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Optimal Design for $SARMA(P,Q)_L$ Process of EWMA Control Chart

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***Abstract*—**The main goal of this paper is to study Statistical Process Control (SPC) with Exponentially Weighted Moving Average (EWMA) control chart when observations are serially-correlated. The characteristic of control chart is Average Run Length (ARL) which is the average number of samples taken before an action signal is given. Ideally, an acceptable ARL of in-control process should be enough large, so-called (ARL_0). Otherwise it should be small when the process is out-of-control, so-called Average of Delay Time (ARL_1) or a mean of true alarm. We find explicit formulas of ARL for EWMA control chart for Seasonal Autoregressive and Moving Average processes (SARMA) with Exponential white noise. The results of ARL obtained from explicit formula and Integral equation are in good agreement. In particular, this formulas for evaluating (ARL_0) and (ARL_1) be able to get a set of optimal parameters which depend on smoothing parameter (λ) and width of control limit (H) for designing EWMA chart with minimum of (ARL_1).

***Keywords*—**Average Run Length, Optimal parameters, Exponentially Weighted Moving Average (EWMA) control chart.

I. Introduction

CONTROL chart is an effective tool in statistical process control for detecting changes in a processes (mean and variance), and uses for measuring, controlling and improving quality in many areas of interest including finance and economics, medicine, sociology, engineering, and others. The SPC charts such as the Shewhart control chart proposed by Shewhart [1], the Cumulative Sum (CUSUM) control chart first presented by Page [2], and the Exponentially Weighted Moving Average (EWMA) control chart was initially introduced by Roberts [3], these are used to monitor product quality and detect the occurrence of special causes that may by indicated to out-of-control situations. Both CUSUM and EWMA charts are based on the assumption that observations being monitored will produce measurements that are independent and identically distribution over time when only the inherent sources of variability are present in the process [4]. However, there are many situations in which the process is serially correlation such as in chemical processes, the manufacture of food and others. Hence, these systems have to be monitored by particular control charts.

A common characteristic used for comparing the performance of control charts is Average Run Length (ARL) defined as the expected number of observations taken from an in-control process until the control chart falsely signals out-of-control is denoted by ARL_0. An ARL_0 will be regarded as acceptable if it is large enough to keep the level of false alarms at an acceptable level. A second common characteristic is the expected number of observations taken from an out-of-control process until the control chart signals that the process is out-of-control is denoted by ARL_1. Ideally, the ARL_1 should be small as possible.

In literatures are many methods for evaluating ARL for CUSUM and EWMA procedures i.e., Monte Carlo simulations (MC), Markov Chain Approximation (MCA) see e.g. Brook and Evans [5]. Integral Equations (IE) (see e.g. [6], Crowder [7]). Using methods to evaluate the ARL of control chart serially-correlated observations have been presented in some processes [8], [9]. Lu and Reynolds [10] used integral equation to compute ARL when the observations can be modeled to AR(1) and ARMA(1,1) processes plus random error. Recently, Suriyakat et al. [11] derived the explicit formulas of ARL for EWMA control chart when process is AR (1) with Exp(1) white noise. In addition, Phanyaem et al. [12] proposed the explicit formulas of ARL for EWMA control chart based on ARMA(1,1) process.

In this paper, we show explicit formulas of ARL for EWMA control chart for Seasonal Autoregressive and Moving Average (SARMA) processes with Exponential white noise and a set of optimal parameters which depend on smoothing parameter (λ) and width of control limit (H) for designing EWMA chart with minimum of ARL_1 are presented.

II. Control Charts and Their Properties

In this paper we consider SPC charts under the assumption that sequential observations $\xi_1, \xi_2, ...,$ are independent random variables with a distribution function $F(x,\beta)$, the parameter $\beta = \beta_0$ before a change-point time $\theta \le \infty$ ("in-control" state; $\theta = \infty$ means that there are no change at all) and $\beta > \beta_0$ after the change-point time θ ("out-of-control" state).

All popular charts are based on use of stopping times τ. The typical condition on choice of the stopping times τ is the following

$$E_\infty(\tau) = T, \tag{1}$$

where T is given (usually large), and $E_\infty(.)$ denote that the expectation under distribution $F(x,\beta_0)$ (in-control) that the

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change-point occurs at point θ (where $\theta \le \infty$). In literature on quality control the quantity $E_\infty(\tau)$ is called as Average Run Length for in-control process (ARL_0) of the algorithm. Then, by definition, $ARL_0 = E_\infty(\tau)$ and the typical practical constraint is

$$ARL_0 = T.$$

Another typical constraint consists in minimizing the quantity

$$Q(\beta_1) = E_\theta(\tau - \theta + 1 | \tau \ge \theta), \qquad (2)$$

where $E_\theta(.)$ is the expectation under distribution $F(x,\beta_1)$ (out-of-control) and β_1 is the value of parameter after the change-point. We restrict on the special case, usually $\theta = 1$. The quantity $E_1(\tau)$ is called as Average Run Length for out-of-control process (ARL_1) and one could expect that a sequential chart has a near optimal performance if ARL_1 is close to a minimal value.

The EWMA statistics based on SARMA(P,Q)$_L$ process is defined by the following recursion:

$$Z_t = (1-\lambda)Z_{t-1} + \lambda X_t \; ; \; t = 1,2,.... \qquad (3)$$

where Z_t is the EWMA statistics, X_t is a sequence of SARMA(P,Q)$_L$ processes and the initial value is a constant (Z_0 = u) and $\lambda \in (0,1)$ is smoothing parameter.

The general Seasonal Autoregressive Moving Average processes, denoted by SARMA(P,Q)$_L$ processes can be written as:

$$X_t = \mu + \phi_1 X_{t-L} + \phi_2 X_{t-2L} + ... + \phi_P X_{t-PL} + \zeta_t - \theta_1\zeta_{t-L} - \theta_2\zeta_{t-2L} - ... - \theta_Q\zeta_{t-QL}$$

where ζ_t is to be a white noise processes assumed with Exponential distribution. An autoregressive coefficient $-1 \le \phi_i \le 1$, a moving average coefficient $-1 \le \theta_i \le 1$, L is a period of time and μ is a constant . We assume the initial value of SARMA(P,Q)$_L$ processes $X_{t-L}, X_{t-2L}, ..., X_{t-PL}$ = 1 and $\zeta_{t-1}, \zeta_{t-2}, ..., \zeta_{QL}$ =1 as the process mean.

The first passage times for the EWMA can be written as:

$$\tau_H = inf\{t > 0 : Z_t > H\}$$

where H is a control limit.

III. Solution for Evaluating ARL_0 and ARL_1 of EWMA Procedure

In this section we present the explicit formulas for ARL which is submitted in Pichit et al.[13]. We obtain the explicit formula for ARL_0 as follows:

$$ARL_0 = 1 - \frac{\lambda e^{\left(\frac{(1-\lambda)u}{\lambda\beta_0}\right)}\left(e^{\left(\frac{-H}{\lambda\beta_0}\right)} - 1\right)}{\lambda e^{\left(\frac{-\mu-\phi_1X_0-\phi_2X_{t-L}-...-\phi_PX_{t-PL}+\theta_1\zeta_0+\theta_2\zeta_{t-2L}+...+\theta_Q\zeta_{t-QL}}{\beta_0}\right)} + e^{\left(-\frac{H}{\beta_0}\right)} - 1} \qquad (4)$$

On the other hand, since the process is out-of-control, parameter $\beta = \beta_1$. The explicit formula for ARL_1 can be written as follows:

$$ARL_1 = 1 - \frac{\lambda e^{\left(\frac{(1-\lambda)u}{\lambda\beta_1}\right)}\left(e^{\left(\frac{-H}{\lambda\beta_1}\right)} - 1\right)}{\lambda e^{\left(\frac{-\mu-\phi_1X_0-\phi_2X_{t-L}-...-\phi_pX_{t-PL}+\theta_1\zeta_0+\theta_2\zeta_{t-2L}+...+\theta_Q\zeta_{t-QL}}{\beta_1}\right)} + e^{\left(-\frac{H}{\beta_1}\right)} - 1} \qquad (5)$$

where $-1 \le \phi_i \le 1$ is an Autoregressive coefficient and $-1 \le \theta_i \le 1$ is a Moving Average coefficient, $\lambda \in (0,1)$ is a smoothing parameter, H is upper control limit and $X_{t-L}, X_{t-2L}, ..., X_{t-PL}$ and $\zeta_{t-1}, \zeta_{t-2}, ..., \zeta_{QL}$ are the initial values.

Using the explicit formulas, we have been able to provide the tables for the optimal smoothing parameter (λ) and width of control limit (H) for designing EWMA chart with minimum of ARL_1. We first describe a procedure for obtaining optimal designs for EWMA chart. The criterion used for choosing optimal values for is smoothing parameter (λ) and width of control limit (H) for designing EWMA chart with minimum of ARL_1 for a given in-control parameter value β_0=1, $ARL_0 = T$ and a given out-of-control parameter value ($\beta = \beta_1$). We compute optimal (λ, H) values for T= 370 and 500 and magnitudes of change. Table of the optimal parameters values are shown in Tables II-III.

A. The Numerical Procedure for Obtaining Optimal Parameters for EWMA Designs

1. Select an acceptable in-control value of ARL and decide on the change parameter value (β_1) for an out-of-control state.
2. For given β_0 and T, find optimal values of λ and H to minimize the ARL_1 values given by (5) subject to the constraint that $ARL_0 = T$ in (4), i.e. λ and H are solutions of the optimality problem

IV. Numerical Results

In this section, the numerical results for ARL_0 and ARL_1 for a EWMA chart were calculated from (4) and (5) as shown in Table I. The parameter values for EWMA chart was chosen by given desired ARL_0 = 370 and 500, λ=0.01, in-control parameter β_0= 1 and out-of-control parameter values β_1 from 1.01 to 1.5 for ARMA(3,2)$_4$ process with $\phi_1 = 0.1$, $\phi_2 = 0.2$, $\phi_3 = 0.1$ and $\theta_1 = \theta_2 = 0.2$ Obviously, the results from suggested formulas are very close to approximation IE. Note that, calculations with explicit formula from (4) and (5) is simple and very fast to calculate which the computational times takes less than 1 second. The numerical results in terms of optimal width of the smoothing parameter (λ), optimal width of control limit (H) and minimum ARL_1 (ARL_1^*) for ARL=370 and 500 are shown in Tables II and III. For example, if we want to detect a parameter change from $\beta = 1$ to $\beta = 1.2$ and the ARL value is 370 then the optimality procedure given above will give optimal parameter values λ = 0.2125 and H = 0.2383. On substituting the values for β, λ and H into (3) we obtain ARL_1^* value = 5.649. As shown in Tables I and III the use of the suggested explicit formulas for ARL_0 and ARL_1 for EWMA chart can greatly reduce the computation times, and are useful to practitioners especially finding optimal parameters of EWMA chart.

TABLE I
Comparison of ARL from Proposed Formulas with Numerical IE Method for Given ARL_0 = 370 and 500, β_0 = 1

β	**Explicit formulas** λ **=0.01,** H **= 0.00794**	**Numerical IE**	**Diff(%)**
1.00	370.885	370.821	0.0173
1.01	338.322	338.315	0.0021
1.03	283.994	283.901	0.0327
1.05	249.89	249.814	0.0304
1.07	206.218	206.213	0.0024
1.09	177.991	177.911	0.0449
1.10	165.839	165.139	0.4221
1.30	53.645	53.6445	0.0009
1.50	24.606	24.605	0.0041
β	**Explicit formulas** λ **=0.01,** H **= 0.008425**	**Numerical IE**	**Diff(%)**
1.00	500.136	500.011	0.0250
1.01	450.180	450.085	0.0211
1.03	369.338	369.128	0.0569
1.05	307.404	307.001	0.1311
1.07	259.023	259.011	0.0046
1.09	220.535	220.045	0.2222
1.10	204.223	204.201	0.0108
1.30	61.688	61.611	0.1248
1.50	27.486	27.468	0.0655

TABLE II
Optimal Design Parameters and Minimum ARL_1 for $SARMA(3,2)_4$ with $\phi_1 = 0.1$, $\phi_2 = 0.2$, $\phi_3 = 0.1$, $\theta_1 = \theta_2 = 0.2$ and $ARL_0 = 370$, 500

β_1	λ	H	ARL_1^*
1.01	0.2227	0.2513	74.481
1.03	0.2215	0.2498	29.324
1.05	0.2204	0.2484	18.583
1.07	0.2193	0.2470	13.769
1.10	0.2177	0.2449	10.069
1.20	0.2125	0.2383	5.649
1.30	0.2077	0.2322	4.146
1.40	0.2031	0.2265	3.387
1.50	0.1989	0.2212	2.928
β_1	λ	H	ARL_1^*
1.01	0.2228	0.2514	78.523
1.03	0.2215	0.2499	29.911
1.05	0.2204	0.2486	18.810
1.07	0.2193	0.2472	13.889
1.10	0.2177	0.2451	10.130
1.20	0.2125	0.2385	5.666
1.30	0.2077	0.2324	4.154
1.40	0.2032	0.2267	3.392
1.50	0.1989	0.2213	2.931

TABLE III
Optimal Design Parameters and Minimum ARL_1 for $SARMA(1,3)_{12}$ with $\phi_1 = 0.2$, $\theta_1 = 0.4$, $\theta_2 = 0.1$, $\theta_3 = 0.1$ and $ARL_0 = 370, 500$

β_1	λ	H	ARL_1^*
1.01	0.1507	0.2541	108.342
1.03	0.1505	0.2537	45.345
1.05	0.1503	0.2533	28.926
1.07	0.1501	0.2529	21.372
1.10	0.1497	0.2522	15.487
1.20	0.1483	0.2496	8.378
1.30	0.1468	0.2466	5.953
1.40	0.1451	0.2434	4.732
1.50	0.1434	0.2401	3.997
β_1	λ	H	ARL_1^*
1.01	0.1507	0.2543	117.156
1.03	0.1505	0.2539	46.784
1.05	0.1503	0.2535	29.491
1.07	0.1501	0.2531	21.672
1.10	0.1497	0.2524	15.639
1.20	0.1484	0.2498	8.418
1.30	0.1468	0.2468	5.971
1.40	0.1451	0.2436	4.742
1.50	0.1434	0.2403	4.004

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