Backstepping Design and Fractional Derivative Equation of Chaotic System

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***Abstract*—**In this paper, Backstepping method is proposed to synchronize two fractional-order systems. The simulation results show that this method can effectively synchronize two chaotic systems.

***Keywords*—**Backstepping method, Fractional order, Synchronization.

I. Introduction

FRACTIONAL calculus is very interesting area, which has been originated from 17th century. For demonstrate the chaotic behavior some fractional-order differential systems such as Chua circuit [1], Duffing system [2], jerk model [3], Chen system [4], the Fractional-order Lü system [5], Rossler system [6], Arneodo system [7] and Newton–Leipnik system [8] have been found. Chaotic systems are difficult to be synchronized. Control of chaotic systems has been considered as an important and challenging problem because of sensitivity of initial conditions. Different control technique as in [9] (FSMC) strategy for synchronization of chaotic systems has been proposed. In [10] an active sliding mode controller has also been presented to synchronize two chaotic systems with parametric uncertainty. An algorithm is designed to determine parameters of active sliding mode controller in synchronizing different chaotic systems have been studied in [11]. A systems with uncertainties of an adaptive sliding mode controller has also been studied in [12].Over the past decade, Backstepping has become the most popular design method for adaptive nonlinear control because it can guarantee global stabilities, tracking, and transient performance for a broad class of strict feedback systems. It has been shown that many chaotic systems as chaos, including Duffing oscillator, van der pol oscillator, Rossler system, Lorenz system, Lü system, Chen system and Chua's circuit, can be transformed into non-autonomous form has been studied in [13]-[15] and the backstepping control schemes have been employed to control these chaotic systems with key parameters unknown. In particular, the output of the controlled chaotic system has been designed to asymptotically track any smooth and bounded reference signals generated from a known reference model which may be a chaotic system. In this paper Backstepping method is applied to synchronize two fractional-order chaotic systems.

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II. Backstepping Method for Fractional System

Consider two dynamical systems. The master system and slave system as:

$$\dot{x}_1 = a(x_2 - x_1)$$

$$\dot{x}_2 = (\Upsilon - a)x_1 - x_1x_3 + \Upsilon x_2$$

$$\dot{x}_3 = -\beta x_3 - \delta x_4 + x_1x_2$$

$$\dot{x}_4 = -dx_4 + fx_3 + x_1x_2 \tag{1}$$

$$\dot{y}_1 = a(y_2 - y_1)$$

$$\dot{y}_2 = by_1 - cy_2 - y_1y_3$$

$$\dot{y}_3 = {y_1}^2 - dy_3$$

$$\dot{y}_4 = -y_1y_3 - \delta y_4 \tag{2}$$

The fractional order derivatives of the systems (1) and (2) are

$$D_*^{\propto} x_1 = a(x_2 - x_1)$$

$$D_*^{\propto} x_2 = (\Upsilon - a)x_1 - x_1x_3 + \Upsilon x_2$$

$$D_*^{\propto} x_3 = -\beta x_3 - \delta x_4 + x_1x_2$$

$$D_*^{\propto} x_4 = -dx_4 + fx_3 + x_1x_2 \tag{3}$$

$$D_*^{\propto} y_1 = a(y_2 - y_1) + u_1$$

$$D_*^{\propto} y_2 = by_1 - cy_2 - y_1y_3 + u_2$$

$$D_*^{\propto} y_3 = {y_1}^2 - dy_3 + u_3$$

$$D_*^{\propto} y_4 = -y_1y_3 - \delta y_4 + u_4 \tag{4}$$

The error dynamics are

$$D_*^{\propto} e_1 = a(e_2 - e_1) + u_1$$

$$D_*^{\propto} e_2 = be_1 + x_1(b - \gamma + a) - ce_2 - x_2(c + \gamma) - e_3(e_1 + x_1) - e_1x_3 + u_2$$

$$D_*^{\propto} e_3 = {e_1}^2 + {x_1}^2 + 2e_1x_1 - \dot{d}e_3 + x_3(\beta - d) + \delta x_4 - x_1x_2 + u_3$$

$$D_*^{\propto} e_4 = -y_1 y_3 - \delta y_4 + u_4 + d x_4 - f x_3 - x_1 x_2 \quad (5)$$

In this section, the backstepping design technique is applied to obtain control laws of error system (5). The design procedure is divided into the following steps.

In the first step we consider the stability of the first equation of system (5)

$$D_*^{\propto} w_1 = a(e_2 - e_1) + u_1$$

$$D_*^{\propto} w_1 = a(e_2 - w_1) + u_1 \quad (6)$$

where $w_1 = e_1$ and e_2 and u_1 are controllers choose the first Lyapunov functional candidate as follow

$$v_1 = \frac{1}{2} w_1^2 > 0 \quad (7)$$

The derivative of the above function is

$$\dot{v}_1 = w_1 \dot{w}_1$$

$$\dot{v}_1 = w_1 D_*^{1-\alpha}(D_*^{\propto} w_1)$$

$$\dot{v}_1 = w_1 D_*^{1-\alpha}(a(e_2 - w_1) + u_1)$$

Assuming controllers,

$$e_2 = \propto_1 (w_1)$$

$$u_1 = a w_1 - k_1 D_*^{1-\alpha} w_1$$

$$\dot{v}_1(e_1) = -k_1 w_1^2 + w_1 D_*^{1-\alpha}(a \propto_1)$$

where k_1 is a positive constant and for

$\propto_1 (w_1) = 0$, the equation is

$$\dot{v}_1(e_1) = -k_1 w_1^2$$

Subsequently the zero solution of (7) is asymptotically stable.

$$w_2 = e_2 - \propto_1 (w_1)$$

When e_2 is considered as an controller, $\propto_1 (e_1)$ is estimative function. Defining

$$w_2 = e_2 - \propto_1 (w_1)$$

$$D_*^{\propto} w_1 = a(w_2 - w_1) + u_1$$

$$D_*^{\propto} w_2 = b w_1 + x_1(b - \gamma + a) - c w_2 - x_2(c + \gamma) - e_3(e_1 + x_1) - w_1 x_3 + u_2$$

Substituting $u_1 = a w_1 - k_1 D_*^{1-\alpha} w_1$

$$D_*^{\propto} w_1 = a w_2 - k_1 D_*^{1-\alpha} w_1$$

$$D_*^{\propto} w_2 = b w_1 + x_1(b - \gamma + a) - c w_2 - x_2(c + \gamma) - e_3(e_1 + x_1) - w_1 x_3 + u_2 \quad (8)$$

where e_3 and u_2 are controllers now, we candidate the second Lyapunov function as

$$v_2(w_1, w_2) = v_1(w_1) + \frac{1}{2} w_2^2 > 0 \quad (9)$$

$$v_2(w_1, w_2) = -k_1 w_1^2 + w_2 D_*^{1-\alpha}(D_*^{\propto} w_2)$$

$$v_2(w_1, w_2) = -k_1 w_1^2 + w_2 D_*^{1-\alpha}(b w_1 + x_1(b - \gamma + a) - c w_2 - x_2(c + \gamma) - e_3(e_1 + x_1) - w_1 x_3) + u_2$$

Assuming controllers

$$u_2 = -[(b w_1 + x_1(b - \gamma + a) - c w_2 - x_2(c + \gamma) - e_3(e_1 + x_1) - w_1 x_3)] - k_2 D_*^{\alpha-1} w_2 \quad (10)$$

and $e_3 = \propto_2 (w_1, w_2)$

$$\dot{v}_2(w_1, w_2) = -k_1 w_1^2 - k_2 w_2^2 + w_2 D_*^{1-\alpha}(-\propto_2(\propto_1 + w_1))$$

where k_2 is a positive constant, and for $\propto_2 (w_1, w_2)$=0, the equation is

$$\dot{v}_2(w_1, w_2) = -k_1 w_1^2 - k_2 w_2^2 < 0$$

This will guaranty that the zero solution of (9) is asymptotically stable.

When e_3 is considered as a controllers in (8), $\propto_2 (w_1, w_2)$ is estimative function. Defining

$$w_3 = e_3 - \propto_2 (w_1, w_2)$$

we study (w_1, w_2, w_3)

$$D_*^{\propto} w_1 = a(w_2 - w_1) + u_1$$

$$D_*^{\propto} w_2 = b w_1 + x_1(b - \gamma + a) - c w_2 - x_2(c + \gamma) - e_3(e_1 + x_1) - w_1 x_3 + u_2$$

$$D_*^{\propto} w_3 = w_1^2 + x_1^2 + 2 w_1 x_1 - d w_3 + x_3(\beta - d) + \delta x_4 - x_1 x_2 + u_3 \quad (11)$$

Substituting $u_1 = a w_1 - k_1 D_*^{1-\alpha} w_1$,

$$u_2 = -[(b w_1 + x_1(b - \gamma + a) - c w_2 - x_2(c + \gamma) - e_3(e_1 + x_1) - w_1 x_3)] - k_2 D_*^{\alpha-1} w_2$$

$$D_*^{\propto} w_1 = a w_1 - k_1 D_*^{1-\alpha} w_1$$

$$D_*^{\propto} w_2 = -k_2 D_*^{\alpha-1} w_2 - w_3(w_1 + x_1)$$

$$D_*^{\propto} w_3 = w_1^2 + x_1^2 + 2 w_1 x_1 - d w_3 + x_3(\beta - d) + \delta x_4 - x_1 x_2 + u_3 \quad (12)$$

where u_3 is a controller. Now the Lyapunov function is

$$v_3(w_1, w_2, w_3) = v_1(w_1) + v_2(w_1, w_2) + \frac{1}{2} w_3^2 > 0$$

$$v_3(w_1, w_2, w_3) = -k_1 w_1^2 - k_2 w_2^2 + w_3 D_*^{1-\alpha}(D_*^{\propto} w_3)$$

$v_3(w_1, w_2, w_3) = -k_1 w_1^2 - k_2 w_2^2 + w_3 D_*^{1-\alpha}(\ w_1^2 + x_1^2 + 2w_1x_1 - dw_3 + x_3(\beta - d) + \delta x_4 \ -\dot{x_1}x_2 + u_3$

Assuming controller

$$u_3 = -[w_1^2 + x_1^2 + 2w_1x_1 - d\dot{w_3} + x_3(\beta - d) + \delta x_4 - x_1x_2] - k_3 D_*^{\alpha-1} w_3 \quad (13)$$

Therefore, the equation

$$\dot{v_3}(w_1, w_2, w_3) = -k_1 w_1^2 - k_2 w_2^2 - k_3 w_3^2 < 0$$

where k_3 a positive constant. The controller is u_1, u_2, u_3 will stabilize the (5).

$$w_4 = e_4 - \propto_3 (w_1, w_2, w_3)$$

$$D_*^{\propto} w_1 = \propto(w_2 - w_1) + u_1$$

$$D_*^{\propto} w_2 = bw_1 + x_1(b - \gamma + \alpha) - cw_2 - x_2(c + \gamma) - e_3(e_1 + x_1) - w_1x_3 + u_2$$

$$D_*^{\propto} w_3 = w_1^2 + x_1^2 + 2w_1x_1 - d\dot{w_3} + x_3(\beta - d) + \delta x_4 - x_1x_2 + u_3$$

$$D_*^{\propto} w_4 = -y_1y_3 - \delta y_4 + u_4 + dx_4 - fx_3 - x_1x_2 \quad (14)$$

Substituting u_1, u_2, u_3

$$D_*^{\propto} w_1 = aw_1 - k_1 D_*^{1-\alpha} w_1$$

$$D_*^{\propto} w_2 = -k_2 D_*^{\alpha-1} w_2 - w_3(w_1 + x_1)$$

$$D_*^{\propto} w_3 = -k_3 D_*^{\alpha-1} w_3$$

$$D_*^{\propto} w_4 = -y_1y_3 - \delta y_4 + u_4 + dx_4 - fx_3 - x_1x_2 \quad (15)$$

where u_4 is a controller the Lyapunov function is

$$v_4(w_1, w_2, w_3, w_4) = -k_1 w_1^2 - k_2 w_2^2 - k_3 w_3^2 + w_4 D_*^{1-\alpha}(w_4)$$

$$v_4(w_1, w_2, w_3, w_4) = -k_1 w_1^2 - k_2 w_2^2 - k_3 w_3^2 + w_4 D_*^{1-\alpha}(y_1y_3 - \delta y_4 + dx_4 - fx_3 - x_1x_2) + u_4$$

Assuming

$$u_4 = -[-y_1y_3 - \delta y_4 + u_4 + dx_4 - fx_3 - x_1x_2] - k_4 D_*^{\alpha-1}(w_4) \quad (16)$$

The time derivative is

$$\dot{v_4}((w_1, w_2, w_3, w_4) = -k_1 w_1^2 - k_2 w_2^2 - k_3 w_3^2 - k_4 w_4^2 < 0$$

where k_4 is a positive constant.

III. Numerical Simulation

Numerical simulations have been carried out using the *MATLAB* to solve the sets of fractional- order differential equations related to the master and slave systems. It has been shown that all of the state variables of the slave system converge to that of the master system. The simulation results verify the performance of the Backstepping controller. We applied Backstepping control to synchronize two fractional-order systems by considering the values $q_1 = q_2 = q_3 = q_4 = 0.95, a = 10, b = 28, c = \frac{8}{3}, d = 1$.The initial conditions for the master system are (1,1,1,1) and for slave system are (3, 4, 6, 5). The values of (k_1, k_2, k_3, k_4), is chosen as (10, 10, 10, 10).

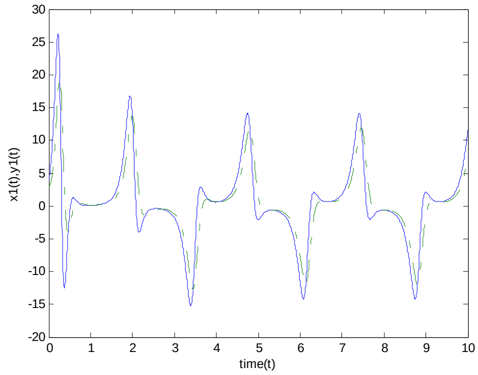

Fig. 1 Synchronization between master and slave system

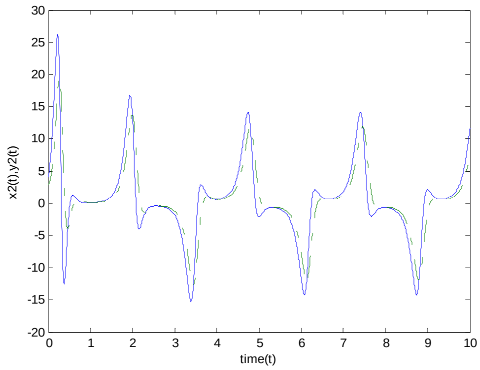

Fig. 2 Synchronization between $x_{2(t)}$ and $y_{2(t)}$

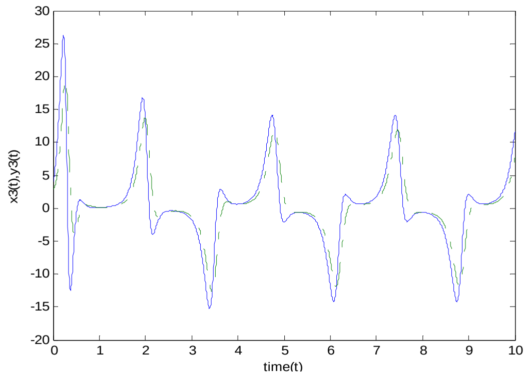

Fig. 3 Simulation result between $x_{3(t)}$ and $y_{3(t)}$

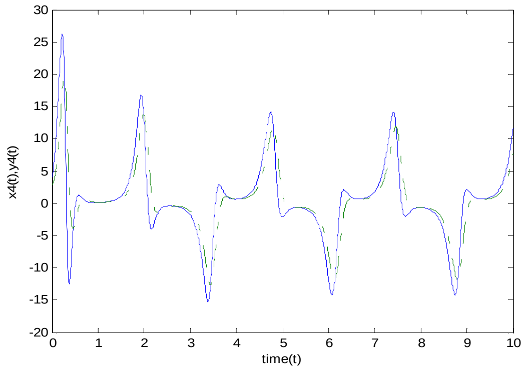

Fig. 4 Simulation result between master and slave system

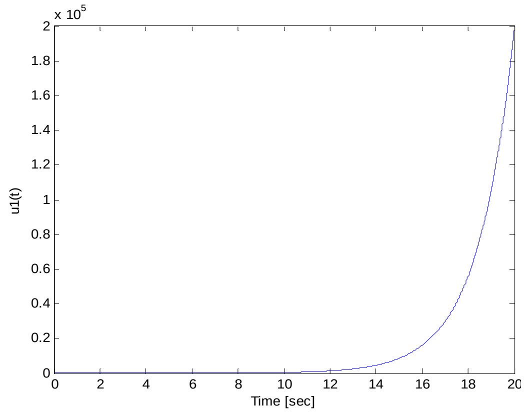

Fig .5 Synchronization of control signal $u_{1(t)}$

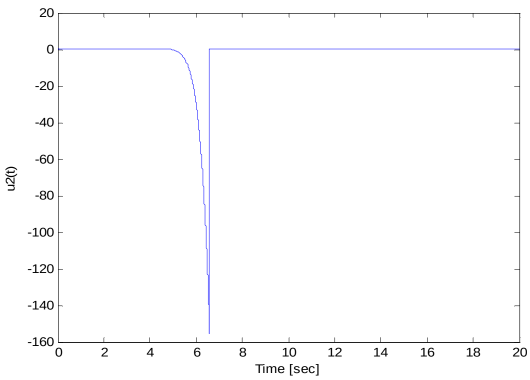

Fig .6 Synchronization of control signal $u_{2(t)}$

IV. Conclusion

In this paper synchronization between fractional order master and slave systems has been investigated by using backstepping control method. Through simulation it has been established that our analytical results and computational results are in excellent agreement.

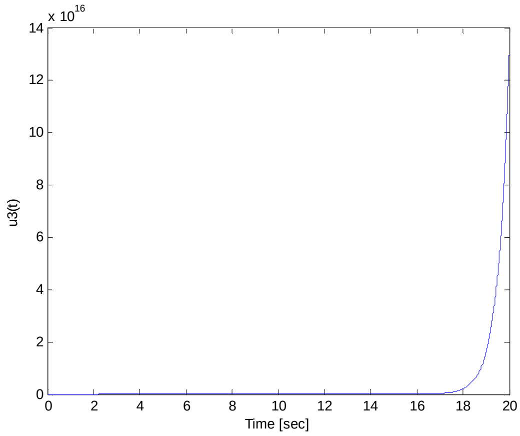

Fig. 7 Synchronization of control signal $u_{3(t)}$

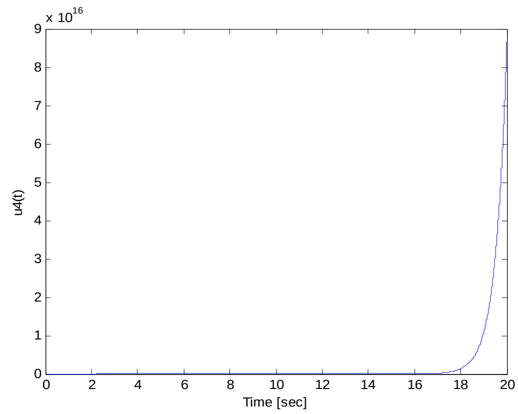

Fig. 8 Synchronization of control signal $u_{4(t)}$

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