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Analytical Solutions for Geodesic Acoustic Eigenmodes in Tokamak Plasmas

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Abstract—The analytical solutions for geodesic acoustic eigenmodes in tokamak plasmas with circular concentric magnetic surfaces are found. In the frame of ideal magnetohydrodynamics the dispersion relation taking into account the toroidal coupling between electrostatic perturbations and electromagnetic perturbations with poloidal mode number $|m| = 2$ is derived. In the absence of such a coupling the dispersion relation gives the standard continuous spectrum of geodesic acoustic modes. The analysis of the existence of global eigenmodes for plasma equilibria with both off-axis and on-axis maximum of the local geodesic acoustic frequency is performed.

Keywords—Tokamak, MHD, geodesic acoustic mode, eigenmode.

I. Introduction

GEODESIC acoustic modes (GAM) in tokamak plasmas are commonly identified as low-frequency toroidally- and poloidally-symmetric ($n = 0$, $m = 0$) oscillations of electrostatic potential attended with oscillations of plasma density on the first poloidal harmonic ($|m| = 1$). Initially, GAM were predicted in [1] where in the limit of large aspect ratio the frequency of their continuous spectrum was determined as

$$\omega = \omega_{\text{geo}}(r) = \frac{c_s(r)}{R}\sqrt{2 + \frac{1}{q^2(r)}}\,, \tag{1}$$

where c_s is the sound frequency, R is the major tokamak radius, q is the safety factor. Nowadays GAM is the most actively studied phenomenon in plasma physics. The measurements of GAMs are performed almost on all leading tokamaks – see, e. g., [2], [3], [4], [5], [6], [7].

One of the most prominent problems in the theory of GAM is the existence of the GAM eigenmode or global GAM (GGAM). It is dictated by some experimental observations of the independence of the frequency of the modes identified as GAM on plasma radius in the whole plasma volume – see [8], [9], [10]. Numerically, GGAM were firstly found in [11] for the equilibria with an off-axis maximum of the local GAM frequency (1) within plasma.

In this paper the results of GGAM analytical solutions search [12], [13] are summarized and further expanded.

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II. The eigenvalue problem

We use the standard one-fluid MHD model with the adiabatic equation of state linearized near the static equilibrium:

$$\rho_0\frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \frac{1}{4\pi}\text{rot}\mathbf{B}\times\mathbf{B}_0 + \frac{1}{4\pi}\text{rot}\mathbf{B_0}\times\mathbf{B}\,, \tag{2}$$

$$\frac{\partial p}{\partial t} + \mathbf{v}\cdot\nabla p_0 + \gamma p_0 \text{div}\mathbf{v} = 0\,, \tag{3}$$

$$\frac{1}{c}\mathbf{v}\times\mathbf{B_0} = \nabla\phi + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\,. \tag{4}$$

Equation (2) is the equation of the motion, (3) is the adiabatic equation with the ratio of the specific heat, γ, (4) describes the electric field. The usual notations are used; subscript "0" denotes the equilibrium (stationary) quantities.

To exclude the magnetic-sound oscillations with predominant perturbation of the longitudinal (along $\mathbf{B}_0$) component of the magnetic field we introduce the perturbation of the magnetic field in the form

$$\mathbf{B} = \text{rot}\left(A_{||}\frac{\mathbf{B_0}}{B_0}\right)\,, \tag{5}$$

where $A_{||}$ is the longitudinal component of the vector potential. The perturbation of the velocity has the form:

$$\mathbf{v} = \frac{c}{B_0^2}\mathbf{B}_0\times\boldsymbol{\nabla}\phi + v_{||}\frac{\mathbf{B}_0}{B_0}\,. \tag{6}$$

In what follows we restrict ourselves with low pressure plasma in the axisymmetric tokamak with large aspect ratio. The absolute value of the equilibrium magnetic field is

$$B_0 = \frac{B_a}{1 + (r/R)\cos\theta}\,. \tag{7}$$

Here $r\ \epsilon\ [0, a]$ is the current radius value counted from the magnetic axis, a is the minor tokamak radius, θ is the poloidal angle, B_a is the magnetic field on the magnetic axis.

The substitution of (5) – (7) into (2) – (4) leads to the following set of equation:

$$\omega^2\nabla_\perp^2\phi + \frac{c_A^2}{qR^2}\nabla_\perp^2\frac{1}{q}\frac{\partial^2\phi}{\partial\theta^2} + \\ +\frac{2i\omega B_a}{c\rho_0 R}\left(\sin\theta\frac{\partial p}{\partial r} + \frac{\cos\theta}{r}\frac{\partial p}{\partial\theta}\right) = 0\,, \tag{8}$$

$$\left(\omega^2 + \frac{\omega_s^2}{q^2}\frac{\partial^2}{\partial\theta^2}\right)\left[\frac{iB_a}{\omega c\rho_0 R}p + \frac{1}{rR\rho_0\omega^2}\frac{dp_0}{dr}\frac{\partial\phi}{\partial\theta}\right] + \\ +2\omega_s^2\left(\sin\theta\frac{\partial\phi}{\partial r} + \frac{\cos\theta}{r}\frac{\partial\phi}{\partial\theta}\right) = 0\,, \tag{9}$$

where the perturbations are taken to be proportional to $\exp(-i\omega t)$, $\omega_s = c_s/R$, $c_s = \sqrt{\gamma p_0/\rho_0}$, $c_A = B_a/\sqrt{4\pi\rho_0}$, and ρ_0 is supposed to be a constant.

The solution to (8) – (9) is convenient to be searched as a sum of the poloidal Fourier-harmonics. Taking $\phi = \phi_0(r)$ (pure electrostatic oscillations), we arrive to

$$\frac{d}{dr}\left\{\frac{r\omega^2(\omega^2-\omega_{geo}^2(r))}{(\omega^2-\omega_s^2(r)/q^2(r))}\frac{d\phi_0}{dr}\right\}=0 \tag{10}$$

that describes continuous GAM spectrum (1).

Taking into account the second poloidal harmonics of the potential, $\phi = \phi_0(r) + \phi_2(r)\exp(2i\theta) + \phi_{-2}(r)\exp(-2i\theta)$ (small magnetic perturbations), which couples with the first harmonics of plasma pressure $m = \pm 1$: $p = p_1(r)\exp(i\theta) + p_{-1}(r)\exp(-i\theta)$, the full system (8) – (9) reduces to

$$\frac{d}{dr}\left[r\left(\frac{d\phi_0}{dr}-P\right)\right]=0, \tag{11}$$

$$\left(\omega^2-\omega_s^2/q^2\right)P+\omega_s^2\left(\frac{a^2}{r^2}\frac{d\Phi_2}{dr}-2\frac{d\phi_0}{dr}\right)=0\,, \tag{12}$$

$$\omega^2\frac{d}{dr}\left[\frac{1}{r}\left(\frac{1}{r^2}\frac{d\Phi_2}{dr}+\frac{1}{a^2}P\right)\right]- -\frac{4c_A^2}{qR^2}\frac{d}{dr}\left[\frac{1}{r^3}\frac{d}{dr}\left(\frac{\Phi_2}{q}\right)\right]=0\,, \tag{13}$$

where variables $\Phi_2 = (r/a)^2(\phi_2 + \phi_{-2})$ and $P = B_a(p_1 - p_{-1})/Rc\rho_0\omega$ are introduced.

System (11) – (13) can be reduced to the one differential equation on Φ_2:

$$\frac{1}{q(\hat r)}\frac{d}{d\hat r}\left[\frac{1}{\hat r^3}\frac{d}{d\hat r}\left(\frac{\Phi_2}{q(\hat r)}\right)\right]+ +\hat\omega^2\frac{d}{d\hat r}\left\{\frac{\epsilon(\hat r)}{\hat r^3 q^2(\hat r)}\left(\frac{1}{\hat\omega^2-\hat\omega_{geo}^2(\hat r)}-\frac{1}{T(\hat r)}\right)\frac{d\Phi_2}{d\hat r}\right\}=0\,. \tag{14}$$

Here the normalized radius $\hat r = r/a$, frequencies $\hat\omega^2 = \omega^2/\omega_s^2|_{\hat r=0}$, $\hat\omega_{geo}^2 = \omega_{geo}^2/\omega_s^2|_{\hat r=0} = T(2+1/q^2)$ and temperature $T(\hat r)$ equals to the unity by $\hat r = 0$ are introduced; $\beta = c_s^2|_{\hat r=0}/c_A^2$, and $\epsilon(\hat r) = \beta T(\hat r)q^2(\hat r)/4$ is the small parameter related to β.

The eigenmode of geodesic acoustic oscillations should satisfy to (14) with two boundary conditions. We suppose zero boundary conditions for Φ_2:

$$\Phi_2|_{\hat r=0}=0,\quad \Phi_2|_{\hat r=1}=0. \tag{15}$$

The first one provides the regularity of the solution on the magnetic axis, ($|\phi_{\pm2}|_{\hat r=0}| < \infty$), the second one – zero radial velocity on the plasma boundary ($v_r|_{\hat r=1} = 0$).

In what follows, we will omit hats on $\hat r$, $\hat\omega$ $\hat\omega_{geo}$, working only with normalized values.

III. Asymptotical solution for equilibria with an off-axis maximum of ω_{geo}

Let us look for the low-frequency asymptotical solution of the (14) with $\omega \sim 1$. We assume that the GAM frequency ω_{geo} has its maximum at the point $r = r_M \in (0,1)$. We look for the eigenfrequency of the problem in the form

$$\omega^2=\omega_{geo}^2(r_M)\left[1+\mathcal{O}(\epsilon_M)\right]\,, \tag{16}$$

where $\epsilon_M = \epsilon(r_M)$. Since $\epsilon(r) \ll 1$, the second term in (14) is important only near the point where $(\omega^2 - \omega_{geo}^2(r)) \to 0$. Thus, we can solve the equation in three regions – in the vicinity of the point $r = r_M$, on the left edge of the calculating area $0 \le r < r_M$ and on the right edge of the calculating area $r_M < r \le 1$.

Near the point $r = r_M$ we use the expansion

$$\omega_{geo}^2(r)\approx\omega_{geo}^2(r_M)-\alpha^2(r-r_M)^2,\ \alpha^2\equiv\left|\frac{d^2\omega_{geo}^2(r_M)}{dr^2}\right|/2\,.$$

Equation (14) is simplified and takes the form

$$\frac{d}{dx}\left[\left(1+\frac{1}{\mu^2(1+x^2)-1}\right)\frac{d\Phi_2}{dx}\right]=0\,, \tag{17}$$

where $x = (r - r_M)/\nu$ and

$$\nu^2 = \frac{1}{\alpha^2}\left(\frac{\epsilon_M\omega^2}{\Lambda}+\omega^2-\omega_{geo}^2(r_M)\right),$$
$$\mu^2 = 1+\frac{\Lambda(\omega^2-\omega_{geo}^2(r_M))}{\epsilon_M\omega^2},\quad \Lambda=1-\frac{\epsilon_M\omega^2}{T(r_M)}. \tag{18}$$

Solution to (17) is described by the expression

$$\Phi_2^{(1)}=C+D\left(x-\frac{1}{\mu^2}\mathrm{arctg}\,(x)\right) \tag{19}$$

with arbitrary constants of integration C and D.

In the regions far from r_M the perturbation is described by the equation

$$\frac{d}{dr}\left[\frac{1}{r^3}\frac{d}{dr}\left(\frac{\Phi_2}{q}\right)\right]=0. \tag{20}$$

Its solutions in these regions satisfying boundary conditions (15) are

$$\Phi_2^{(2)}=Eqr^4,\quad 0\le r<r_M,$$
$$\Phi_2^{(3)}=Fq(1-r^4),\quad r_M<r\le 1, \tag{21}$$

where E and F are constants.

Let us concretize the values of C, D, E, F from the condition of asymptotic matching of $\Phi_2^{(1)}$ with $\Phi_2^{(2)}$ and $\Phi_2^{(3)}$. Namely, on the left edge of the calculating area $\Phi_2^{(1)}|_{x\ll -1} = \Phi_2^{(2)}|_{x\to -0}$, so

$$C+D(x+\pi/2\mu^2)=Eq(r_M)r_M^3\left[r_M+\nu x(4+s)\right]; \tag{22}$$

on the right edge of the calculating area $\Phi_2^{(1)}|_{x\gg +1} = \Phi_2^{(3)}|_{x\to +0}$, so

$$C+D(x-\pi/2\mu^2)= =Fq(r_M)\left\{1-r_M^4-4r_M^3\nu x(4+s-s/r_M^4)\right\}. \tag{23}$$

Here the local shear of the magnetic field $s = r_M dq/dr(r_M)/q(r_M)$ is introduced at the point $r = r_M$. The matching of (22), (23) gives the dispersion relation for GGAM eigenfrequency:

$$\omega^2 = \omega^2_{GAM}(r_M) + \\ +2\epsilon^2_M \omega^2_{geo}(r_M)\left(\frac{\pi\Delta}{r_M}\right)^2 \frac{\omega^2_{geo}(r_M)}{|d^2\omega^2_{geo}(r_M)/dr^2|}. \quad (24)$$

Here

$$\omega^2_{GAM}(r_M) = \omega^2_{geo}(r_M)\left[1 - \epsilon_M - \epsilon^2_M\left(1 + \frac{1}{q^2(r_M)}\right)\right]$$

$$\Delta = 4\left(1 + \frac{s}{4}\right)\left[r^4_M + \frac{s}{4}\left(r^4_M - 1\right)\right].$$

The first term in the expression of the right-hand side of (24) is nothing else but the frequency of continuous GAM spectrum at the point $r = r_M$ taking into account the effects of toroidal coupling with $m = 2$ poloidal harmonics of perturbed electromagnetic field. It is evident from (24) that the eigenfrequency is slightly higher than the maximum of continuous GAM spectrum $\omega_{GAM}(r_M)$ but lower than the maximum of continuous GAM spectrum $\omega_{geo}(r_M)$.

IV. Exact Solutions for the Equilibria with Positive Magnetic Shear

A. Eigenmode Existence Condition

To find the exact solution for GGAM let us come back to the (14) and rewrite it in the form

$$\frac{d}{dr}\left(\frac{1}{r^3q^2(r)}\frac{d\Phi_2}{dr}\left[1 + \frac{\omega^2\epsilon(r)}{\omega^2 - \omega^2_{geo}(r)} - \frac{\omega^2\epsilon(r)}{T(r)}\right]\right) + \\ +\frac{\Phi_2}{r^3q^3(r)}\left\{\frac{3}{r}\frac{dq}{dr} - \frac{d^2q}{dr^2} + \frac{2}{q(r)}\left(\frac{dq}{dr}\right)^2\right\} = 0. \quad (25)$$

It is easy to see that under condition $3qdq/dr - rqd^2q/dr^2 + 2r(dq/dr)^2 = 0$ (25) can be integrated in elementary way. This condition uniquely determines the profile of the safety factor

$$q(r) = \frac{q_0q_1}{q_1 - (q_1 - q_0)r^4}. \quad (26)$$

Here $q_0 = q|_{r=0}$, $q_1 = q|_{r=1}$. By $q_1 > q_0$ profile (26) describes the monotonic growth of q with small gradient near the magnetic axis.

After the integration of (25) we have

$$\frac{d\Phi_2}{dr} = \frac{Kr^3q^2(r)(\omega^2 - \omega^2_{geo}(r))}{\omega^2(1+\epsilon(r)) - \omega^2_{geo}(r)}, \quad (27)$$

where K is the integration constant. The zero of the denominator in (27) determines the frequency of the continuous spectrum:

$$\omega^2_{GAM}(r) = \frac{\omega^2_{geo}(r)}{1+\epsilon(r)}. \quad (28)$$

Let us rewrite the boundary conditions (15) in the form of one integral requirement

$$\int_0^1 \frac{d\Phi_2}{dr}dr = 0. \quad (29)$$

Changing in (29) the integrational variable from r to q and using (27), we have

$$\int_{q_0}^{q_1} \frac{(\omega^2 - \omega^2_{geo})dq}{\omega^2(1+\epsilon) - \omega^2_{geo}} = 0. \quad (30)$$

Equation (30) determines the eigenfrequency of GGAM in tokamak plasmas with safety factor $q(r)$ described by (26).

B. Eigenmodes

Let us rewrite (30) with substitution of ω^2_{geo} and ϵ:

$$(q_1 - q_0) - \frac{\beta}{4}\int_{q_0}^{q_1}\frac{\omega^2Tq^2dq}{\omega^2(1+\beta Tq^2/4) - T(2+1/q^2)} = 0. \quad (31)$$

To integrate (31) let us consider the temperature profile in the form:

$$T = q^2/(D_0 + D_2q^2 + D_4q^4), \quad (32)$$

where D_0, D_2 and $D_4 \neq 0$ are constants which determine the temperature at three reference points.

For the chosen $T(q)$, (31) reduces to the combination of two integrals:

$$q_1 - q_0 + \frac{\beta}{4D_4}(cI_0 + bI_2) = 0, \quad (33)$$

where

$$I_0 = \int_{q_0}^{q_1}\frac{dq}{q^4 + bq^2 + c}, \; I_2 = \int_{q_0}^{q_1}\frac{q^2dq}{q^4 + bq^2 + c}, \quad (34)$$

$b = (D_2 - 2/\omega^2)/(D_4 + \beta/4)$, $c = (D_0 - 1/\omega^2)/(D_4 + \beta/4)$. The values of integrals I_0 and I_2 are determined by the sign of the parameter $\delta = b^2 - 4c$, which is negative for $\omega_{geo}(r)$ with off-axis maximum and positive for $\omega_{geo}(r)$ with on-axis maximum

1) GGAM for w_{geo} with off-axis maximum ($\delta < 0$): In this case (33) reduces to

$$q_1 - q_0 - \frac{\beta}{8D_4\sqrt{-\delta}}\left\{\frac{(b-\sqrt{c})c_+}{2}\ \ln\left|\frac{q^2 + qc_- + \sqrt{c}}{q^2 - qc_- + \sqrt{c}}\right|\Bigg|_{q_0}^{q_1} \right. \\ - c_-\left[b\left(\text{arctg}\left(\frac{2q + c_-}{c_+}\right)\Bigg|_{q_0}^{q_1} + \text{arctg}\left(\frac{2q - c_-}{c_+}\right)\Bigg|_{q_0}^{q_1}\right)\right. \\ \left.\left. + \sqrt{c}\ \text{arctg}\left(\frac{q^2 - \sqrt{c}}{qc_+}\right)\Bigg|_{q_0}^{q_1}\right]\right\} = 0, \quad (35)$$

where the following notations are used: $c_- = \sqrt{2\sqrt{c} - b}$, $c_+ = \sqrt{2\sqrt{c} + b}$. The solution to (35) is found near $\delta = 0$.

On Fig. 1 the example of the eigenfunctions of the GGAM is shown.

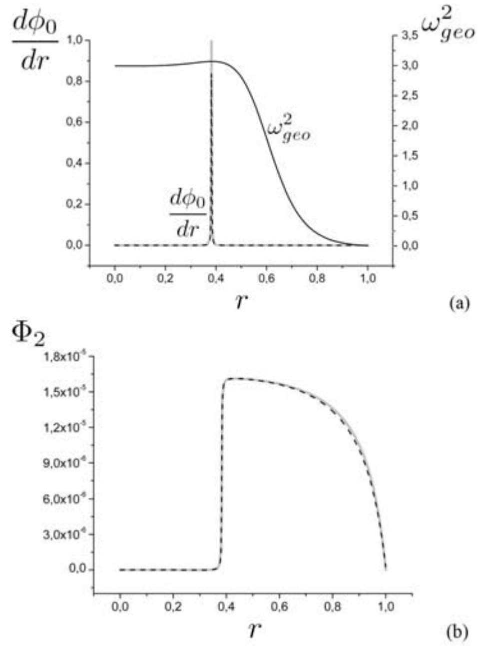

Fig. 1. Radial profiles of $d\phi_0/dr$, ω_{geo}^2 (a) and of Φ_2 (b) for the equilibrium with off-axis maximum of local GAM frequency. Here $\beta = 0.04$.

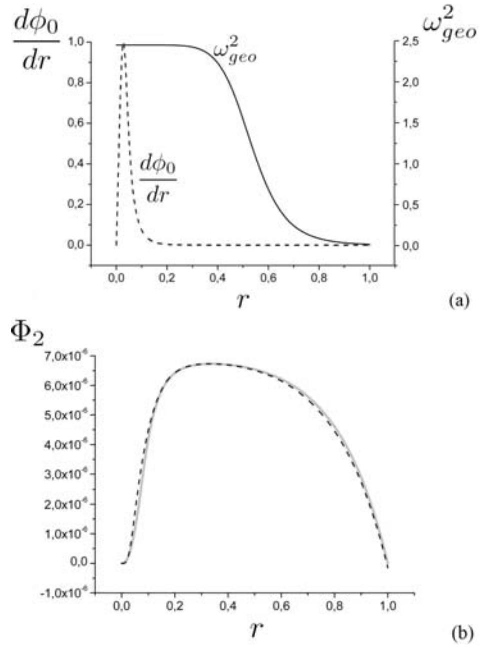

Fig. 2. Radial profiles of $d\phi_0/dr$, ω_{geo}^2 (a) and of Φ_2 (b) for the equilibrium with on-axis maximum of local GAM frequency. Here $\beta = 0.04$.

2) GGAM for w_{geo} with on-axis maximum ($\delta > 0$): In this case instead of (33) we have

$$q_1 - q_0 - \frac{\beta}{8D_4\sqrt{\delta}}\left\{b_-^2 I(b_-) - b_+^2 I(b_+)\right\} = 0\,, \quad (36)$$

where $b_- = (b - \sqrt{\delta})/2$, $b_+ = (b + \sqrt{\delta})/2$, and function $I(y)$ is defined as

$$I(y) = \begin{cases} \frac{2}{\sqrt{y}}\mathrm{arctg}\left(\frac{q}{\sqrt{y}}\right)\Big|_{q_0}^{q_1}, & y > 0 \\ \frac{1}{\sqrt{-y}}\ln\left|\frac{\sqrt{-y}-q}{\sqrt{-y}+q}\right|\Big|_{q_0}^{q_1}, & y < 0 \quad . \end{cases} \quad (37)$$

The solution to (36) is located near the point satisfying the condition $|\sqrt{-b_-} - q_0| \ll 1$.

On Fig. 2 the example of the eigenfunctions of the GGAM is shown.

V. Conclusion

The analytical solutions for global geodesic acoustic mode in tokamak plasmas are found. The frequency of such mode lies slightly higher than the upper boundary of the continuous GAM spectrum calculated with taking into account electromagnetic plasma perturbations. Two types of the solution can be yielded. The first type exists if the local GAM frequency has an off-axis maximum. Although this mode is global by definition it is strongly peaked near the point of ω_{geo} maximum. The second type exists when ω_{geo} is monotonic and has respectively small gradient near the axis. The eigenfunctions of such a solution are not localized but have the significant amplitude in the whole plasma volume where the gradient of ω_{geo} is small. The solution demonstrates analytically the possibility of GGAM formation in the discharges with monotonic profiles of the local GAM frequency typical for the present-day tokamak experiments [8], [9], [10].

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