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Stabilization of the Bernoulli-Euler Plate Equation: Numerical Analysis

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Abstract—The aim of this paper is to study the internal stabilization of the Bernoulli-Euler equation numerically. For this, we consider a square plate subjected to a feedback/damping force distributed only in a subdomain. An algorithm for obtaining an approximate solution to this problem was proposed and implemented. The numerical method used was the Finite Difference Method. Numerical simulations were performed and showed the behavior of the solution, confirming the theoretical results that have already been proved in the literature. In addition, we studied the validation of the numerical scheme proposed, followed by an analysis of the numerical error; and we conducted a study on the decay of the energy associated.

Keywords—Bernoulli-Euler Plate Equation, Numerical Simulations, Stability, Energy Decay, Finite Difference Method.

I. Introduction

THE aim of this paper is to study the internal stabilization of the Bernoulli-Euler plate equation numerically. So, we consider a square plate subjected to a feedback/damping force distributed only in a subdomain.

The stabilization of this equation has been studied by many researchers and we can mention, for example, the following articles: [1] and [2].

However, constructing finite dimensional systems that are accurate is not a simple task. The approximate systems obtained using the Finite Element Method (FEM) or Finite Difference Method (FDM), in general, are not uniformly stable with respect to the discretization parameter [3]. Some works have been done proposing new ideas to avoid this problem, for example, [4] and [5].

To overcome this adversity, an idea proposed in the literature is to add a numerical viscosity term to the numerical problem [6].

Thus, we proposed and implemented an algorithm, based on the Finite Difference Method, for obtaining an approximate solution to this problem.

II. Problem

First of all, it is necessary to define the domain. Consider the square $\Omega = (0,\pi)\times(0,\pi)$ and let $\mathcal{O}\subset\Omega$ be the rectangle $[a,b]\times[c,d]$, $0<a<b<\pi$ and $0<c<d<\pi$, as shown in Fig 1.

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Denoting the characteristic function of $\mathcal{O}$ by $\chi_{\mathcal{O}}$, we will study the problem modelled by:

$$\begin{cases} \ddot{\omega}(t)+\Delta^2\omega(t)+\chi_{\mathcal{O}}\dot{\omega}(t)=0 \quad , \quad x\in\Omega, \quad t\geq 0, \\ \omega(t)=\Delta\omega(t)=0, \ x\in\partial\Omega, \ t\geq 0, \\ \omega(x,0)=\omega_0(x), \quad \dot{\omega}(x,0)=\omega_1(x), \quad \forall x\in\Omega, \end{cases} \tag{1}$$

where $(\dot{})$ denotes the derivative with respect to time and $\Delta^2\omega$ is the bilaplacian operator. Notice that the damping term $\chi_{\mathcal{O}}\dot{\omega}(t)$ acts only in the subset $\mathcal{O}\subset\Omega$ and the last two relations of (1) are the initial and boundary conditions of the problem.

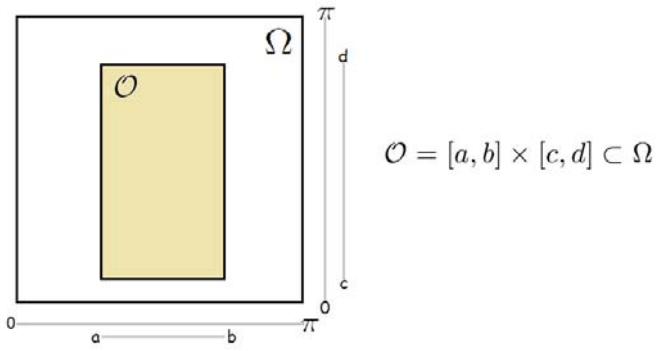


Fig. 1. Problem Domain

The proof of existence and uniqueness of solution can be found in [6]. And, also, it is known that the energy of this system at instant t is given by:

$$E(t)=\frac{1}{2}\left\{\|\dot{\omega}(t)\|^2_{L^2(\Omega)}+\|\Delta\omega(t)\|^2_{L^2(\Omega)}\right\}. \tag{2}$$

III. Numerical Aspects

Since we are interested in studying (1) numerically, as in [6], to discretize the spatial domain, we consider a uniform mesh:

$$h=\frac{\pi}{m+1}, \quad m\in\mathbb{N}.$$

To represent the rectangle $\mathcal{O}$, assume that there exist integers $1\leq a(h),b(h),c(h),d(h)\leq m$ such that:

$$a=a(h)h,\ b=b(h)h,\ c=c(h)h,\ d=d(h)h.$$

We denote by $\omega_{j,k}$ the approximate solution ω of the above system at the point $x_{j,k}=(jh,kh)$, $j,k=0,\ldots,m+1$.

We consider the second order finite difference approximation for the Laplacian, $\forall j,k\in\{1,2,\ldots,m\}$:

$$\Delta\omega(jh,kh)\approx\frac{1}{h^2}\left(\omega_{j+1,k}+\omega_{j-1,k}+\omega_{j,k+1}+\omega_{j,k-1}-4\omega_{j,k}\right).$$

Let $\omega_h \in V_h = \mathbb{R}^{m^2}$ define the vector whose components are $\omega_{j,k}$, $1 \leq j,k \leq m$, i.e., the mesh nodes in which we need to calculate an approximation for the solution.

We define the matrix A_{0h} that represents the discretization of the Bilaplacian operator, with hinged boundary conditions, by its square root $A_{0h}^{\frac{1}{2}}$:

$$\left(A_{0h}^{\frac{1}{2}}\omega_h\right)_{j,k} = \frac{1}{h^2}\left(\omega_{j+1,k} + \omega_{j-1,k} + \omega_{j,k+1} + \omega_{j,k-1} - 4\omega_{j,k}\right),$$

for $1 \leq j,k \leq m$.

Thus, the semi-discrete system that will be studied is given by:

$$\begin{cases} \ddot{\omega}_{j,k} + (A_{0h}\omega_h)_{j,k} + (\chi_{\mathcal{O}}\dot{\omega}_h)_{j,k} + h^2(A_{0h}\dot{\omega}_h)_{j,k} = 0, \\ \qquad 1 \leq j,k \leq m\ , \quad t \geq 0\ , \\ \omega_h(0) = \omega_{0h} \quad , \quad \dot{\omega}_h(0) = \omega_{1h}, \end{cases} \tag{3}$$

where A_{0h} represents the second-order discretization of the bilaplacian operator, considering the boundary conditions given in (1). Notice that the term $h^2(A_{0h}\dot{\omega}_h)$, that is, the numerical viscosity term, has been added to the system (3).

Since the term $\chi_{\mathcal{O}}$ denotes the characteristic function of $\mathcal{O}$, the term $(\chi_{\mathcal{O}}\dot{\omega}_h)_{j,k}$ in (3) corresponds to the vector whose components are:

$$(\chi_{\mathcal{O}}\dot{\omega}_h)_{j,k} = \begin{cases} \dot{\omega}_{j,k},\ a(h) \leq j \leq b(h)\ \textit{and}\ c(h) \leq k \leq d(h), \\ 0, \quad \textit{otherwise}. \end{cases}$$

According to [6], the family of systems defined by (3) is uniformly exponentially stable and it is known that the energy of the semi-discretized system at time t is given by:

$$E_h(t) = \frac{1}{2}\left\{\|\dot{\omega}_h(t)\|^2 + \|A_{0h}^{1/2}\omega_h(t)\|^2\right\}. \tag{4}$$

Let Δt be the time step:

$$t_n = n\Delta t\ ,\ \ n = 0, 1, \ldots, N,$$

i.e., $t_0 = 0 < t_1 = \Delta t < t_2 < \ldots < t_N = T = N\Delta t$.

Using the second order approximations [7]:

$$\ddot{\omega}_{j,k}^n = \frac{\omega_{j,k}^{n+1} - 2\omega_{j,k}^n + \omega_{j,k}^{n-1}}{(\Delta t)^2} + O(\Delta t^2), \tag{5}$$

$$\dot{\omega}_{j,k}^n = \frac{\omega_{j,k}^{n+1} - \omega_{j,k}^{n-1}}{2\Delta t} + O(\Delta t^2), \tag{6}$$

and the Newmark Method with $\theta = \frac{1}{4}$, that is, the mean below, for the terms in which derivates don't appear.

$$\omega_{j,k}^n = \frac{1}{4}\omega_{j,k}^{n+1} + \frac{1}{2}\omega_{j,k}^n + \frac{1}{4}\omega_{j,k}^{n-1}. \tag{7}$$

From (3), (5), (6) and (7) , it comes that:

$$2\omega_{j,k}^{n+1} - 4\omega_{j,k}^n + 2\omega_{j,k}^{n-1} + 2(\Delta t)^2 A_{0h}\left(\frac{1}{4}\omega_{j,k}^{n+1} + \frac{1}{2}\omega_{j,k}^n + \frac{1}{4}\omega_{j,k}^{n-1}\right)$$
$$+ 2(\Delta t)^2(\chi_{\mathcal{O}}\dot{\omega})_{j,k}^n + h^2\Delta t\left(A_{0h}\left(\omega^{n+1} - \omega^{n-1}\right)_{j,k}\right) = 0. \tag{8}$$

So, there are two possibilities:

- If $a(h) \leq j \leq b(h)$ and $c(h) \leq k \leq d(h)$:

$$\left(2 + \frac{(\Delta t)^2}{2}A_{0h} + \Delta t + h^2\Delta t A_{0h}\right)\omega_{j,k}^{n+1} = -(\Delta t)^2 A_{0h}\omega_{j,k}^n$$
$$+ 4\omega_{j,k}^n + \left(-2 - \frac{(\Delta t)^2}{2}A_{0h} + \Delta t + h^2\Delta t A_{0h}\right)\omega_{j,k}^{n-1}. \tag{9}$$

- Otherwise:

$$\left(2 + \frac{(\Delta t)^2}{2}A_{0h} + h^2\Delta t A_{0h}\right)\omega_{j,k}^{n+1} = -(\Delta t)^2 A_{0h}\omega_{j,k}^n$$
$$+ 4\omega_{j,k}^n + \left(-2 - \frac{(\Delta t)^2}{2}A_{0h} + h^2\Delta t A_{0h}\right)\omega_{j,k}^{n-1}. \tag{10}$$

This way, from (9) and (10), we obtain a linear system that can be solved for each discrete time $t_n = n\Delta t$ and thus, it is possible to obtain an approximate solution for the system (1).

To inicialize the method, considering $n = 0$, we used the following approximation:

$$\omega_{j,k}^{-1} \approx \omega_{j,k}^1 - 2\Delta t\dot{\omega}_{j,k}^0, \tag{11}$$

IV. Results and Discussion

After implementing the algorithm mentioned in the previous chapter, we are able to do numerical simulations. So, in this section, we will present and discuss the results of some simulations. It's important to mention that the mesh nodes were numbered from the left to the right, from the bottom to the top.

A. Validation

Let $\omega_e(\bar{x}, t) = \cos(t)\sin(x)\sin(y)$, where $\bar{x} = (x, y)$, i.e., the usual coordinates x and y, be the exact solution of the problem:

$$\begin{cases} \ddot{\omega}(t) + \Delta^2\omega(t) + \chi_{\mathcal{O}}\dot{\omega}(t) = f(\bar{x}, t), \quad \bar{x} \in \Omega, \quad t \geq 0, \\ \omega(t) = \Delta\omega(t) = 0,\ \ \bar{x} \in \partial\Omega,\ \ t \geq 0, \\ \omega(\bar{x}, 0) = \omega_0(\bar{x}) = \sin x \sin y,\ \ \forall \bar{x} \in \Omega, \\ \dot{\omega}(\bar{x}, 0) = \omega_1(\bar{x}) = 0,\ \ \forall \bar{x} \in \Omega, \end{cases} \tag{12}$$

where $f = f(\bar{x}, t)$ is obtained substituting $\omega_e(\bar{x}, t)$ in (3) and using the ideas presented in (7), that is,

$$f(t_n) = \frac{1}{4}f^{n+1} + \frac{1}{2}f^n + \frac{1}{4}f^{n-1}.$$

Since the exact solution of (12) is known, it is possible to compare the analytical and the numerical solutions. For these comparisons, we used the norm $\|E\|_{L^\infty(0,T;L^2(\Omega))}$.

Now, we will present the results obtained: first, we fixed the area $\mathcal{O}$ where the damping acts, and then, in the next subsection, we changed its size and position.

1) Simulations considering $\mathcal{O} = \left[\frac{\pi}{4}, \frac{\pi}{2}\right] \times \left[\frac{\pi}{4}, \frac{3}{4}\pi\right]$.

We defined: $h = \frac{\pi}{2^{i+2}},\ \ i = 0, 1, 2, 3, 4.$

So, considering $T = 1$, $\mathcal{O} = \left[\frac{\pi}{4}, \frac{\pi}{2}\right] \times \left[\frac{\pi}{4}, \frac{3}{4}\pi\right]$, the results of some simulations can be seen in Table I and Table II below.

To illustrate, considering the simulation on the penultimate row of Table II, Fig. 2 shows a comparison between the exact and approximate solutions at the point $x_{1461} = \left(\frac{3}{16}\pi, \frac{3}{8}\pi\right)$, inside $\mathcal{O}$, and at the point $x_{1469} = \left(\frac{5}{31}\pi, \frac{3}{8}\pi\right)$, outside the area $\mathcal{O}$. It can be noticed that one solution is near the other.

From the simulations made, we can conclude that the numerical solution is being calculated correctly.

TABLE I
VALIDATION: ERROR TABLE CONSIDERING $\Delta t = 0.1$

i	h_i	Δt	$\|E\|_{L^\infty(0,1;L^2(\Omega))}$
0	0.7853982	0.1	0.0775903
1	0.3926991	0.1	0.0563078
2	0.1963495	0.1	0.0256471
3	0.0981748	0.1	0.0115063
4	0.0490874	0.1	0.0068410

TABLE II
VALIDATION: ERROR TABLE CONSIDERING $\Delta t = 0.01$

i	h_i	Δt	$\|E\|_{L^\infty(0,1;L^2(\Omega))}$
0	0.7853982	0.01	0.0758241
1	0.3926991	0.01	0.0550678
2	0.1963495	0.01	0.0239501
3	0.0981748	0.01	0.0090214
4	0.0490874	0.01	0.0032785

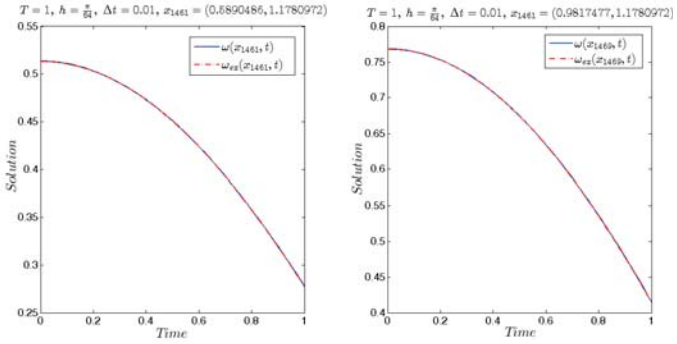


Fig. 2. Simulation Results: Validation

2) Simulations considering different rectangles $\mathcal{O}$.

The aim of this subsection is to verify what happens to the approximate solution when the position of the subdomain $\mathcal{O}$ changes.

For this, we fixed $h = \frac{\pi}{32}$ and $\Delta t = 0.01$. In each numerical simulation, the subdomian $\mathcal{O}$ has the same size, but different positions. The results can be seen in Table III.

TABLE III
VALIDATION: ERROR TABLES CONSIDERING $h = \frac{\pi}{32}$, $\Delta t = 0.01$ AND DIFFERENT SUBDOMAINS $\mathcal{O}$ THAT HAVE THE SAME SIZE, BUT DIFFERENT POSITIONS

h	Δt	$\mathcal{O}$	$\|E\|_{L^\infty(0,1;L^2(\Omega))}$
$\frac{\pi}{32}$	0.01	$\left[\frac{\pi}{4}, \frac{\pi}{2}\right] \times \left[\frac{\pi}{4}, \frac{3}{4}\pi\right]$	0.0090214
$\frac{\pi}{32}$	0.01	$\left[\frac{\pi}{32}, \frac{9}{32}\pi\right] \times \left[\frac{\pi}{4}, \frac{3}{4}\pi\right]$	0.0097619
$\frac{\pi}{32}$	0.01	$\left[\frac{\pi}{32}, \frac{9}{32}\pi\right] \times \left[\frac{15}{32}\pi, \frac{31}{32}\pi\right]$	0.0098695
$\frac{\pi}{32}$	0.01	$\left[\frac{\pi}{32}, \frac{9}{32}\pi\right] \times \left[\frac{\pi}{32}, \frac{17}{32}\pi\right]$	0.0098695
$\frac{\pi}{32}$	0.01	$\left[\frac{23}{32}\pi, \frac{31}{32}\pi\right] \times \left[\frac{15}{32}\pi, \frac{31}{32}\pi\right]$	0.0098695
$\frac{\pi}{32}$	0.01	$\left[\frac{\pi}{32}, \frac{9}{32}\pi\right] \times \left[\frac{15}{32}\pi, \frac{31}{32}\pi\right]$	0.0098695
$\frac{\pi}{32}$	0.01	$\left[\frac{5}{32}\pi, \frac{13}{32}\pi\right] \times \left[\frac{11}{32}\pi, \frac{27}{32}\pi\right]$	0.0093700
$\frac{\pi}{32}$	0.01	$\left[\frac{15}{32}\pi, \frac{23}{32}\pi\right] \times \left[\frac{4}{32}\pi, \frac{20}{32}\pi\right]$	0.0090706

Furthermore, varying the position of $\mathcal{O}$, the error does not change significantly and the numerical solution is also obtained correctly.

3) Validation: Conclusion

Notice that analyzing the simulation results, we can conclude that the numerical solution is close to the exact one. In other words, the numerical solution is being calculated correctly. Then, considering $f(\bar{x}, t) = 0$ on the right side of (12), we return to the original problem (1).

B. Decay of Energy

Using the same initial conditions $\omega_0(\bar{x})$ and $\omega_1(\bar{x})$ previously used, the problem that will be studied is given by:

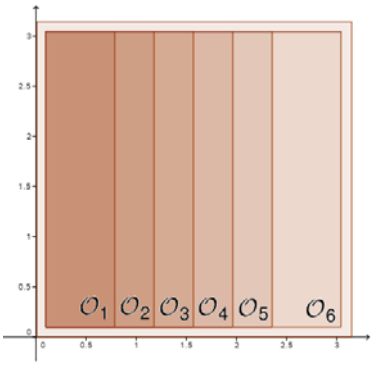
$$\begin{cases} \ddot{\omega}(t) + \Delta^2\omega(t) + \chi_{\mathcal{O}}\dot{\omega}(t) = 0, \quad \bar{x} \in \Omega, \quad t \geq 0, \\ \omega(t) = \Delta\omega(t) = 0, \ \bar{x} \in \partial\Omega, \ t \geq 0, \\ \omega(\bar{x}, 0) = \omega_0(\bar{x}) = \sin x \sin y, \ \forall \bar{x} \in \Omega, \\ \dot{\omega}(\bar{x}, 0) = \omega_1(\bar{x}) = 0, \ \forall \bar{x} \in \Omega. \end{cases} \tag{13}$$

Our goal from now on is to verify the energy decay that has already been proved in the literature. We know that the energy of the semi-discretized system is given by (4). For simplicity, using the Composite Trapezoidal Rule and second-order approximations for the derivates, we obtained a formula to calculate numerically the energy associated with the problem at each discrete time.

Then, in the following subsections, the results of some tests will be presented.

1) Increasing Intervals

In this simulation, we set $T = 10$, $h = \frac{\pi}{32}$, $\Delta t = \frac{10}{100} = 0.1$ and tests varying only $\mathcal{O}$ were performed. These differents subdomains can be seen in Fig. 3.



$\mathcal{O}_1 = \left[\frac{\pi}{32}, \frac{\pi}{4}\right] \times \left[\frac{\pi}{32}, \frac{31}{32}\pi\right]$
$\mathcal{O}_2 = \left[\frac{\pi}{32}, \frac{3}{8}\pi\right] \times \left[\frac{\pi}{32}, \frac{31}{32}\pi\right]$
$\mathcal{O}_3 = \left[\frac{\pi}{32}, \frac{\pi}{2}\right] \times \left[\frac{\pi}{32}, \frac{31}{32}\pi\right]$
$\mathcal{O}_4 = \left[\frac{\pi}{32}, \frac{5}{8}\pi\right] \times \left[\frac{\pi}{32}, \frac{31}{32}\pi\right]$
$\mathcal{O}_5 = \left[\frac{\pi}{32}, \frac{3}{4}\pi\right] \times \left[\frac{\pi}{32}, \frac{31}{32}\pi\right]$
$\mathcal{O}_6 = \left[\frac{\pi}{32}, \frac{31}{32}\pi\right] \times \left[\frac{\pi}{32}, \frac{31}{32}\pi\right]$

Fig. 3. Increasing Intervals: Subdomains $\mathcal{O}$

The results of energy study for each of these regions can be seen in Fig. 4. Note that the energy decreases in all cases. Furthermore, the bigger the area $\mathcal{O}$ is, the faster the energy decreases.

2) Nested Intervals

In the next simulation, an increasing sequence of nested subdomains was constructed and our aim is to try to understand the influence of the *damping* acting in it.

We set $T = 8$, $h = \frac{\pi}{32}$ and $\Delta t = \frac{8}{100} = 0.08$. Also, we have, for $i = 1, 2, 3, 4, 5$:

$$\mathcal{O}_i = \left[\frac{16}{32}\pi - 3ih, \frac{16}{32}\pi + 3ih\right] \times \left[\frac{16}{32}\pi - 3ih, \frac{16}{32}\pi + 3ih\right].$$

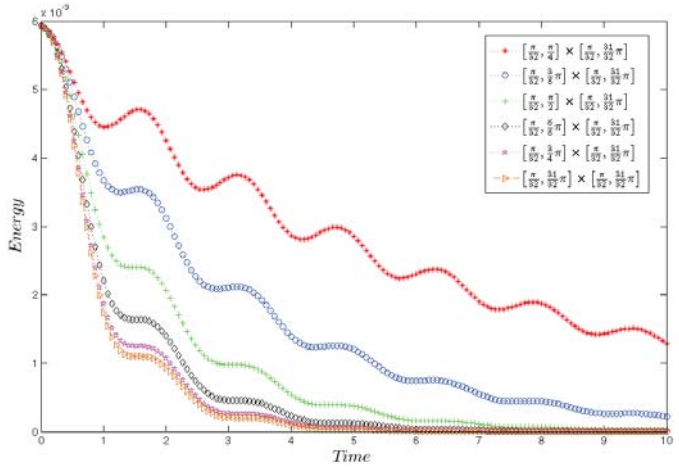


Fig. 4. Increasing Intervals: Energy Decay

From Fig. 5, we can conclude the energy also decays in each of the simulations. Furthermore, again, the larger the region $\mathcal{O}$, the energy decreases faster.

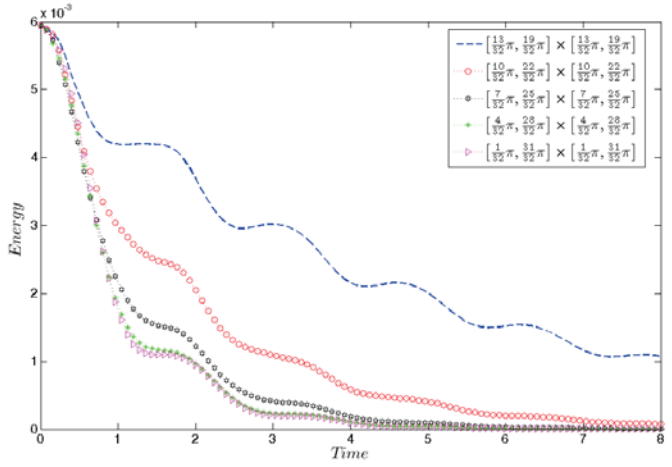


Fig. 5. Nested Intervals: Energy Decay

3) Influence of the Numerical Viscosity Term

In this example, we will investigate the necessity of adding the numerical viscosity term. For this, some simulations, whose graphs can be seen in Fig. 6 and Fig. 7, were performed. Note that different subdomains $\mathcal{O}$ were considered and there was energy decay in both cases, whether considering the numerical viscosity or not. Moreover, with respect to energy, we noticed little difference when added this term.

As we can see in Fig. 8, in the absence of the region of *damping*, considering the scheme with the numerical viscosity term, the energy decays slowly. However, when we remove that term, there was no energy dissipation. That is, in the case without the subdomain $\mathcal{O}$ (area where the *damping* acts), the term numerical viscosity is essential to have energy decay.

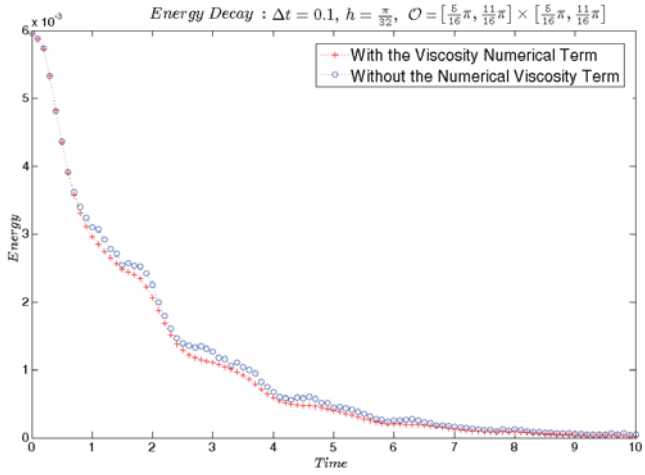


Fig. 6. Energy Decay: $\Delta t = 0.1$, $h = \frac{\pi}{32}$, $\mathcal{O} = \left[\frac{5}{16}\pi, \frac{11}{16}\pi\right] \times \left[\frac{5}{16}\pi, \frac{11}{16}\pi\right]$.

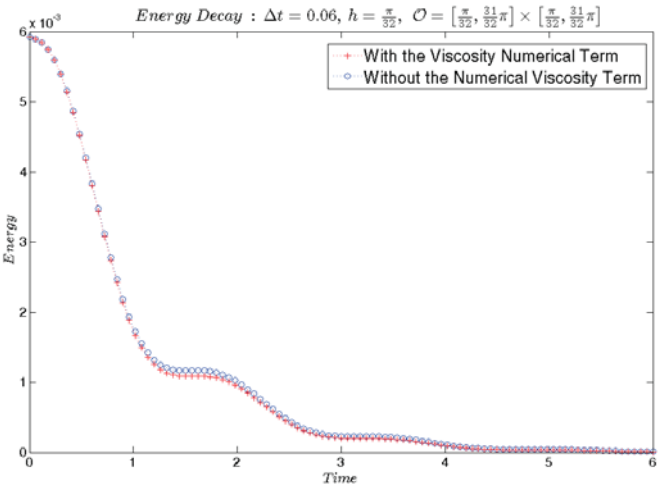


Fig. 7. Energy Decay: $\Delta t = 0.06$, $h = \frac{\pi}{32}$, $\mathcal{O} = \left[\frac{\pi}{32}, \frac{31}{32}\pi\right] \times \left[\frac{\pi}{32}, \frac{31}{32}\pi\right]$.

V. Conclusions

In this paper, we studied the stabilization of Bernoulli-Euler plate equation numerically. So, we proposed and implemented an algorithm, based on the Finite Difference Method, for obtaining an approximate solution to this problem. Then, numerical simulations, which showed the behavior of the solution, were performed.

Moreover, we conducted a study on the decay of the energy associated with the problem, confirming the theoretical results that have already been proved in the literature. Furthermore, we realised that this decay is related to the region $\mathcal{O}$: the larger the area where the damping acts is, the faster the energy decreases. Analyzing the results of the simulations performed in the presence of *damping*, considering or not the viscosity term did not change significantly the energy decay. However, the same does not happen in the absence of a region $\mathcal{O}$.

According to the simulations, we can conclude that either the damping term or the numerical viscosity is essential to

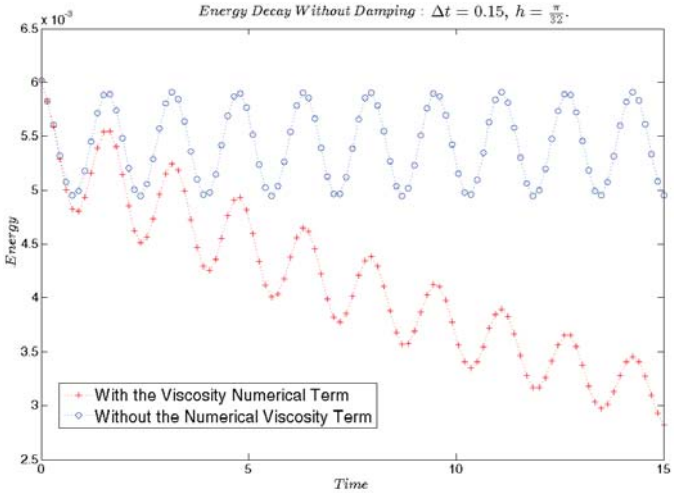


Fig. 8. Energy Decay Without Damping: $\Delta t = 0.15,\ h = \frac{\pi}{32}$.

have energy decay. However, the action of both terms isn't required for this decay.

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