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Existence of Positive Solutions for Second-Order Difference Equation with Discrete Boundary Value Problem

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Abstract—We study the existence of positive solutions to the three points difference-summation boundary value problem. We show the existence of at least one positive solution if f is either superlinear or sublinear by applying the fixed point theorem due to Krasnoselskii in cones.

Keywords—Positive solution, Boundary value problem, Fixed point theorem, Cone.

I. Introduction

THE study of the existence of solutions of multipoint boundary value problems for linear second-order ordinary differential and difference equations was initiated by Ilin [1]. Then Gupta [2] studied three-point boundary value problems for nonlinear second-order ordinary differential equations. Since then, nonlinear second-order three-point boundary value problems have also been studied by many authors, one may see the text books [3-4] and the papers [6-11]. However, all these papers are concerned with problems with three-point boundary condition restrictions on the difference of the solutions and the solutions themselves, for example,

$$\begin{aligned}
u(0)&=0, & u(T+1)&=0\\
u(0)&=0, & au(s)&=u(T+1),\\
u(0)&=0, & u(T+1)-au(s)&=b.\\
u(0)-\alpha\Delta u(0)&=0, & u(T+1)&=\beta u(s).\\
u(0)-\alpha\Delta u(0)&=0, & \Delta u(T+1)&=0\\
u(0)&=0, & u(T+1)&=\alpha\sum_{s=1}^{\eta}u(s)\\
u(0)&=\beta\sum_{s=1}^{\eta}u(s), & u(T+1)&=\alpha\sum_{s=1}^{\eta}u(s)
\end{aligned}$$

and so forth.

In [6], Leggett-Williams developed a fixed point theorem to prove the existence of three positive solutions for Hammerstein integral equations. Since then, this theorem has been reported to be a successful technique for dealing with the existence of three solutions for the two-point boundary value problems of differential and difference equations; see [7,8]. In [9], X. Lin and W. Liu using the properties of the associate Green's function and Leggett-Williams fixed point theorem, studied the existence of positive solutions of the problem.

G. Zhang and R. Medina [10], T. Sitthiwirattham and J.Tariboon [11], studied the existence of positive solutions for second order boundary value problems of difference equations by applying the Krasnoselskii's fixed point theorem. In [12], J. Henderson and H.B. Thompson used lower and upper solution methods.

In this paper, we consider the existence of positive solutions to the equation

$$\Delta^2 u(t-1)+a(t)f(u)=0, \qquad t\in\{1,2,...,T\}, \tag{1}$$

with difference-summation boundary condition

$$u(0)=\beta\Delta u(0), \qquad u(T+1)=\alpha\sum_{s=1}^{\eta}u(s), \tag{2}$$

where f is continuous.

The aim of this paper is to give some results for existence of positive solutions to (1)-(2).

Let $\mathbb{N}$ be the nonnegative integer, we let $\mathbb{N}_{i,j}=\{k\in\mathbb{N}|\ i\le k\le j\}$ and $\mathbb{N}_p=\mathbb{N}_{0,p}$. By the positive solution of (1)-(2) we mean that a function $u(t):\mathbb{N}_{T+1}\to[0,\infty)$ and satisfies the problem (1)-(2).

Throughout this paper, we suppose the following conditions hold:

$(H1)$ $T\ge 3$ is a fixed positive integer, $\eta\in\{1,2,...,T-1\}$, constant $\alpha,\beta>0$ such that $0<\alpha<\frac{2(T+1)}{\eta(\eta+1)}$ and $0<\beta<\frac{2(T+1)-\alpha\eta(\eta+1)}{2(\alpha\eta-1)}$.

$(H2)$ $f\in C([0,\infty),[0,\infty))$, f is either superlinear or sublinear. Set

$$f_0=\lim_{u\to 0^+}\frac{f(u)}{u}, \qquad f_\infty=\lim_{u\to\infty}\frac{f(u)}{u}.$$

Then $f_0=0$ and $f_\infty=\infty$ correspond to the superlinear case, and $f_0=\infty$ and $f_\infty=0$ correspond to the sublinear case.

$(H3)$ $a\in C(\mathbb{N}_{T+1},[0,\infty))$ and there exists $t_0\in\mathbb{N}_{\eta,T+1}$ such that $a(t_0)>0$.

The proof of the main theorem is based upon an application of the following Krasnoselskii's fixed point theorem in a cone.

Theorem 1. ([5]). *Let E be a Banach space, and let $K\subset E$ be a cone. Assume Ω_1, Ω_2 are open subsets of E with $0\in\Omega_1$, $\overline{\Omega}_1\subset\Omega_2$, and let*

$$A:K\cap(\overline{\Omega}_2\setminus\Omega_1)\longrightarrow K$$

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be a completely continuous operator such that
(i) $\|Au\| \leqslant \|u\|, \quad u \in K \cap \partial\Omega_1$, *and* $\|Au\| \geqslant \|u\|, \quad u \in K \cap \partial\Omega_2$; *or*
(ii) $\|Au\| \geqslant \|u\|, \quad u \in K \cap \partial\Omega_1$, *and* $\|Au\| \leqslant \|u\|, \quad u \in K \cap \partial\Omega_2$.
Then A *has a fixed point in* $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

II. Preliminaries

We now state and prove several lemmas before stating our main results.

Lemma 1. *The problem*

$$\Delta^2 u(t-1) + y(t) = 0, \qquad t \in \mathbb{N}_{1,T}, \tag{3}$$

$$u(0) = \beta\Delta u(0), \qquad u(T+1) = \alpha\sum_{s=1}^{\eta} u(s), \tag{4}$$

has a unique solution

$$\begin{aligned} u(t) =& \frac{2(t+\beta)}{2(T+1) - \alpha\eta(\eta+1) - 2\beta(\alpha\eta-1)} \times \\ & \sum_{s=1}^{T}(T-s+1)y(s) \\ & - \frac{\alpha(t+\beta)}{2(T+1) - \alpha\eta(\eta+1) - 2\beta(\alpha\eta-1)} \times \\ & \sum_{s=1}^{\eta-1}(\eta-s)(\eta-s+1)y(s) \\ & - \sum_{s=1}^{t-1}(t-s)y(s), \quad t \in \mathbb{N}_{T+1}. \end{aligned}$$

Proof. From $\Delta^2 u(t-1) = \Delta u(t) - \Delta u(t-1)$ and the first equation of (3), we get

$$\begin{aligned} \Delta u(t) - \Delta u(t-1) &= -y(t), \\ \Delta u(t-1) - \Delta u(t-2) &= -y(t-1), \\ &\vdots \\ \Delta u(1) - \Delta u(0) &= -y(1). \end{aligned}$$

We sum the above equations to obtain

$$\Delta u(t) = \Delta u(0) - \sum_{s=1}^{t} y(s), t \in \mathbb{N}_T. \tag{5}$$

We define $\sum_{s=p}^{q} y(s) = 0$; if $p < q$. Similarly, we sum (5) from $t = 0$ to $t = h$, and by using the boundary condition $u(0) = \beta\Delta u(0)$ in (4), we obtain

$$u(h+1) = (h+1+\beta)\Delta u(0) - \sum_{s=1}^{h}(h+1-s)y(s), h \in \mathbb{N}_T,$$

by changing the variable from $h+1$ to t, we have

$$u(t) = (t+\beta)\Delta u(0) - \sum_{s=1}^{t-1}(t-s)y(s), t \in \mathbb{N}_{T+1}. \tag{6}$$

From (6) ,

$$\begin{aligned} \sum_{s=1}^{\eta} u(s) =& \left(\frac{1}{2}\eta(\eta+1) + \beta\eta\right)\Delta u(0) - \sum_{s=1}^{\eta-1}\sum_{l=1}^{\eta-s} ly(s) \\ =& \left(\frac{1}{2}\eta(\eta+1) + \beta\eta\right)\Delta u(0) \\ & - \frac{1}{2}\sum_{s=1}^{\eta-1}(\eta-s)(\eta-s+1)y(s) \end{aligned}$$

Again using the boundary condition $u(T+1) = \alpha\sum_{s=1}^{\eta} u(s)$ in (4), we obtain
$(T+1+\beta)\Delta u(0) - \sum_{s=1}^{T}(T-s+1)y(s) =$
$\alpha\left(\frac{1}{2}\eta(\eta+1) + \beta\eta\right)\Delta u(0) - \frac{\alpha}{2}\sum_{s=1}^{\eta-1}(\eta-s)(\eta-s+1)y(s)$
Thus,

$$\begin{aligned} \Delta u(0) =& \frac{2}{2(T+1) - \alpha\eta(\eta+1) - 2\beta(\alpha\eta-1)} \times \\ & \sum_{s=1}^{T}(T-s+1)y(s) \\ & - \frac{\alpha}{2(T+1) - \alpha\eta(\eta+1) - 2\beta(\alpha\eta-1)} \times \\ & \sum_{s=1}^{\eta-1}(\eta-s)(\eta-s+1)y(s). \end{aligned}$$

Therefore, (3)-(4) has a unique solution

$$\begin{aligned} u(t) =& \frac{2(t+\beta)}{2(T+1) - \alpha\eta(\eta+1) - 2\beta(\alpha\eta-1)} \times \\ & \sum_{s=1}^{T}(T-s+1)y(s) \\ & - \frac{\alpha(t+\beta)}{2(T+1) - \alpha\eta(\eta+1) - 2\beta(\alpha\eta-1)} \times \\ & \sum_{s=1}^{\eta-1}(\eta-s)(\eta-s+1)y(s) \\ & - \sum_{s=1}^{t-1}(t-s)y(s), \quad t \in \mathbb{N}_{T+1}. \end{aligned}$$

□

Lemma 2. *The function*

$$G(t,s) = \frac{1}{\Lambda}\begin{cases} (s+\beta)[2(T+1) - \alpha\eta(\eta+1) + 2t(\alpha\eta-1)] \\ \quad + \alpha s(t+\beta)(1-s), s \in \mathbb{N}_{1,t-1} \cap \mathbb{N}_{1,\eta-1} \\ 2(s+\beta)(T+1-t) + \alpha\eta(t-s)(\eta+1+2\beta), \\ \quad s \in \mathbb{N}_{\eta,t-1} \\ (t+\beta)[2(T+1) - \alpha\eta(\eta+1) + 2s(\alpha\eta-1) + \\ \quad \alpha s(1-s)], s \in \mathbb{N}_{t,\eta-1} \\ 2(T+\beta)(T+1-s), s \in \mathbb{N}_{t,T} \cap \mathbb{N}_{\eta,T} \end{cases} \tag{7}$$

where

$$\Lambda = 2(T+1) - \alpha\eta(\eta+1) - 2\beta(\alpha\eta-1) > 0,$$

is the Green's function of the problem

$$
\begin{aligned}
&-\Delta^2 u(t-1)=0, \qquad t\in\mathbb{N}_{1,T},\\
&u(0)=\beta\Delta u(0), \qquad u(T+1)=\alpha\sum_{s=1}^{\eta}u(s).
\end{aligned} \tag{8}
$$

Proof. Suppose $t<\eta$. The unique solution of problem (3)-(4) can be written

$$
\begin{aligned}
u(t)=&-\sum_{s=1}^{t-1}(t-s)y(s)+\frac{2(t+\beta)}{\Lambda}\Big[\sum_{s=1}^{t-1}(T-s+1)y(s)\times\\
&+\sum_{s=t}^{\eta-1}(T-s+1)y(s)+\sum_{s=\eta}^{T}(T-s+1)y(s)\Big]\\
&-\frac{\alpha(t+\beta)}{\Lambda}\Big[\sum_{s=1}^{t-1}(\eta-s)(\eta-s+1)y(s)\\
&+\sum_{s=t}^{\eta-1}(\eta-s)(\eta-s+1)y(s)\Big]\\
=&\frac{1}{\Lambda}\sum_{s=1}^{t-1}\Big[(s+\beta)[2(T+1)-\alpha\eta(\eta+1)]\\
&+\alpha s(t+\beta)(1-s)\Big]y(s)\\
&+\frac{1}{\Lambda}\sum_{s=t}^{\eta-1}\Big[(t+\beta)[2(T+1)-\alpha\eta(\eta+1)+2s(\alpha\eta-1)\\
&+\alpha s-\alpha s^2]\Big]y(s)\\
&+\frac{1}{\Lambda}\sum_{s=\eta}^{T}2(T+\beta)(T+1-s)y(s)\\
=&\sum_{s=1}^{T}G(t,s)y(s).
\end{aligned}
$$

Suppose $t\geq\eta$. The unique solution of problem (3)-(4) can be written

$$
\begin{aligned}
u(t)=&-\sum_{s=1}^{\eta-1}(t-s)y(s)-\sum_{s=\eta}^{t-1}(t-s)y(s)\\
&+\frac{2(t+\beta)}{\Lambda}\Big[\sum_{s=1}^{\eta-1}(T-s+1)y(s)+\sum_{s=\eta}^{t-1}(T-s+1)y(s)\\
&+\sum_{s=t}^{T}(T-s+1)y(s)\Big]\\
&-\frac{\alpha(t+\beta)}{\Lambda}\sum_{s=1}^{\eta-1}(\eta-s)(\eta-s+1)y(s)
\end{aligned}
$$

$$
\begin{aligned}
u(t)=&\frac{1}{\Lambda}\sum_{s=1}^{\eta-1}\Big[(s+\beta)[2(T+1)-\alpha\eta(\eta+1)]\\
&+\alpha s(t+\beta)(1-s)\Big]y(s)+\frac{1}{\Lambda}\sum_{s=\eta}^{t-1}\Big[2(s+\beta)(T+1-t)\\
&+\alpha\eta(t-s)(\eta+1+2\beta)\Big]y(s)\\
&+\frac{1}{\Lambda}\sum_{s=t}^{T}2(T+\beta)(T+1-s)\\
=&\sum_{s=1}^{T}G(t,s)y(s).
\end{aligned}
$$

Then the unique solution of problem (3)-(4) can be written as $u(t)=\sum_{s=1}^{T}G(t,s)y(s)$. The proof is complete. □

We observe that the condition $0<\alpha<\frac{2(T+1)}{\eta(\eta+1)}$ and $0<\beta<\frac{2(T+1)-\alpha\eta(\eta+1)}{2(\alpha\eta-1)}$. implies $G(t,s)$ is positive on $\mathbb{N}_{1,T}\times\mathbb{N}_{1,T}$, which mean that the finite set

$$\left\{\frac{G(t,s)}{G(t,t)}:t\in\mathbb{N}_{T+1},s\in\mathbb{N}_{1,T}\right\},$$

take positive values. Then we let

$$M_1=\min\left\{\frac{G(t,s)}{G(t,t)}:t\in\mathbb{N}_{T+1},s\in\mathbb{N}_{1,T}\right\} \tag{9}$$

$$M_2=\max\left\{\frac{G(t,s)}{G(t,t)}:t\in\mathbb{N}_{T+1},s\in\mathbb{N}_{1,T}\right\} \tag{10}$$

Lemma 3. *Let* $(t,s)\in\mathbb{N}_{1,T}\times\mathbb{N}_{1,T}$. *Then we have*

$$G(t,s)\geq M_1G(t,t.) \tag{11}$$

where $0<M_1<1$ *is a constant given by*

$$
M_1=\begin{cases}
\min\Big\{\frac{(1+\beta)[2T-\alpha\eta(\eta+4)+3\alpha]}{(\eta+\beta-1)[2(T+2)+\alpha\eta(\eta-3)-2\eta]},\ \frac{2(T+2+\alpha)-\alpha\eta(\eta+4)}{2(T+2)+\alpha\eta(\eta-3)-2\eta},\ \frac{(1+\beta)[2(T+1-\eta)-\alpha\eta(3\eta+1)]+\alpha(\eta+\beta)(2-\eta)}{2(T+\beta)(T+1-\eta)},\\
\qquad \frac{2}{2(T+2)+\alpha\eta(\eta-3)-2\eta},\ \frac{1}{2(T+\beta)(T+1-\eta)}\Big\}; \quad \text{if}\ \ \alpha>\frac{1}{\eta}\\
\min\Big\{\frac{(1+\beta)[2(T+2)+\alpha\eta(\eta-4)-2\eta+3\alpha]}{(\eta+\beta-1)[2T-\alpha\eta(\eta-1)]},\ \frac{2}{2T-\alpha\eta(\eta-1)},\ \frac{(1+\beta)[\alpha\eta(2T-\eta-1)+2]+\alpha(\eta+\beta)(2-\eta)}{2(T+\beta)(T+1-\eta)},\\
\qquad \frac{1}{2(T+\beta)(T+1-\eta)}\Big\}; \quad \text{if}\ \ 0<\alpha<\frac{1}{\eta}
\end{cases} \tag{12}
$$

Proof. In order that (11) holds, it is sufficient that M_1 satisfies

$$M_1\leq\min_{(t,s)\in\mathbb{N}_{1,T}\times\mathbb{N}_{1,T}}\frac{G(t,s)}{G(t,t)}. \tag{13}$$

Then we may choose

$$M_1\leq\min\left\{\min_{(t,s)\in\mathbb{N}_{1,\eta-1}\times\mathbb{N}_{1,T}}\frac{G(t,s)}{G(t,t)},\min_{(t,s)\in\mathbb{N}_{\eta,T}\times\mathbb{N}_{1,T}}\frac{G(t,s)}{G(t,t)}\right\}. \tag{14}$$

since

$$\min_{(t,s)\in\mathbb{N}_{1,\eta-1}\times\mathbb{N}_{1,T}} \frac{G(t,s)}{G(t,t)}$$

$$\ge \begin{cases} \min_{t\in\mathbb{N}_{1,\eta-1}} \\ \Big\{ \frac{(1+\beta)[2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)]+\alpha(t+\beta)(2-t)}{(t+\beta)[2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)+\alpha t(1-t)]}, \\ \frac{2(T+1)-\alpha\eta(\eta+1)-2(\eta-1)+\alpha(2-\eta)}{2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)+\alpha t(1-t)}, \\ \frac{2}{2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)+\alpha t(1-t)} \Big\}; \quad \text{if } \alpha > \frac{1}{\eta} \\ \min_{t\in\mathbb{N}_{1,\eta-1}} \\ \Big\{ \frac{(1+\beta)[2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)]+\alpha(t+\beta)(2-t)}{(t+\beta)[2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)+\alpha t(1-t)]}, \\ \frac{2(T+1)-\alpha\eta(\eta+1)-2(\eta-1)(\alpha\eta-1)+\alpha(2-\eta)}{2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)+\alpha t(1-t)}, \\ \frac{2}{2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)+\alpha t(1-t)} \Big\}; \quad \text{if } 0 < \alpha < \frac{1}{\eta} \end{cases}$$

$$\ge \begin{cases} \min \Big\{ \frac{(1+\beta)[2T-\alpha\eta(\eta+4)+3\alpha]}{(\eta+\beta-1)[2(T+2)+\alpha\eta(\eta-3)-2\eta]}, \frac{2(T+2+\alpha)-\alpha\eta(\eta+4)}{2(T+2)+\alpha\eta(\eta-3)-2\eta}, \\ \frac{2}{2(T+2)+\alpha\eta(\eta-3)-2\eta]} \Big\}; \quad \text{if } \alpha > \frac{1}{\eta} \\ \min \Big\{ \frac{(1+\beta)[2(T+2)+\alpha\eta(\eta-4)-2\eta+3\alpha]}{(\eta+\beta-1)[2T-\alpha\eta(\eta-1)]}, \frac{2(T+\eta+\alpha)-3\alpha\eta^2}{2T-\alpha\eta(\eta-1)}, \\ \frac{2}{2T-\alpha\eta(\eta-1)} \Big\}; \quad \text{if } 0 < \alpha < \frac{1}{\eta} \end{cases}$$

Similarly, we get

$$\min_{(t,s)\in\mathbb{N}_{\eta,T}\times\mathbb{N}_{1,T}} \frac{G(t,s)}{G(t,t)}$$

$$\ge \begin{cases} \min \Big\{ \frac{(1+\beta)[2(T+1-\eta)-\alpha\eta(3\eta+1)]+\alpha(\eta+\beta)(2-\eta)}{2(T+\beta)(T+1-\eta)}, \\ \frac{2(\eta+\beta)+\alpha\eta(\eta+1+2\beta)}{2(T+\beta)(T+1-\eta)}, \\ \frac{1}{2(T+\beta)(T+1-\eta)} \Big\}; \quad \text{if } \alpha > \frac{1}{\eta} \\ \min \Big\{ \frac{(1+\beta)[\alpha\eta(2T-\eta-1)+2]+\alpha(\eta+\beta)(2-\eta)}{2(T+\beta)(T+1-\eta)}, \\ \frac{2(\eta+\beta)+\alpha\eta(\eta+1+2\beta)}{2(T+\beta)(T+1-\eta)}, \\ \frac{1}{2(T+\beta)(T+1-\eta)} \Big\}; \quad \text{if } 0 < \alpha < \frac{1}{\eta} \end{cases} \tag{15}$$

The (12) is immediate from (15)-(16) □

Lemma 4. *Let $(t,s) \in \mathbb{N}_{T+1} \times \mathbb{N}_{1,T}$. Then we have*

$$G(t,s) \le M_2 G(t,t.) \tag{16}$$

where $M_2 \ge 1$ is a constant given by

$$M_2 = \begin{cases} \max \Big\{ \frac{2(T+1-\eta)}{2(T+\alpha)-\alpha\eta^2}, \frac{(\eta-1+\beta)[\alpha\eta(2T-\eta-1)+2]}{2(\eta+\beta)}, \\ \frac{2(T-1+\beta)+\alpha\eta(T-\eta)(\eta+1+2\beta)}{2(\eta+\beta)}, \ 1 \Big\}; \ \textit{if } \alpha > \frac{1}{\eta} \\ \max \Big\{ \frac{2(T+1-\eta)}{2(T+2-\eta-\alpha)+\alpha\eta(\eta-4)}, \\ \frac{(\eta-1+\beta)[2(T+1-\eta)+\alpha\eta(\eta-1)]}{2(\eta+\beta)}, \\ \frac{2(T-1+\beta)+\alpha\eta(T-\eta)(\eta+1+2\beta)}{2(\eta+\beta)}, \ 1 \Big\}; \ \textit{if } 0 < \alpha < \frac{1}{\eta} \end{cases} \tag{17}$$

Proof. For $k = 0$, from (7) we get

$$G(0,s) = 2\beta(T+1-s) < 2\beta(T+1) = G(0,0).$$

Then we may choose $M_2 = 1$. For $k \in \mathbb{N}_{1,T}$, if (16) holds, it is sufficient that M_2 satisfies

$$M_2 \ge \max_{(t,s)\in\mathbb{N}_{1,T}\times\mathbb{N}_{1,T}} \frac{G(t,s)}{G(t,t)}. \tag{18}$$

Then we may choose

$$M_2 \le \min\left\{ \max_{(t,s)\in\mathbb{N}_{1,\eta-1}\times\mathbb{N}_{1,T}} \frac{G(t,s)}{G(t,t)}, \max_{(t,s)\in\mathbb{N}_{\eta,T}\times\mathbb{N}_{1,T}} \frac{G(t,s)}{G(t,t)} \right\}. \tag{19}$$

since

$$\max_{(t,s)\in\mathbb{N}_{1,\eta-1}\times\mathbb{N}_{1,T}} \frac{G(t,s)}{G(t,t)}$$

$$\ge \begin{cases} \max_{t\in\mathbb{N}_{1,\eta-1}} \\ \Big\{ \frac{(t-1+\beta)[2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)]}{(t+\beta)[2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)+\alpha t(1-t)]}, \\ \frac{2(T+1)-\alpha\eta(\eta+1)+2(\eta-1)(\alpha\eta-1)+\alpha(\eta-1)(1-t)}{2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)+\alpha t(1-t)}, \\ \frac{2(T+1-\eta)}{2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)+\alpha t(1-t)} \Big\}; \quad \text{if } \alpha > \frac{1}{\eta} \\ \max_{t\in\mathbb{N}_{1,\eta-1}} \\ \Big\{ \frac{(t-1+\beta)[2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)]}{(t+\beta)[2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)+\alpha t(1-t)]}, \\ \frac{2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)+\alpha(\eta-1)(1-t)}{2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)+\alpha t(1-t)}, \\ \frac{2(T+1-\eta)}{2(T+1)-\alpha\eta(\eta+1)+2t(\alpha\eta-1)+\alpha t(1-t)} \Big\}; \quad \text{if } 0 < \alpha < \frac{1}{\eta} \end{cases}$$

$$\ge \begin{cases} \max \Big\{ \frac{(\eta-2+\beta)[2(T+2-\eta)+\alpha\eta(\eta-3)]}{(1+\beta)[2(T+\alpha)-\alpha\eta^2]}, \frac{2(T+2-\eta)+\alpha\eta(\eta-3)}{2(T+\alpha)-\alpha\eta^2}, \\ \frac{2(T+1-\eta)}{2(T+\alpha)-\alpha\eta^2} \Big\}; \quad \text{if } \alpha > \frac{1}{\eta} \\ \max \Big\{ \frac{(\eta-2+\beta)[2T-\alpha\eta(\eta-1)]}{(1+\beta)[2(T+2-\eta-\alpha)+\alpha\eta(\eta-4)]}, \frac{2T-\alpha\eta(\eta-1)}{2(T+2-\eta-\alpha)+\alpha\eta(\eta-4)}, \\ \frac{2(T+1-\eta)}{2(T+2-\eta-\alpha)+\alpha\eta(\eta-4)} \Big\}; \quad \text{if } 0 < \alpha < \frac{1}{\eta} \end{cases}$$

Similarly, we get

$$\max_{(t,s)\in\mathbb{N}_{\eta,T}\times\mathbb{N}_{1,T}} \frac{G(t,s)}{G(t,t)}$$

$$\ge \begin{cases} \max \Big\{ \frac{(\eta-1+\beta)[\alpha\eta(2T-\eta-1)+2]}{2(\eta+\beta)}, \\ \frac{2(T-1+\beta)+\alpha\eta(T-\eta)(\eta+1+2\beta)}{2(\eta+\beta)}, \ 1 \Big\}; \quad \text{if } \alpha > \frac{1}{\eta} \\ \max \Big\{ \frac{(\eta-1+\beta)[2(T+1-\eta)+\alpha\eta(\eta-1)]}{2(\eta+\beta)}, \\ \frac{2(T-1+\beta)+\alpha\eta(T-\eta)(\eta+1+2\beta)}{2(\eta+\beta)}, \ 1 \Big\}; \quad \text{if } 0 < \alpha < \frac{1}{\eta} \end{cases}$$

For $k = T+1$m from (7) we get,

$$\begin{aligned} G(T+1,s) &= \alpha\eta(s+\beta)[2(T+1)-(\eta+1)] \\ &\quad + 2s(T+1+\beta)(1-s) \\ &< \alpha\eta(T+1+\beta)[2(T+1)-(\eta+1)] \\ &\quad + 2T(T+1+\beta)(T+1) \\ &= G(T+1,T+1). \end{aligned}$$

Then we choose $M_2 = 1$. So (18) is immediate from (21)-(22). □

III. Main Results

Now we are in the position to establish the main result.

Theorem 2. *Assume* $(H1)$ - $(H3)$ *hold. Then the problem* (1)-(2) *has at least one positive solution.*

Proof. In the following, we denote

$$m = \min_{t\in\mathbb{N}_{\eta,T}} G(t,t), \quad M = \max_{t\in\mathbb{N}_{T+1}} G(t,t).$$

Then $0 < m < M$.

Let Ebe the Banach's space defined by $E = \{u : \mathbb{N}_{T+1} \to R\}$. Define

$$K = \{u \in E : u \geqslant 0, t \in \mathbb{N}_{T+1} \ and \ \min_{t\in\mathbb{N}_{1,T}} u(t) \geq \sigma \parallel u \parallel\}.$$

where $\sigma = \frac{M_1 m}{M_2 M} \in (0,1), \parallel u \parallel = \max_{t\in\mathbb{N}_{T+1}} \mid u(t) \mid$. It is obvious that K is a cone in E.

We define the operator $F : K \to E$ by

$$(Fu)(t) = \sum_{s=1}^{T} G(t,s)a(s)f(u(s)), t \in \mathbb{N}_{T+1}.$$

It is clear that problem (1)-(2) has a solution u if and only if $u \in K$ is a fixed point of operator F. We shall now show that the operator F maps K to itself. For this, let $u \in K$, from $(H_2) - (H_3)$, we get

$$(Fu)(t) = \sum_{s=1}^{T} G(t,s)a(s)f(u(s)) \geq 0, t \in \mathbb{N}_{T+1}. \quad (20)$$

from (10), we obtain

$$\begin{aligned}(Fu)(t) =& \sum_{s=1}^{T} G(t,s)a(s)f(u(s)) \leq M_2 \sum_{s=1}^{T} G(t,t)a(s)f(u(s)) \\ \leq& M_2 M \sum_{s=1}^{T} a(s)f(u(s)), \quad t \in \mathbb{N}_{T+1}.\end{aligned}$$

Therefore

$$\parallel Fu \parallel \leq M_2 M \sum_{s=1}^{T} a(s)f(u(s)). \quad (21)$$

Now from $(H_2), (H_3)$, (2.7) and (3.2), for $t \in \mathbb{N}_{\eta,T}$, we have

$$\begin{aligned}(Fu)(t) \geq& M_1 \sum_{s=1}^{T} G(t,t)a(s)f(u(s)) \geq M_1 m \sum_{s=1}^{T} a(s)f(u(s)) \\ \geq& \frac{M_1 m}{M_2 M} \parallel Fu \parallel = \sigma \parallel u \parallel .\end{aligned}$$

Then

$$\min_{t\in\mathbb{N}_{\eta,T}} (Fu)(t) \geq \sigma \parallel u \parallel . \quad (22)$$

From (20)-(21), we obtain $Fu \in K$, Hence $F(K) \subseteq K$. So $F : k \to K$ is completely continuous.

Superlinear case. $f_0 = 0$ and $f_\infty = \infty$. Since $f_0 = 0$, we may choose $H_1 > 0$ so that $f(u) \leqslant \epsilon_1 u$, for $0 < u \leqslant H_1$, where $\epsilon_1 > 0$ satisfies

$$\epsilon_1 M_2 M \sum_{s=1}^{T} a(s) \leq 1. \quad (23)$$

Thus, if we let

$$\Omega_1 = \{u \in E : \|u\| < H_1\},$$

then for $u \in K \cap \partial\Omega_1$, we get

$$\begin{aligned}(Fu)(t) \leq& M_2 \sum_{s=1}^{T} G(t,t)a(s)f(u(s)) \leq \epsilon_1 M_2 M \sum_{s=1}^{T} a(s)u(s) \\ \leq& \epsilon_1 M_2 M \sum_{s=1}^{T} a(s)\|u\| \leq \|u\|.\end{aligned}$$

Thus $\|Fu\| \leq \|u\|$, $u \in K \cap \partial\Omega_1$.

Further, since $f_\infty = \infty$, there exists $\widehat{H}_2 > 0$ such that $f(u) \geq \epsilon_2 u$, for $u \geq \widehat{H}_2$, where $\epsilon_2 > 0$ satisfies

$$\epsilon_2 M_1 \sigma \sum_{s=\eta}^{T} G(\eta,\eta)a(s) \geq 1. \quad (24)$$

Let $H_2 = \max\{2H_1, \frac{\widehat{H}_2}{\sigma}\}$ and $\Omega_2 = \{u \in E : \|u\| < H_2\}$. Then $u \in K \cap \partial\Omega_2$ implies

$$\min_{t\in\mathbb{N}_{\eta,T}} u(t) \geq \sigma\|u\| \geq \widehat{H}_2.$$

Applying (9) and (24), we get

$$\begin{aligned}(Fu)(\eta) =& M_1 \sum_{s=1}^{T} G(\eta,s)a(s)f(u(s)) \geq M_1 \sum_{s=\eta}^{T} G(\eta,\eta)a(s)f(u(s)) \\ \geq& \varepsilon_2 M_1 \sum_{s=\eta}^{T} G(\eta,\eta)a(s)y(s) \geq \varepsilon_2 M_1 \sigma \sum_{s=\eta}^{T} G(\eta,\eta)a(s)\|u\| \\ \geq& \|u\|.\end{aligned}$$

Hence, $\|Fu\| \geq \|u\|$, $u \in K \cap \partial\Omega_2$. By the first part of Theorem 1, F has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$ such that $H_1 \leqslant \|u\| \leqslant H_2$.

Sublinear case. $f_0 = \infty$ and $f_\infty = 0$. Since $f_0 = \infty$, choose $H_3 > 0$ such that $f(u) \geqslant \epsilon_3 u$ for $0 < u \leqslant H_3$, where $\varepsilon_3 > 0$ satisfies

$$\epsilon_3 M_1 \sigma \sum_{s=\eta}^{T} G(\eta,\eta)a(s) \geqslant 1. \quad (25)$$

Let

$$\Omega_3 = \{u \in E : \|u\| < H_3\},$$

then for $u \in K \cap \partial\Omega_3$, we get

$$\begin{aligned}(Fu)(\eta) \geq& M_1 \sum_{s=\eta}^{T} G(\eta,\eta)a(s)f(u(s)) \geq \epsilon_3 M_1 \sum_{s=\eta}^{T} G(\eta,\eta)a(s)y(s) \\ \geq& \varepsilon_3 M_1 \sigma \sum_{s=\eta}^{T} G(\eta,\eta)a(s)\|u\| \geq \|u\|.\end{aligned}$$

Thus, $\|Fu\| \geqslant \|u\|$, $u \in K \cap \partial\Omega_3$.

Now, since $f_\infty = 0$, there exists $\widehat{H}_4 > 0$ so that $f(u) \leqslant \epsilon_4 u$ for $u \geqslant \widehat{H}_4$, where $\epsilon_4 > 0$ satisfies

$$\epsilon_4 M_2 M \sum_{s=\eta}^{T} a(s) \geqslant 1. \tag{26}$$

Subcase 1. Suppose f is bounded, $f(u) \leq L$ for all $u \in [0,\infty)$ for some $L > 0$. Let $H_4 = \max\{2H_3, LM_2M\sum_{s=1}^{T} a(s)\}$.

Then for $u \in K$ and $\|u\| = H_4$, we get

$$\begin{aligned}(Fu)(\eta) \leq & M_2 \sum_{s=1}^{T} G(t,t)a(s)f(u(s)) \leq LM_2M\sum_{s=1}^{T} a(s)\\ \leq & H_4 = \|u\|\end{aligned}$$

Thus $(Fu)(t) \leq \|u\|$.

Subcase 2. Suppose f is unbounded, there exist $H_4 > \max\{2H_3, \frac{\widehat{H}_4}{\sigma}\}$ such that $f(u) \leq f(H_4)$ for all $0 < u \leq H_4$. Then for $u \in K$ with $\|u\| = H_4$ from (10) and (26), we have

$$\begin{aligned}(Fu)(t) \leq & M_2 \sum_{s=1}^{T} G(t,t)a(s)f(u(s)) \leq M_2M\sum_{s=1}^{T} a(s)f(H_4)\\ \leq & \epsilon_4 M_2 M \sum_{s=1}^{T} a(s)H_4 \leq H_4 = \|u\|.\end{aligned}$$

Thus in both cases, we may put $Omega_4 = \{u \in E : \|u\| < H_4\}$. Then

$$\|Fu\| \leqslant \|u\|, u \in K \cap \partial\Omega_4.$$

By the second part of Theorem 1, A has a fixed point u in $K \cap (\overline{\Omega}_4 \setminus \Omega_3)$, such that $H_3 \leqslant \|u\| \leqslant H_4$. This completes the sublinear part of the theorem. Therefore, the problem (1)-(2) has at least one positive solution. □

IV. Some examples

In this section, in order to illustrate our result, we consider some examples.

Example 1 Consider the BVP

$$\Delta^2 u(t-1) + t^2 u^k = 0, \qquad t \in N_{1,4}, \tag{27}$$

$$u(0) = \frac{1}{4}\Delta u(0), \qquad u(5) = \frac{2}{3}\sum_{s=1}^{2} u(s). \tag{28}$$

Set $\alpha = \frac{2}{3},\ \beta = \frac{1}{4},\ \eta = 2,\ T = 4,\ a(t) = t^2,\ f(u) = u^k$. We can show that

$$2(T+1) - \alpha\eta(\eta+1) - 2\beta(\alpha\eta - 1) = \frac{40}{6} > 0.$$

Case I : $k \in (1,\infty)$. In this case, $f_0 = 0,\ f_\infty = \infty$ and (i) of theorem 2 holds. Then BVP (27)-(28) has at least one positive solution.
Case II : $k \in (0,1)$. In this case, $f_0 = \infty,\ f_\infty = 0$ and (ii) of theorem 2 holds. Then BVP (27)-(28) has at least one positive solution.

Example 2 Consider the BVP

$$\Delta^2 u(t-1) + e^t t^e \left(\frac{\pi \sin u + 2\cos u}{u^2}\right) = 0, \qquad t \in N_{1,4}, \tag{29}$$

$$u(0) = \frac{2}{5}\Delta u(0), \qquad u(5) = \frac{1}{3}\sum_{s=1}^{3} u(s), \tag{30}$$

Set $\alpha = \frac{1}{3},\ \beta = \frac{2}{5},\ \eta = 3,\ T = 4,\ a(t) = e^t t^e,\ f(u) = \frac{\pi \sin u + 2\cos u}{u^2}$.
We can show that

$$\Lambda = 2(T+1) - \alpha\eta(\eta+1) - 2\beta(\alpha\eta - 1) = 6 > 0,$$

Through a simple calculation we can get $f_0 = \infty,\ f_\infty = 0$. Thus, by (ii) of theorem 2, we can get BVP (29)-(30) has at least one positive solution.

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