta^{\frac{2^n-1}{5}} \neq 1$ is a 5-th primitive root of unity in $\mathbb{F}_{2^n}$. If $f_{a,b}^{(r)}$ is a bent function,

$$\widehat{\chi}_{f_{a,b}^{(r)}}(0) = 1 + (K_m(a') - 1)(\frac{2^m - 1}{5}) \sum_{i=0}^{4} \chi(\mathrm{Tr}_1^4(b\gamma^i)) = \pm 2^m.$$

By Lemma 2,

(1) if $\sum_{i=0}^{4} \chi(\mathrm{Tr}_1^4(b\gamma^i)) = -3$, then $K_m(a') = \frac{8}{3}$ or $3(2^m - 1)(K_m(a') - 1) = -5(2^m + 1)$. Since $K_m(a')$ is an integer, however $(\frac{2^m-1}{5}, 2^m + 1) = 1$, Neither of the two equations stands, thus $f_{a,b}^{(r)}$ is not a bent function.

(2) if $\sum_{i=0}^{4} \chi(\mathrm{Tr}_1^4(b\gamma^i)) = 1$, which means $K_m(a') = -4$, $\widehat{\chi}_{f_{a,b}^{(r)}}(0) = 2^m$, or $(2^m - 1)(K_m(a') - 1) = 5(2^m + 1)$, $\widehat{\chi}_{f_{a,b}^{(r)}}(0) = -2^m$. Since $(\frac{2^m-1}{5}, 2^m + 1) = 1$, the last group of equations can not stand. By Lemma 1, if $a' \in \mathbb{F}_{2^{m_1}}$, then $K_m(a') \neq -4$.

If $(r(2^m-1), 2^m+1) = d > 1$. Since $5 \mid 2^m-1$, $5 \nmid d$. By (17),

$$\widehat{\chi}_{f_{a,b}^{(r)}}(0) =$$

$$1 + \sum_{u\in U} \chi(\mathrm{Tr}_1^n(au^{r(2^m-1)})) \sum_{i=0}^{4} \sum_{v\in V} \chi(\mathrm{Tr}_1^4(b(v\beta^i)^{\frac{2^n-1}{5}}))$$

$$= 1 + d \sum_{u'\in U^d} \chi(\mathrm{Tr}_1^n(au)) \frac{2^m-1}{5} \sum_{i=0}^{4} \chi(\mathrm{Tr}_1^4(b\gamma^i))$$

$$= 1 + dh \frac{2^m-1}{5} \sum_{i=0}^{4} \chi(\mathrm{Tr}_1^4(b\gamma^i)),$$

where $U^d = \{u^d \mid u \in U\}$, $u \mapsto u^{r(2^m-1)}$ is a d *to* 1 morphism from U to U^d, $h = \sum_{u'\in U^d} \chi(\mathrm{Tr}_1^n(au))$. If $f_{a,b}^{(r)}$ is a bent function,

$$\widehat{\chi}_{f_{a,b}^{(r)}}(0) = 1 + dh(\frac{2^m-1}{5}) \sum_{i=0}^{4} \chi(\mathrm{Tr}_1^4(b\gamma^i)) = \pm 2^m.$$

By Lemma 2,

(1) if $\sum_{i=0}^{4} \chi(\mathrm{Tr}_1^4(b\gamma^i)) = -3$, then $3dh = -5$ or $3dh(2^m-1) = 5(2^m+1)$.

(2) if $\sum_{i=0}^{4} \chi(\mathrm{Tr}_1^4(b\gamma^i)) = 1$, then $dh = 5$ or $dh(2^m-1) = -5(2^m+1)$.

Notice that $d > 1$, $5 \nmid d$, $3 \nmid 2^m+1$, $(2^m-1, 2^m+1) = 1$, all of the above equations can not stand.

Above all, the results follow. ■

V. Conclusion

This paper considers the bent and hyper-bent properties of the Boolean functions $f_{a,b}^{(r)}$ of the form $f_{a,b}^{(r)} := \mathrm{Tr}_1^n(ax^{r(2^m-1)}) + \mathrm{Tr}_1^4(bx^{\frac{2^n-1}{5}})$, where $n = 2m$, $m = 2k \pmod 4$, $k \in \{0,1\}$, $a \in \mathbb{F}_{2^n}$ and $b \in \mathbb{F}_{16}$. When $m = 2 \pmod 4$, we give a detailed description of the hyper-bent properties of $f_{a,b}^{(r)}$, and prove that the hyper-bent properties of $f_{a,b}^{(r)}$ can be characterized by that of $f_{a',b'}^{(r)}$, where $a = a'\xi^i \in \mathbb{F}_{2^n}$, $a' \in \mathbb{F}_{2^m}$, b, $b' = b\alpha^{-\frac{i}{r}\frac{2^n-1}{5}} \in \mathbb{F}_{16}$. We also prove that $f_{a,b}^{(r)}$ is not a hyper-bent function unless $n = 12$ or $n = 28$ when $a \in \mathbb{F}_{2^{\frac{m}{2}}}$. Further, we give all the hyper-bent functions for $n = 12$ or $n = 28$. When $m = 0 \pmod 4$, we give a necessary condition for $f_{a,b}^{(r)}$ to be a bent function. To those strict restrictions, it seems $f_{a,b}^{(r)}$ can not be bent. In fact with the help of computer, we have checked all of the functions which satisfy Proposition 13 for $m = 4, 8$, and find that none of them is bent. Thus we guess when $m = 0 \pmod 4$, $f_{a,b}^{(r)}$ can not be bent.

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