Stability Analysis of Fractional Order Systems with Time Delay

Hong Li, Shou-Ming Zhong, Hou-Biao Li

Abstract—In this paper, we mainly study the stability of linear and interval linear fractional systems with time delay. By applying the characteristic equations, a necessary and sufficient stability condition is obtained firstly, and then some sufficient conditions are deserved. In addition, according to the equivalent relationship of fractional order systems with order $0 < \alpha \le 1$ and with order $1 \le \beta < 2$, one may get more relevant theorems. Finally, two examples are provided to demonstrate the effectiveness of our results.

Keywords—Fractional order systems, Time delay, Characteristic equation.

I. INTRODUCTION

RECENTLY, fractional order systems have considerable importance mainly due to the following two facts. First, fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes, such as dielectric [1], electrolyte polarization [2] and electromagnetic wave [3]. The advantages of the fractional order systems are that we have more degrees of freedom in the model and that a memory is included in the model. Second, fractional order controllers such as CRONE controller [4], TID controller [5] and fractional PID controller [6] have so far been implemented to enhance the robustness and the performance of the closed loop control system.

For commensurate fractional order systems, some powerful criteria have been proposed. The most well known one is Matignon's stability theorem [7]. It permits us to check the system stability through the location in the complex plane of the dynamic matrix eigenvalues of the state space like system representation. Matignon's theorem is the starting point of several results in the field. In addition, LMI approach ([8]-[10]) and Lyapunov approach ([11,12]) are also used to investigate the stability of fractional order linear time invariant systems.

Recently, a finite time stability analysis of fractional time delay systems is firstly presented and reported on paper [13]. In addition, in [14], a stability test procedure is proposed for linear nonhomogeneous fractional order systems with a pure time delay by using a recently obtained generalized Gronwall's inequality.

In this paper, we mainly research the stability of linear fractional systems with time delay. By using the characteristic

Hong Li is with the School of Mathematical Science, University of Electronic Science and Technology, Chengdu, Sichuan, 610054 P. R. China (e-mail: sichuanhong@163.com).

Shou-ming Zhong is with Key Laboratory for NeuroInformation of Ministry of Education, University of Electronic Science and Technology of China, Chengdu 611731, P.R. China.

Hou-biao Li is with the School of Mathematical Science, University of Electronic Science and Technology, Chengdu, Sichuan, 610054 P. R. China (e-mail: lihoubiao0189@163.com).

equation for the system, we firstly give a necessary and sufficient stability condition, then, a sufficient condition is deserved. We also studied the stability of interval linear fractional system with time delay. Finally, two examples are provided to demonstrate the effectiveness of our results.

II. PROBLEM FORMULATION AND PRELIMINARIES

The differ-integral operator, denoted by $_aD_t^\alpha$, is a combined differentiation and integration operator commonly used in fractional calculus which is defined by:

$$_aD_t^\alpha = \begin{cases} \frac{d^\alpha}{dt^\alpha}, & \alpha > 0 \\ 1, & \alpha = 0 \\ \int_a^t (d\tau)^{-\alpha}, & \alpha < 0. \end{cases}$$

However, there exist some different definitions for fractional derivatives [15]. The most commonly used definitions are the Grünwald-Letnikov, Riemann-Liouville and Caputo ones. The Caputo definition is sometimes called smooth fractional derivative in literature because it is suitable to be treated by the Laplace transform technique, while the Riemann-Liouville definition is unsuitable.

In the rest of the paper, D^α is used to denote the Caputo fractional derivative of order α

$$D^\alpha f(t) = \frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(\alpha - m)} \int_0^t \frac{f^{(m)}(\tau)}{(t - \tau)^{\alpha+1-m}} d\tau, \quad (1)$$

where m is an integer satisfying $m - 1 < \alpha \le m$.

This paper focuses on the case that the fractional order is $0 < \alpha < 1$.

Firstly, let us consider the linear fractional system with time delay described by the following form:

$$\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + Bx(t - \tau) \qquad (2)$$

with the initial condition $x(t_0 + t) = \psi(t) \in C[-\Delta, 0]$. Here $0 < \alpha < 1$ is the fractional commensurate order, $x(t) \in \mathbb{R}^n$ denotes the state vector, A and $B \in \mathbb{R}^{n \times n}$ are the system matrices, and $\tau > 0$ is pure time delay.

If the matrix A, B is uncertain, then the FO-LTI interval system can be described by the state space equation of the form

$$\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) - Bx(t - \tau) \qquad (3)$$

where

$$A \in [A^m, A^M] = \{[a_{ij}] : a^m_{\ ij} \le a_{ij} \le a^M_{\ ij}, 1 \le i, j \le n\},$$

$$B \in [B^m, B^M] = \{[b_{ij}] : b^m_{\ ij} \le b_{ij} \le b^M_{\ ij}, 1 \le i, j \le n\}.$$

World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:8, No:4, 2014

If the fractional commensurate order α is uncertain, then the fractional-order interval system without delay can be described by the state space equation of the form

$$\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) \qquad (4)$$

where $\alpha \in [\alpha_1, \alpha_2], \alpha_1, \alpha_2 \in \mathbb{R}$.

To prove the main results in the next section, we need the following lemma.

Lemma 1 ([16]). Let R, T, and $V \in \mathbb{C}^{\kappa \times \kappa}$. If $|R| \leq v$, then

$$|RT| \leq |R||T| \leq V|T|,$$

$$|R + T| \leq |R| + |T| \leq V + |T|,$$

$$\rho(R) \leq \rho(|R|) \leq \rho(V),$$

$$\rho(R + T) \leq \rho(|R + T|) \leq \rho(|R| + |T|) \leq \rho(V + |T|),$$

$$\rho(RT) \leq \rho(|R||T|) \leq \rho(V|T|).$$

III. MAIN RESULTS

Theorem 1. If all the roots of the characteristic equation $D(s) = det(s^\alpha I - A - Be^{-s\tau}) = 0$ have negative real parts, then the zero solution of system (2) is Lyapunov asymptotically stable.

Proof. Taking Laplace transform on both sides of system (2), we obtain

$$s^\alpha X(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^k(0) = \qquad (5)$$
$$AX(s) + Be^{-s\tau} X(s) + Be^{-s\tau} \int_{-\tau}^{0} e^{-st} x(t) dt,$$

where $X(s)$ is the Laplace transform of $x(t)$ with $X(s) = \mathfrak{L}(x(t))$. Next, we rewrite (5) as follows:

$$D(s)X(s) = b(s) \qquad (6)$$

in which

$$D(s) = det(s^\alpha I - A - Be^{-s\tau}),$$

$$b(s) = \sum_{k=0}^{n-1} s^{\alpha-k-1} f^k(0) + Be^{-s\tau} \int_{-\tau}^{0} e^{-st} x(t) dt.$$

then, according to the paper [17], we can easily prove the theorem. ∎

In addition, let $F(s) = (s^\alpha I - A)^{-1}$, and F_M is the matrix formed by taking the maximum magnitude of each element of $F(s)$ for $\mathfrak{R}(s) > 0$. Then we have the following theorem.

Theorem 2. The system (2) is Lyapunov asymptotically stable, if the following inequalities are satisfied:

$$(1). \quad det(s^\alpha I - A) \neq 0, \text{for } \mathfrak{Re}(s) \geq 0; \qquad (7)$$
$$(2). \quad \rho(F_M|B|) < 1.$$

Proof. The characteristic equation of the system (2) is $D(s) = 0$. Since $det(s^\alpha I - A) \neq 0$, for $Re(s) \geq 0$, so $(s^\alpha I - A)^{-1}$ exists. We have

$$D(s) = det(s^\alpha I - A - Be^{-s\tau})$$
$$= det(s^\alpha I - A)det(I - (s^\alpha I - A)^{-1}Be^{-s\tau}). \qquad (8)$$

Therefore, if we can show that

$$det(I - (s^\alpha I - A)^{-1}Be^{-s\tau}) \neq 0, \quad \text{for } \mathfrak{R}(s) > 0, \qquad (9)$$

then the system (2) is Lyapunov asymptotically stable. In fact, if we can show that $\rho((s^\alpha I - A)^{-1}Be^{-s\tau}) < 1$, for $\mathfrak{R}(s) > 0$, (9) is satisfied by Lemma 1.

Now, using the Lemma 1, we obtain

$$\rho((s^\alpha I - A)^{-1}Be^{-s\tau}) \leq \rho(|F(s)||Be^{-s\tau}|)$$
$$\leq \rho(F_M|B|) < 1, \text{ for } \mathfrak{R}(s) > 0. \qquad (10)$$

Thus the proof is completed. ∎

Next, we will discuss the system (3). Let

$$A \in [A_1, A_2], \ B \in [B_1, B_2]; \ \bar{A} = \frac{1}{2}(A_1 + A_2), \ \bar{B} = \frac{1}{2}(B_1 + B_2);$$

$$\Delta A = A - \bar{A}, \ \Delta B = B - \bar{B}; \ A_M = A_2 - \bar{A}, \ B_M = B_2 - \bar{B},$$

we can get

$$|\Delta A| \leq A_M, \quad |\Delta B| \leq B_M.$$

Theorem 3. The system (3) is Lyapunov asymptotically stable, if the following inequalities are satisfied:

$$(1). \quad det(s^\alpha I - A) \neq 0, \text{for } \mathfrak{Re}(s) \geq 0; \qquad (11)$$
$$(2). \quad \rho(F_M(A_M + |\bar{B}| + B_M)) < 1.$$

Proof. The characteristic equation of the system (3) is

$$D(s) = det(s^\alpha I - (A + \Delta A) - (B + \Delta B)e^{-s\tau}) = 0.$$

Since $det(s^\alpha I - A) \neq 0$, for $\mathfrak{Re}(s) \geq 0$ exists. We have

$$D(s) = det(s^\alpha I - (A + \Delta A) - (B + \Delta B)e^{-s\tau})$$
$$= det(s^\alpha I - A)det(I - (s^\alpha I - A)^{-1}(\Delta A + (B + \Delta B)e^{-s\tau})). \qquad (12)$$

In the view of $|\arg \lambda(A)| > \frac{\pi\alpha}{2}$, $(s^\alpha I - A)^{-1}$ exists. Therefore, if we can show that, for $\mathfrak{R}(s) > 0$,

$$det(I - (s^\alpha I - A)^{-1}(\Delta A + (B + \Delta B)e^{-s\tau})) \neq 0, \qquad (13)$$

then the system (3) is Lyapunov asymptotically stable. In fact, if we can show that $\rho((s^\alpha I - A)^{-1}(\Delta A + (B + \Delta B)e^{-s\tau})) < 1$, for $\mathfrak{R}(s) > 0$, the equation (13) is satisfied by Lemma 1.

Now, by Lemma 1, we may obtain

$$\rho((s^\alpha I - A)^{-1}(\Delta A + (B + \Delta B)e^{-s\tau}))$$
$$\leq \rho(|F(s)||\Delta A + (B + \Delta B)e^{-s\tau}|) \qquad (14)$$
$$\leq \rho(F_M(A_M + |\bar{B}| + B_M)) < 1, \text{for } \mathfrak{R}(s) > 0,$$

This completes the proof. ∎

In addition, according to the following equivalent theorem of fractional order systems with order $0 < \alpha \leq 1$ and with order $1 \leq \beta < 2$, one can obtain some other analogical conclusions on the order $1 \leq \beta < 2$ systems by the corresponding ones on the order $0 < \alpha \leq 1$ systems. Here we do not describe in detail.

Theorem 4. ([18]) All eigenvalues of the FO-LTI system (4) with order $0 < \alpha \leq 1$ and output $u(t) = 0$ settle in the unstable region if and only if the fractional-order system

$$\frac{d^\beta x(t)}{dt^\beta} = -Ax(t), \quad 1 \leq \beta \triangleq 2 - \alpha < 2 \qquad (15)$$
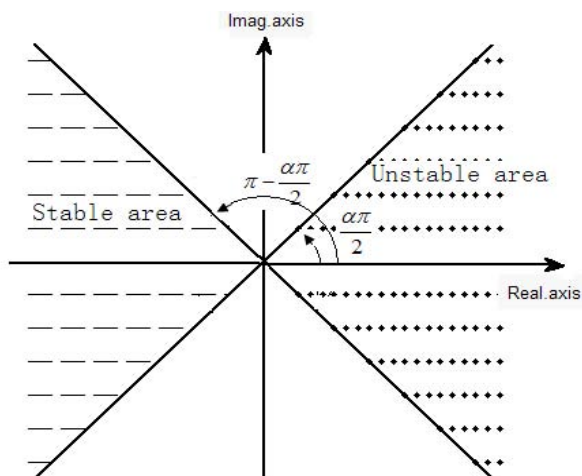
is asymptotically stable, see Fig. 1.

World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:8, No:4, 2014

Fig. 1   Relation of stability domain and unstability domain

IV. Numerical Examples

Example 1. Consider the stability of the following FO-LTI system with delay

$$\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + Bx(t - \tau) \tag{16}$$

where $\alpha = \frac{1}{2}$, and

$$A = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0.25 & -0.25 \\ -0.25 & 0.25 \end{bmatrix}.$$

Firstly, note that $det(s^\alpha I - A) = s^{2\alpha} + 2$, so $det(s^\alpha I - A) \neq 0$ for $\Re e(s) \geq 0$.

$$(s^\alpha I - A)^{-1} = \begin{bmatrix} \frac{s^\alpha}{2+s^{2\alpha}} & -0.5 + \frac{s^{2\alpha}}{4+2s^{2\alpha}} \\ \frac{2}{2+s^{2\alpha}} & \frac{s^{2\alpha}}{2+s^{2\alpha}} \end{bmatrix},$$

so

$$F_M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Since the eigenvalues of matrix $F_M|B|$ are $\lambda_1 = 0, \lambda_2 = 0.75$. Therefore, from Theorem 2, we know that the fractional system (15) is Lyapunov stable.

Example 2. Consider the stability of the following interval FO-LTI system with delay

$$\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + Bx(t - \tau) \tag{17}$$

where $\alpha = \frac{1}{2}$, and $B \in [B_1, B_2]$

$$A = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} -0.1 & -0.2 \\ -0.5 & -0.2 \end{bmatrix}, B_2 = \begin{bmatrix} 0.1 & 0.2 \\ -0.3 & 0.4 \end{bmatrix}.$$

Firstly, we have

$$(s^\alpha I - A)^{-1} = \begin{bmatrix} \frac{s^\alpha}{2+s^{2\alpha}} & -0.5 + \frac{s^{2\alpha}}{4+2s^{2\alpha}} \\ \frac{2}{2+s^{2\alpha}} & \frac{s^{2\alpha}}{2+s^{2\alpha}} \end{bmatrix},$$

so

$$F_M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \bar{B} = \begin{bmatrix} 0 & 0 \\ -0.4 & 0.1 \end{bmatrix},$$

$$\Delta A = 0, B_M = B_2 - \bar{B} = \begin{bmatrix} 0.1 & 0.2 \\ 0.1 & 0.3 \end{bmatrix},$$

$$F_M(|\bar{B}| + B_M) = \begin{bmatrix} 0.1 & 0.2 \\ 0.6 & 0.6 \end{bmatrix}.$$

Since eigenvalues of the matrix $F_M(|\bar{B}| + B_M)$ are $\lambda_1 = 0.4$, $\lambda_2 = -0.1$, so $\rho(F_M(|\bar{B}| + B_M)) < 1$. That is, from Theorem 3, the interval FO-LTI system (16) is Lyapunov stable.

V. Conclusions

In summary, this paper mainly presents a brief necessary and sufficient condition and some sufficient conditions for the stability of a class of FO-LTI system with uncertain parameters. The proposed method here is quite different from other ones in literature. Two simple examples also demonstrate that this method is feasible.

References

[1] I. Podlubuy, Fractional differential equations. New York: Academic Press, 1999.
[2] B. O'Neill, R. Jeanloz, Geophys. Res. Lett. 17 (1990) 1477.
[3] J.J. Ita, L. Stixrude, J. Geophys. Res. 97 (1992) 6849.
[4] A. Oustaloup, X. Morean, M. Nouiuant, The CRONE suspension. Control Eng. Pract. 4(8)(1996) 1101-1108.
[5] B.J. Lurie, Tunable TID controller. US Patent 5, 371, 630, December 6, 1944.
[6] I. Podlubuy, Fractional-order systems and $PI^\lambda D^u$-controllers, IEEE Trans. Automat. Control, 44(1)(1999) 208-214.
[7] D. Matignon, Stability results on fractional differential equations to control processing, in: peocessings of Computational Engineering in Syatems and Application Multiconference, vol.2, IMACS, IEEE-SMC, 1996, 963-968.
[8] Y.X. Sheng, J.G. Lu, Robust stability and stabilization of fractional-order linear systems with nonlinear ncertain parameters: An LMI approach, Chaos, Solitons and Fractals, 42(2009) 1163-1169.
[9] J. Sabatier, M. Moze, C. Farges, LMI stability conditions for fractional order systems. Computers and Mathematics with Applications, 59(2010) 1594-1609.
[10] J.G. Lu, Y.Q. Chen, Robust stability and stabilization of fractional-order interval systems with the Fractional Order α: The $0 < \alpha < 1$ Case, IEEE Transactions on Automatic Control, 55(1) 2010, 152-158.
[11] H.S. Ahn, Y.Q. Chen, I.Podlubny, Robust stability test of a class of linear time-invariant interval fractional-order system using Lypunov inequality. Applied Mathematics and Computation, 187(1) 2007, 27-34.
[12] J.C. Trigeassou, N. Maamri, J. Sabatier, A. Oustaloup, A Lyapunov approach to the stability of fractional differential equations. Signal Processing, 91(2011) 437-44.
[13] M.P. Lazarević, Finite time stability analysis of fractional control of robotic time-delay systems: Gronwall's approach, Mathematical and Computer Modelling, 49(2009) 475-481.
[14] M.P. Lazarević, A.M. Spasić, Finite-time stability analysis of fractional order time-delay systems, San Diego, 1999.
[15] Z. Liao, C. Peng, W. Li, Y. Wang, Robust Stability Analysis for a class of Fractional order Systems with Uncertain Parameters, Journal of the Franklin Institute, 348(2011) 1101-1113.
[16] Tingzhu Huang, Shouming Zhong and Zhengliang Li, Matrix Theory. Beijing, Higher Education Press, 2003.

World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:8, No:4, 2014

[17] Weihua Deng, Changpin Li, Jinhu Lü, Stability analysis of linear fractional differential system with multiple time delays, Nonlinear Dyn., 48(2007) 409-416.

[18] Hong Li, Shou-ming Zhong and Hou-biao Li. Stability of Interval Fractional-order systems with order $0 < \alpha < 1$. International Journal of Computational and Mathematical Sciences, 6 (2012): 89-93.

Hong Li was born in 1979 in Sichuan, China. She received B.S. degree from China University of Geosciences(CUG), Wuhan, China, in 2002, and M.S. degrees in applied mathematics from university of electronic science and technology of China (UESTC), China, in 2006. She is currently as a lecturer in the school of mathematics sciences in UESTC and is pursuing a Ph.D degree. Her current research interests is mainly fractional-order systems. Email: sichuanhong@163.com.

Shou-ming Zhong was born in 1955 in Sichuan, China. He received B.S. degree in applied mathematics from UESTC, Chengdu, China, in 1982. From 1984 to 1986, he studied at the department of mathematics in Sun Yatsen University, Guangzhou, China. From 2005 to 2006, he was a visiting research associate with the department of mathematics in university of Waterloo, Waterloo, Canada. He is currently as a full professor with school of mathematics sciences, UESTC. His current research interests include differential equations, neural networks, biomathematics and robust control. He has authored more than 80 papers in reputed journals such as the International Journal of Systems Science, Applied Mathematics and Computation, Chaos, Solitons and Fractals, Dynamics of Continuous, Discrete and Impulsive Systems, Acta Automatica Sinica, Journal of Control Theory and Applications, Acta Electronica Sinica, Control and Decision, and Journal of Engineering Mathematics etc.

Hou-biao Li received the M.Sc. and Ph.D. degrees in computational and applied mathematics from university of electronic science and technology of China (UESTC), China, in 2005 and 2007, respectively. He currently is an associate professor with the School of Mathematics Sciences, UESTC. His research interests involve numerical linear algebra, preconditioning technology and computational mathematics, etc. Email: lihoubiao0189@163.com