Optimal Control of Volterra Integro-Differential Systems Based On Legendre Wavelets and Collocation Method

Khosrow Maleknejad, Asyieh Ebrahimzadeh

Abstract—In this paper, the numerical solution of optimal control problem (OCP) for systems governed by Volterra integro-differential (VID) equation is considered. The method is developed by means of the Legendre wavelet approximation and collocation method. The properties of Legendre wavelet together with Gaussian integration method are utilized to reduce the problem to the solution of nonlinear programming one. Some numerical examples are given to confirm the accuracy and ease of implementation of the method.

Keywords—Collocation method, Legendre wavelet, optimal control, Volterra integro-differential equation.

I. Introduction

In the present paper, a collocation approach based on Legendre wavelets is utilized for numerical approximation of optimal control u^* and corresponding optimal state x^* that minimizes the quadratic performance index

$$J = \int_0^T \Big(x^2(t) + u^2(t) + f(t)x(t) + g(t)u(t)\Big)dt, \qquad (1)$$

subject to dynamic system

$$x'(t) = a(t)x(t) + b(t)u(t) + \int_0^t \Big(K(t,s)\varphi(x(s))\Big)ds, \quad (2)$$

where $x(t)$ and $u(t)$ are real valued functions belong to $L^2[0,T]$. In (2), φ can be a nonlinear or linear operator. A wide class of control systems can be described by Volterra integral (VI) or VID equations. These Optimal control problems can be used to model many classes of phenomena, such as population dynamics, continuum mechanics of materials with memory, economic problem, the spread of epidemics, non-local problems of diffusion and heat conduction problem [1]. The problem of optimal control for systems governed by VI or VID systems has been studied by many authors [1]- [15]. In the rest of the paper and without loss of generality, it is assumed that $T = 1$.

The approximate optimal solution of both control and state functions together with the value of performance index in this paper is obtained by applying a direct method. The direct approach transforms the control problem after discretization to an optimization problem [16]. The nonlinear optimization problem can be solved by means of optimization algorithms such as sequential quadratic programming or gradient methods [17]. The advantage of direct methods over the indirect methods, which is based upon solving the necessary conditions derived from Pontryagin' s minimum principle, is their broader radius of convergence to an optimal solution. In addition, since the necessary conditions do not have to be derived, the direct methods can quickly be utilized to solve a number of practical trajectory optimization [16].

In recent years, Wavelets lead to a huge number of applications in numerical approximations. A survey of some of their usages in various sciences can be found in [18]. Legendre wavelet has been used by many researchers because of its good accuracy in approximations. An excellent survey on applications of Legendre wavelets on solving different problems can be found in [19]- [26].

In recent decades, numerical schemes which are based on operational matrix of integration have been widely utilized for solving different equations. The idea of these methods is based upon the integral expression

$$\int_0^t \psi(\tau)d\tau \approx P\psi(x) \qquad (3)$$

where ψ is an arbitrary basis vector and P is operational matrix of integration. The operational matrix of integration can be uniquely determined on the use of particular basis functions. In other words,

$$\psi'(x) = B\psi(x), \qquad (4)$$

where B is the operational matrix of derivative for any selected basis. The advantage of using this matrix is that, in the matrix relation, there is not any approximation symbol; meanwhile, in the integration form (3), the approximation symbol could be seen obviously [26]. In [26], Mohammadi and Hosseini introduced an operational matrix of derivative for Legendre wavelets for solving the singular ordinary differential equations. Several papers have appeared in the literature concerned with the applications of operational matrix of derivative [27], [28].

In the present paper, we introduce a new direct computational method to solve the optimal control of nonlinear VID system with quadratic performance index. Our method consists of reducing the OCP to a nonlinear programming (NLP) by first expanding the state rate $x'(t)$, state and control functions as Legendre wavelet functions with unknown coefficients. Many well developed optimization algorithms can be used to solve the resulted NLP **[17]**. The operational matrix of derivative D, integration P and the

K. Maleknejad is with school of Mathematics, Iran University of Science and Technology, Tehran, 16844, Iran, (e-mail:maleknejad@iust.ac.ir)(http://webpages.iust.ac.ir/maleknejad/).

A. Ebrahimzadeh is with school of Mathematics, Iran University of Science and Technology.(e-mail:ebrahimzadeh@iust.ac.ir).

integration of the cross product of two Legendre wavelet functions are also given.

The outline of this paper is as follows: In Section II, we will review the basic properties of Legendre wavelet which has been used in our approximations. In Section III, the proposed method is used to approximate the solution of the optimal control problem. As a result, a NLP problem is obtained . At the end of this section, the convergence analysis of Legendre wavelets is stated. For confirming the effectiveness of the presented method, several illustrative examples are provided in Section IV. Section V ends this paper with a brief conclusion.

II. Wavelets and their properties

Recently, wavelets have found their way into many different fields of science and engineering. Wavelets constitute a family of functions constructed from dilation and translation of a single function called mother wavelet. When the dilation parameter a and the translation parameter b vary continuously, we have the following family of continuous wavelets

$$\psi_{a,b}(t) = |a|^{-\frac{1}{2}}\psi\Big(\frac{t-b}{a}\Big), \quad a, b \in R, a \neq 0.$$

where, $\psi_{a,b}$ forms a wavelet basis for $L^2(R)$. If we consider the parameters a and b as discrete values $a = a_0^{-k}$, $b = nb_0a_0^{-k}$, where $a_0 > 0$, $b_0 > 0$ and n and k are positive integers, then we have the following discrete wavelets:

$$\psi_{n,k}(t) = |a_0|^{\frac{k}{2}}\psi(a_0^k t - nb_0).$$

In particular, when $a_0 = 2$ and $b_0 = 1$ then ψ_{nk} form an orthonormal basis [18].

A. Legendre Wavelet

For any positive integer k, the Legendre wavelets are defined as follows [25]:

$$\phi_{nm}(t) = \begin{cases} \sqrt{m+\frac{1}{2}}\,2^{\frac{k}{2}}L_m(2^k t - 2n + 1), & t \in [\frac{2n-2}{2^k}, \frac{2n}{2^k}), \\ 0, & otherwise, \end{cases}$$

where $n = 1, \ldots, 2^{k-1}$ and $m = 0, \ldots, M-1$. Here, $L_m(t)$ are the Legendre polynomials of order m. For $m = 0, 1, 2, \cdots$, Legendre polynomials can be calculated by using the following relation [25]:

$$L_0(t) = 1, \quad L_1(t) = t,$$

$$L_{m+1}(t) = \Big(\frac{2m+1}{m+1}\Big)tL_m(t) - \Big(\frac{m}{m+1}\Big)L_{m-1}(t).$$

A function $f(t)$ defined over $[0, 1)$ may be expanded in terms of Legendre wavelets as

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}\phi_{nm}(t), \tag{5}$$

where $c_{nm} = \langle f(t), \phi_{nm}(t)\rangle$, in which $\langle\cdot,\cdot\rangle$ denotes the inner product. If the infinite series in (5) is truncated, then it can be written as

$$f(t) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{nm}\phi_{nm}(t) = C^T\phi(t), \tag{6}$$

where C and $\phi(t)$ are $2^{k-1}M \times 1$ vectors given by

$$C = [C_{10}, C_{11}, \ldots, C_{1M-1}, \ldots, C_{2^{k-1}0}, \ldots, C_{2^{k-1}M-1}]^T,$$

$$\phi = [\phi_{10}, \phi_{11}, \ldots, \phi_{1M-1}, \ldots \phi_{2^k-10}, \ldots, \phi_{2^{k-1}M-1}]^T. \tag{7}$$

B. Operational Matrices of Legendre Wavelet

In the following section, we introduce the operational matrix of derivative and integration for Legendre wavelet, and state the operational matrix of product for Legendre wavelets.

Theorem 1 [26]: Let $\phi(t)$be the Legendre wavelet vector defined in (7). Then, the first derivative of the vector $\phi(t)$can be expressed as:

$$\frac{d\phi}{dt} = D\phi(t),$$

where D is $2^{k-1}M$ operational matrix of derivative and defined as follows:

$$D = \begin{pmatrix} F & 0 & \ldots & 0 \\ 0 & F & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & F \end{pmatrix}, \tag{8}$$

in which F is $M \times M$ square matrix with entries

$$F_{rs} = \begin{cases} 2^k\sqrt{(2r-1)(2s-1)}, & r = 2, \ldots, M, s = 1, \ldots, r-1, \\ & and (r+s) is odd, \\ 0, & otherwise. \end{cases} \tag{9}$$

The integration of vector ϕ defined in (7) can be obtained as:

$$\int_0^t \phi(t')dt' = P\phi(t),$$

where P is the $2^{k-1}M \times 2^{k-1}M$ operational matrix for integration and is given in [20] as:

$$P = \begin{pmatrix} U & F & \cdots & F & F \\ 0 & U & \cdots & F & F \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & U & F \\ 0 & 0 & \cdots & 0 & U \end{pmatrix}, \tag{10}$$

In (10), U and F are $M \times M$ matrices given by:

$$U = \frac{1}{2^k}$$

$$\begin{pmatrix} 1 & \frac{1}{\sqrt{3}} & 0 & \cdots & 0 & 0 \\ \frac{-\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3\sqrt{5}} & \cdots & 0 & 0 \\ 0 & -\frac{\sqrt{5}}{5\sqrt{3}} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \frac{\sqrt{2M-3}}{(2M-3)\sqrt{2M-1}} \\ 0 & 0 & 0 & \cdots & \frac{-\sqrt{2M-1}}{(2M-1)\sqrt{2M-3}} & 0 \end{pmatrix}.$$

and

$$F = \frac{1}{2^k}\begin{pmatrix} 2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

The integration of the product of two Legendre function vectors is obtained as

$$I = \int_0^1 \left(\phi(t)\phi^T(t)\right)dt \tag{11}$$

where I is an identity matrix.

C. Convergence of Legendre Wavelet

In the following theorem, the error bound of approximation with Legendre wavelet for a function $f \in L^2[0,1]$, a nonnegative integer k, $m = 0,1,\cdots$, $n = 0,1,\cdots,2^{k-1}$ is given.

Theorem 2 [29]: Suppose that the function $f : [0,1] \longrightarrow R$ is m times continuously differentiable, $f \in C^m[0,1]$. Then $C^T\phi$ approximate f with mean error bounded as follows:

$$\|f - C^T\phi\| \leq \frac{1}{m!2^{mk}} \sup_{x\in[0,1]} |f^m(x)|.$$

proof: see **[29]**.

III. Legendre wavelet collocation method

In this section, we use Legendre wavelet for discretization of considering OCP. Let N be the number of basis functions. The nodal point arrangement for the Legendre wavelet collocation method (LWCM) is given as:

$$t_i = \frac{2i-1}{2^kM} \qquad i = 1,\ldots,2^{k-1}M. \tag{12}$$

Solving the considered OCP consists of two stages: the discretization of quadrature performance index (1) and the controlled VID system (2). In our scheme, we use the matrix of product integration to approximate the quadrature performance index (1). In the discretization of the controlled VIE, we utilize both of the quadrature rule and the Legendre wavelet approximation of control, state and state rate functions together with collocating the system over the given nodes in (12).

A. The System Dynamic Approximation

For discretization of the integro-differential dynamic system (2), we suppose

$$u(t) \simeq U^T\phi(t), \quad x(t) \simeq X^T\phi(t), \quad x'(t) \simeq X^TD\phi(t). \tag{13}$$

where X, U and $\phi(t)$ are introduced in (7). By substituting (13) in dynamic system (2), we gain

$$X^TD\phi(t) - a(t)X^T\phi(t) - b(t)U^T\phi(t) - \int_0^t \left(k(t,s)\varphi(X^T\phi(s))\right)ds = 0, \quad X^T\phi(0) = X_0. \tag{14}$$

By collocating (14) in points (12), we gain

$$X^TD\phi(t_i) - a(t_i)X^T\phi(t_i) - b(t_i)U^T\phi(t_i) - \int_0^{t_i} \left(k(t_i,s)\varphi(X^T\phi(s))\right)ds = 0, \quad X^T\phi(0) = X_0. \tag{15}$$

By using transformation $s = \frac{t_i}{2}(\tau+1)$, (15) is converted to

$$X^TD\phi(t_i) - a(t_i)X^T\phi(t_i) - b(t_i)U^T\phi(t_i) - \int_{-1}^1 \left(k(t_i,\frac{t_i}{2}(\tau+1))\varphi(X^T\phi(\frac{t_i}{2}(\tau+1)))\right)d\tau = 0,$$

$$X^T\phi(0) = X_0. \tag{16}$$

By utilizing Gauss-Legendre (GL) quadrature formula, we obtain

$$X^TD\phi(t_i) - a(t_i)X^T\phi(t_i) - b(t_i)U^T\phi(t_i) - \frac{t_i}{2}\sum_{j=0}^{N} \left(w_j\left(k(t_i,\frac{t_i}{2}(\tau_j+1))\varphi(X^T\phi(\frac{t_i}{2}(\tau_j+1)))\right)\right) = 0, \quad X^T\phi(0) = X_0. \tag{17}$$

where τ_js are GL nodes, zeros of Legendre polynomials $L_M(t)$ in the interval $[-1,1]$ and w_js are the corresponding weights. While explicit formulas for quadrature nodes are not known, the weights can be expressed in closed form by the following relation [1]:

$$w_j = \frac{2}{(1-\tau_j^2)(L'_M(\tau_j))^2}.$$

Finally, the controlled VID (2) is reduced to $2^{k-1}M$ nonlinear algebraic equations given in (17).

B. The Performance Index Approximation

For discritization of the performance index stated in (1), we use the following approximation methodology. Firstly, the real valued functions $f(t)$ and $g(t)$ are approximated

$$f(t) = F^T\phi(t), \qquad g(t) = G^T\phi(t). \tag{18}$$

where $F = [f_{10},\ldots,f_{2^{k-1}M}]$, $G = [g_{10},\ldots,g_{2^{k-1}M}]$ and ϕ is defined in (7).
By substituting (18) in (1), we get

$$J = \int_0^1 \Big(X^T\phi(t)\phi^T(t)X + U^T\phi(t)\phi^T(t)U + F^T\phi(t)\phi^T(t)X + G^T\phi(t)\phi^T(t)U\Big)dt. \tag{19}$$

We obtain from (11)

$$J(X,U) = X^TX + U^TU + F^TX + G^TU. \tag{20}$$

If f and g be constant functions, so (1) is converted to

$$J(X,U) = X^TX + U^TU + fPX + gPU, \tag{21}$$

where P is given in (10).

IV. Numerical experiment

In this section, we examine the accuracy of the new methods on several examples. These problems are considered in order to demonstrate the efficiency and accuracy of our method. In order to analysis the error of the method, the following notations are introduced:

$$\|E_x(t)\|_\infty = \max_{1\leq i\leq 2^{k-1}M} |E_x(t_i)|,$$

$$\|E_u(t)\|_\infty = \max_{1\leq i\leq 2^{k-1}M} |E_u(t_i)|. \tag{22}$$

where $E_x(t) = x^*(t) - X^T\phi(t)$, $E_u(t) = u^*(t) - U^T\phi(t)$ and t_i are given in (12). We applied the method presented in the previous section on following three examples and obtained the results for $k = 2$ and different values of M.

TABLE I
NUMERICAL RESULTS OF EXAMPLE 1

M	E_x	E_u	J^*
2	1.4202E-01	3.7357E-01	1.0511E-02
4	1.6433E-04	1.1330E-02	1.9821E-07
6	6.0008E-06	7.4246E-05	1.1673E-11
8	2.1710E-07	2.5093E-07	2.7293E-12

TABLE II
NUMERICAL RESULTS OF EXAMPLE 2

M	E_x	E_u	J^*
2	8.2349E-02	7.4267E-02	9.3354E-03
4	2.9025E-02	2.5947E-03	5.1861E-04
6	8.0233E-05	4.5393E-04	7.4667E-08
8	1.4878E-07	4.8128E-06	4.5791E-12

A. Example 1

Consider the minimization of functional

$$J = \int_0^1 \Big((x(t) - e^t)^2 + (u(t) - e^{3t})^2 \Big) dt, \qquad (23)$$

subject to nonlinear VID system

$$x'(t) - \frac{3}{2}x(t) + \frac{1}{2}u(t) - \int_0^t \Big(e^{t-s}x^3(s) \Big) ds = 0, \quad x(0) = 1. \ (24)$$

The exact optimal control and state are $x^*(t) = e^t$ and $u^*(t) = e^{3t}$. Trivially, the optimal value of cost functional is $J^* = 0$. The results of solving this example with LWCM for $k = 2$ and different values of M are given in Table I.

B. Example 2

Find the optimal control u^* and corresponding optimal state x^* that minimizes the quadratic performance index

$$J = \int_0^1 \Big((x(t) - e^{t^2})^2 + (u(t) - (1+2t))^2 \Big) dt, \qquad (25)$$

subject to VID system

$$x'(t) + x(t) - u(t) - \int_0^t \Big(t(1+2t)e^{s(t-s)}x(s) \Big) ds = 0. \qquad (26)$$

The optimal control u^* and corresponding optimal state x^* are respectively $1 + 2t$ and e^{t^2}. Table II presents the results of LWCM for $k = 2$ and various values of M.

C. Example 3

Consider the minimization of functional

$$J = \int_0^1 \Big((x(t) - t)^2 + (u(t) - (1 - te^{t^2}))^2 \Big) dt, \qquad (27)$$

subject to dynamic state

$$x'(t) - x(t) - u(t) + 2\int_0^t \Big(tse^{-x^2(s)} \Big) ds = 0. \qquad (28)$$

The optimal value of cost functional is $J^* = 0$. The optimal control $u^*(t)$ and corresponding optimal state $x^*(t)$ are as follows:

$$\begin{cases} x^*(t) = t, \\ u^*(t) = 1 - te^{-t^2}, \end{cases} \qquad (29)$$

Table III exhibits the results of example 3 with LWCM.

TABLE III
NUMERICAL RESULTS OF EXAMPLE 3

M	E_x	E_u	J^*
2	7.4716E-04	4.5276E-03	3.9430E-06
4	1.9633E-06	1.7998E-04	5.9811E-10
6	5.5157E-08	1.8581E-06	2.4736E-14
8	3.1731E-09	8.4100E-09	8.6735E-17

V. CONCLUSION

A collocation Legendre wavelet-based method was developed to obtain the approximate optimal control and state of controlled VID system with quadratic performance index. The proposed approach is based on converting the OCP into a finite dimensional mathematical programming problem. Many effective algorithms can be applied to solve the NLP [17]. Since, the integration of the product of two Legendre wavelet function vectors is an identity matrix and the operational matrix of derivative is rather sparse, so Legendre wavelet method is easy to implement and computationally very attractive. The method is in the case of optimal control of systems governed by VID equation which is applicable in the field of practical science and engineering for systems with memory effect [1], [9].

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Khosrow Maleknejad: Professor Khosrow Maleknejad obtained his MSc degree in Applied Mathematics from Tehran University, Iran, in 1972 and received his PhD degree in Applied Mathematics in Numerical Analysis area from the University of Wales, Aberystwyth, UK in 1980. In September 1972, he joined the School of Mathematics at Iran University of Science and Technology(IUST), Narmak, Tehran,Iran. Where he was previously lecturer , assistant Professor and associate professor at IUST. He has been a professor since 2002 at IUST. He was a Visiting Professor at the University of California at Los Angeles in 1990. His research interests include numerical analysis in solving ill-posed problems and solving Fredholm and Volterra integral equations. He has authored as a Editor-in-chief of the International Journal of Mathematical Sciences, which publishers by Springer.He is a member of the AMS. External link: http://webpages.iust.ac.ir/maleknejad/.

Asyieh Ebrahimzadeh: Asyieh Ebrahimzadeh is a Ph.D student in Department of Applied Mathematics at Iran university of Science and Technology, Tehran, Iran. She obtained M.Sc in Mathematics in 2009 at Khaje Nasir Toosi University of Technology, Tehran, Iran. She also obtained Bs.C in Mathematics at ValiAsr University of Rafsanjan, Kerman, Iran. Her scientific interests includes: Optimal control, Wavelets, Integral Equations and Orthogonal functions.