Nonlinear Equations with N-dimensional Telegraph Operator Iterated K-times

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Abstract—In this article, using distribution kernel, we study the nonlinear equations with n-dimensional telegraph operator iterated k-times.

Keywords—Telegraph operator, Elementary solution, Distribution kernel.

I. INTRODUCTION

The telegraph equation arises in the study of propagation of electrical signals in a cable of transmission line and wave phenomena. The interaction of convection and diffusion or reciprocal action of reaction and diffusion describes a number of nonlinear phenomena in physics, chemistry and biology. Further, the telegraph equation is more suitable than ordinary diffusion in modeling reaction-diffusion for such branches of applied sciences. We refer the reader to [1]-[4] and the references therein.

Kamathai [5]-[6] has studied some properties and results of the distribution \( e^{\alpha x} \partial^k \delta \) and solved the convolution equation

\[
e^{\alpha x} \partial^k \delta \ast u(x) = e^{\alpha x} \sum_{r=0}^{m} C_r \partial^r \delta,
\]

which is related to the ultra-hyperbolic equation, where \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \), \( \alpha x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n \). \( C_r \) are given constants for \( r = 1, 2, \ldots, m \). \( \partial^k \delta \) is the n-dimensional ultra-hyperbolic operator iterated \( k \) times defined by

\[
\partial^k = \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k
\]

with \( p + q = n \) and \( \delta \) is the Dirac-delta distribution with \( \partial^k \delta = \delta, \partial^1 \delta = \delta \).

In this work, by applying the distribution \( e^{\alpha x} \partial^k \delta \), we study the elementary solution of the following n-dimensional telegraph equation

\[
\left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k u(x, t) := T^k u(x, t) = \delta(x, t),
\]

where \( \Delta \) is the n-dimensional Laplacian operator iterated \( k \) times defined by

\[
\Delta^k = \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right)^k.
\]

\( \beta \) is a positive constant. As an application, we solve the nonlinear equation with n-dimensional telegraph operator iterated \( k \)-times of the form

\[
\left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \beta \frac{\partial}{\partial t} + \beta^2 - \Delta \right)^k u(x, t) = f(x, t),
\]

where \( f(t, x) \) is a generalized function.

II. SOME DEFINITIONS AND LEMMAS

Definition 1. Let \( x = (x_1, x_2, \ldots, x_n) \) be a point of \( \mathbb{R}^n \) and write

\[
v = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2, \quad p + q = n.
\]

Define by \( \Gamma_+ = \{ x \in \mathbb{R}^n : x_1 > 0 \text{ and } v > 0 \} \) designating the interior of forward cone and \( \Gamma_+ \) designating its closure.

For any complex number \( \gamma \), we define the function

\[
R^{\gamma}_n(v) = \begin{cases} \frac{\Gamma(n/2 + \gamma)}{\Gamma(n/2)} & \text{if } x \in \Gamma_+, \\ 0 & \text{if } x \notin \Gamma_+, \end{cases}
\]

where the constant \( K_n(\gamma) \) is given by the formula

\[
K_n(\gamma) = \frac{\pi^{(n-1)/2} \Gamma \left( \frac{2+n}{2} \right)}{\Gamma(\frac{n}{2})}.
\]

Let \( \text{supp } R^{\gamma}_n(v) \subset \Gamma_+ \) where \( \text{supp } R^{\gamma}_n(v) \) denotes the support of \( R^{\gamma}_n(v) \). The function \( R^{\gamma}_n(v) \) is first introduced by Nozaki [7] and is called the ultra-hyperbolic kernel of Marcel Riesz. Moreover, \( R^{\gamma}_n(v) \) is an ordinary function if \( \text{Re}(\gamma) \geq n \) and is a distribution of \( \gamma \) if \( \text{Re}(\gamma) < n \).

Definition 2. Let \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \) and write

\[
s = x_1^2 + x_2^2 + \cdots + x_n^2.
\]

For any complex number \( \beta \), define the function

\[
R^{\beta}_n(s) = 2^{-\beta} \pi^{-n/2} \Gamma \left( \frac{n-\beta}{2} \right) \frac{s^{(\beta-n)/2}}{\Gamma \left( \frac{\beta}{2} \right)}
\]

The function \( R^{\beta}_n(s) \) is called the elliptic kernel of Marcel Riesz and is ordinary function if \( \text{Re}(\beta) \geq n \) and is a distribution of \( \beta \) if \( \text{Re}(\beta) < n \).

Lemma 1. [5] Let \( L \) be the partial differential operator defined by

\[
L = \square - 2 \left( \sum_{i=1}^{p} \alpha_i \frac{\partial}{\partial x_i} - \sum_{j=p+1}^{p+q} \alpha_j \frac{\partial}{\partial x_j} \right) + \left( \sum_{i=1}^{p} \alpha_i^2 - \sum_{j=p+1}^{p+q} \alpha_j^2 \right).
\]
Theorem 1. Let $T^k$ be the partial differential operator which iterated $k$-times defined by

$$T^k = \left(\frac{\partial^2}{\partial x^2} + 2\beta \frac{\partial}{\partial t} + \beta^2 - \Delta\right)^k,$$

where $\Delta$ is the n-dimensional Laplacian operator and $\beta$ is a given positive constant. Then, $u(x,t) = e^{-\beta t}M_{2k}(w)$ is a unique elementary solution of (1), where $M_n(w)$ is defined by

$$M_n(w) = \frac{w^{(n-n/2)}}{\Gamma(n+1/2)} \text{ if } t \in \Gamma_+,$$

$$0 \text{ if } t \notin \Gamma_+,$$

where $w = t^2 - x^2_1 - x^2_2 - \cdots - x^2_n$, $t$ is the time and $H_{n+1}(n) = \pi^{(n-1)/2} \frac{\Gamma(n+1/2)}{\Gamma(n/2)}$. (10)

Proof. Firstly, we define the $n+1$-dimensional ultra-hyperbolic operator as

$$\square_{n+1} = \left(\frac{\partial^2}{\partial x^2} - \Delta\right).$$

Setting $\alpha_2 = \alpha_3 = \cdots = \alpha_n = 0$, we have

$$e^{\alpha_1 t} (\frac{\partial^2}{\partial x^2} - \Delta)^k \delta(x,t).$$

Applying Lemma 2 for $p = 1$, $q = n$ and $p + q = n + 1$, (3) and (4) are reduced to (9) and (10), respectively. Indeed, we have $\delta(x,t) = \delta(x)\delta(t)$ and $e^{\alpha_1 t}\delta(x) = \delta(x).$

Using Lemma 2, we get

$$e^{\alpha_1 t} (\frac{\partial^2}{\partial x^2} - \Delta) \delta(x,t) = e^{\alpha_1 t} \frac{\partial^2}{\partial x^2} \delta(x,t) - e^{\alpha_1 t} \Delta \delta(x,t).$$

$$= \left(\frac{\partial}{\partial t} - \alpha_1\right)^2 \delta(x,t) - \Delta e^{\alpha_1 t} \delta(x,t)$$

$$= \left(\frac{\partial^2}{\partial x^2} - 2\alpha_1 \frac{\partial}{\partial t} + \alpha_1^2 - \Delta\right) \delta(x,t).$$

Substituting $\alpha_1 = -\beta$, it follows that

$$e^{-\beta t} (\frac{\partial^2}{\partial x^2} - \Delta) \delta(x,t) = \left(\frac{\partial^2}{\partial x^2} + 2\beta \frac{\partial}{\partial t} + \beta^2 - \Delta\right) \delta(x,t)$$

$$= T \delta(x,t).$$

Convolving $k$-times for both sides of the above equation by $e^{-\beta t}(\frac{\partial^2}{\partial t^2} - \Delta)\delta(x,t)$, we have

$$e^{-\beta t} (\frac{\partial^2}{\partial x^2} - \Delta) \delta(x,t) = 0.$$

Then (1) can be written as

$$T^k u(x,t) = e^{-\beta t} (\frac{\partial^2}{\partial x^2} - \Delta) \delta(x,t) = \delta(x,t).$$

Convolving both sides of the above equation by $e^{-\beta t}M_{2k}(w)$ and applying Lemma 1, we have

$$u(x,t) = e^{-\beta t}M_{2k}(w),$$

where $M_{2k}(w)$ is defined by (9) with $\eta = 2k$.

Theorem 2. Given the equation

$$\left(\frac{\partial^2}{\partial x^2} + 2\beta \frac{\partial}{\partial t} + \beta^2 - \Delta\right) u(x,t) = f(x,t),$$

where $f(x,t)$ is a given generalized function and $u(x,t)$ is an unknown function. Then,

$$u(x,t) = e^{-\beta t}M_{2k}(w) \ast f(x,t),$$

where $M_{2k}(w)$ is defined by (9) with $\eta = 2k$.

Proof. Convoluting both sides of (11) by $e^{-\beta t}M_{2k}(w)$ and applying Theorem 1, we obtain (12) as required.

Remark 3. By using the method of proving Theorem 1 together with suitable modifications, we have $u(x,t) = e^{-\beta t}(-1)^k R_{2k}^n(s)$ is a unique elementary solution of the following equation

$$\left(\frac{\partial^2}{\partial x^2} + 2\beta \frac{\partial}{\partial t} + \beta^2 - \Delta\right) u(x,t) = \delta(x,t),$$

where $R_{2k}^n(s)$ is defined by Definition 2 with $\beta = 2k$, $s = t^2 + x^2_1 + x^2_2 + \cdots + x^2_n$ and a constant $n$ in (5) is replaced by $n + 1$.

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REFERENCES