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Unit Root Tests Based On the Robust Estimator

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***Abstract*—**The unit root tests based on the robust estimator for the first-order autoregressive process are proposed and compared with the unit root tests based on the ordinary least squares (OLS) estimator. The percentiles of the null distributions of the unit root test are also reported. The empirical probabilities of Type I error and powers of the unit root tests are estimated via Monte Carlo simulation. Simulation results show that all unit root tests can control the probability of Type I error for all situations. The empirical power of the unit root tests based on the robust estimator are higher than the unit root tests based on the OLS estimator.

***Keywords*—**Autoregressive, Ordinary least squares, Type I error, Power of the test, Monte Carlo simulation.

I. Introduction

HAMINTON [1] described the econometric applications for the first-order autoregressive process in time series analysis. He also discussed the necessity of using the unit root test in order to find the correct model for the nominal interest rate and real GNP of the United States from the period of 1947 to 1989. Hamilton [1] indicated that there are no guarantees in economic theory to suggest that nominal interest rates should exhibit a deterministic time trend, although the time series data display an upward trend over the sample data. Consequently, the model for these time series may be a random walk process without trend or a stationary process with a constant term. In order to answer this question, the unit root test can be applied to select between these two processes. The unit root test has drawn much attention over the past three decades, especially in economics and other related fields. Statisticians and econometricians are interested in the unit root test since economic time series data may be non-stationary. Contributions to the unit root literature include the works of Fuller [2], Dickey and Fuller [3], [4], Said and Dickey [5], [6], Phillips [7], Phillips and Perron [8], Hall [9], Pantula and Hall [10], Lucas [11], Park [12], Paparoditis and Politis [13], among others.

The first-order autoregressive process $\{y_t, t = 1, 2, ..., n\}$ denoted as AR (1) is given by

$$y_t = \delta + \rho y_{t-1} + e_t, \qquad (1)$$

where $\delta = \mu(1-\rho)$, μ is the mean of the process, ρ is the autoregressive coefficient and e_t are a sequence of independent and identically random variables from a normal distribution with zero mean and variance σ_e^2. Defining the AR polynomial by $\rho(z) = 1 - \rho z$, we can rewrite the process as

$$\rho(B)(y_t - \mu) = e_t,$$

where B is the backward shift operator such that $B^k y_t = y_{t-k}$. Equation (1) is called a stationary AR(1) process if and only if the root of the AR characteristic equation $(\rho(z) = 0)$ exceeds 1 in absolute value, i.e., $|\rho| < 1$, otherwise it is called a non-stationary process or random walk process. In the case of a near non-stationary process, i.e., $|\rho| \to 1$, the mean and variance of this process change over time.

The null hypothesis; H_0 and the alternative hypothesis; H_a for the unit root tests are as follows:

$$H_0 : \rho = 1 \text{ and } H_a : \rho < 1.$$

A common feature of almost all unit root tests is that they make use of the ordinary least squares (OLS) estimator. Although the OLS estimator has asymptotic normality for $|\rho| < 1$ (see [14]; [15], it has long been known that the OLS estimator can have large bias and is sensitive to the occurrence of outliers in the data; see, for example, [16]-[18]. There have been useful improvements in parameter estimation so as to reduce the bias of the OLS estimator. Denby and Martin [19] presented the robust estimator for an autoregressive model. Gonzalez-Farias and Dickey [20] considered maximum likelihood (ML) estimation for the parameters of the autoregressive process and suggested tests for unit roots on the basis of these estimators. Park and Fuller [21] proposed the weighted symmetric estimator of an autoregressive parameter. Fuller [2] presented a modification of the weighted symmetric estimator. Shin and So [22] developed an adaptive maximum likelihood procedure. Guo [23] developed the simple and robust estimator for an AR(1) model. However, Guo [23] and others do not develop the testing for a unit root based on the robust estimator proposed by Guo [23]. Thus, the main objective of this paper is to develop a unit root test based on the estimator presented by Guo [23] and to evaluate the efficiency of the unit root test based on OLS estimator.

The organization of the paper is as follows. In the next section, we explain the details of the estimators and unit root tests. The performance of the unit root tests based on the robust estimator is examined and compared with those of the unit root tests based on the OLS estimator through Monte Carlo experiments in Section III. Section IV is devoted to conclusions.

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II. Detailed Description of the Estimators and Unit Root Tests

The ordinary least square (OLS) estimator for (δ, ρ), can be obtained by regressing y_t on y_{t-1} as in (1). So, these estimators are given by

$$\hat{\rho}_{ols} = \frac{\sum_{t=2}^{n}(y_{t-1} - \bar{y}_{(-1)})y_t}{\sum_{t=2}^{n}(y_{t-1} - \bar{y}_{(-1)})^2}, \qquad (2)$$

and

$$\hat{\delta}_{ols} = \bar{y}_{(0)} - \hat{\rho}_{ols}\bar{y}_{(-1)}, \qquad (3)$$

where

$$\bar{y}_{(0)} = (n-1)^{-1}\sum_{t=2}^{n} y_t \quad \text{and} \quad \bar{y}_{(-1)} = (n-1)^{-1}\sum_{t=2}^{n} y_{t-1}.$$

The simple and robust estimator of ρ is proposed by Guo [23], which is denoted by $\hat{\rho}_r$. The robust estimator is defined as follows

$$\hat{\rho}_r = \text{median}(a_i), \quad i = 2,3,...,n, \qquad (4)$$

where $a_i = y_i / y_{i-1}$. Guo [23] also showed that the estimator $\hat{\rho}_r$ is unbiased.

The Dickey and Fuller unit root tests associated with $\hat{\rho}_{ols}$ are $\hat{\kappa}_{ols}$ and $\hat{\tau}_{ols}$ where

$$\hat{\kappa}_{ols} = n(\hat{\rho}_{ols} - 1), \qquad (5)$$

and

$$\hat{\tau}_{ols} = \left[\hat{V}(\hat{\rho}_{ols})\right]^{-1/2}(\hat{\rho}_{ols} - 1), \qquad (6)$$

and $\hat{V}(\hat{\rho}_{ols})$ is the estimated variance of $\hat{\rho}_{ols}$ defined as

$$\hat{V}(\hat{\rho}_{ols}) = \frac{\hat{\sigma}_{ols}^2}{\sum_{t=2}^{n}(y_{t-1} - \bar{y}_{(-1)})^2},$$

where

$$\hat{\sigma}_{ols}^2 = (n-3)^{-1}\sum_{t=2}^{n}(y_{t-1} - \hat{y}_t)^2, \text{ and } \hat{y}_t = \hat{\delta}_{ols} + \hat{\rho}_{ols}y_{t-1}.$$

Equivalent $\hat{\kappa}_{ols}$ and $\hat{\tau}_{ols}$, unit root tests based on the robust estimator are $\hat{\kappa}_r$ and $\hat{\tau}_r$ where

$$\hat{\kappa}_r = n(\hat{\rho}_r - 1), \qquad (7)$$

and

$$\hat{\tau}_r = \left[\hat{V}(\hat{\rho}_r)\right]^{-1/2}(\hat{\rho}_r - 1), \qquad (8)$$

and $\hat{V}(\hat{\rho}_r)$ is the estimated variance of $\hat{\rho}_r$ defined as

$$\hat{V}(\hat{\rho}_r) = \frac{\hat{\sigma}_r^2}{\sum_{t=2}^{n}(y_{t-1} - \bar{y}_{(-1)})^2},$$

where

$$\hat{\sigma}_r^2 = (n-3)^{-1}\sum_{t=2}^{n}(y_{t-1} - \tilde{y}_t)^2, \text{ and } \tilde{y}_t = \hat{\delta}_r + \hat{\rho}_r y_{t-1}.$$

The percentiles of the null distributions of the unit root tests are shown in Tables I and II. These values are based on the average of percentiles of the 100 sets of percentiles from 10,000 independent simulated test statistics.

TABLE I
PERCENTILES OF THE NULL DISTRIBUTIONS OF THE $\hat{\kappa}_{ols}$ AND $\hat{\kappa}_r$ TESTS

n	Probability that $\hat{\kappa}_{ols}$ is less than entry							
	0.01	0.025	0.05	0.10	0.90	0.95	0.975	0.99
25	-17.17	-14.59	-12.49	-10.24	-0.75	0.00	0.64	1.40
50	-18.75	-15.67	-13.23	-10.71	-0.80	-0.07	0.52	1.22
100	-19.66	-16.29	-13.66	-10.98	-0.83	-0.11	0.46	1.12
250	-20.15	-16.63	-13.90	-11.15	-0.84	-0.14	0.43	1.07
500	-20.41	-16.77	-13.99	-11.20	-0.84	-0.14	0.43	1.07
n	Probability that $\hat{\kappa}_r$ is less than entry							
	0.01	0.025	0.05	0.10	0.90	0.95	0.975	0.99
25	-9.33	-6.29	-4.17	-2.35	1.12	1.64	2.17	2.91
50	-10.73	-7.45	-5.16	-3.11	1.35	1.89	2.42	3.15
100	-12.02	-8.54	-6.07	-3.80	1.55	2.08	2.61	3.32
250	-13.56	-9.85	-7.18	-4.67	1.78	2.33	2.86	3.56
500	-14.49	-10.65	-7.84	-5.18	1.93	2.48	3.02	3.75

The probability shown at the head of the column is the area in the left-hand tail.

TABLE II
PERCENTILES OF THE NULL DISTRIBUTIONS OF THE $\hat{\tau}_{ols}$ AND $\hat{\tau}_r$ TESTS

n	Probability that $\hat{\tau}_{ols}$ is less than entry							
	0.01	0.025	0.05	0.10	0.90	0.95	0.975	0.99
25	-3.73	-3.32	-2.99	-2.63	-0.37	0.01	0.33	0.72
50	-3.57	-3.22	-2.92	-2.60	-0.41	-0.04	0.28	0.65
100	-3.50	-3.17	-2.89	-2.58	-0.42	-0.06	0.26	0.63
250	-3.46	-3.14	-2.87	-2.57	-0.44	-0.07	0.25	0.61
500	-3.44	-3.13	-2.87	-2.57	-0.44	-0.08	0.24	0.61
n	Probability that $\hat{\tau}_r$ is less than entry							
	0.01	0.025	0.05	0.10	0.90	0.95	0.975	0.99
25	-2.82	-2.13	-1.54	-0.91	0.44	0.68	0.92	1.24
50	-3.02	-2.35	-1.78	-1.15	0.56	0.83	1.10	1.44
100	-3.17	-2.50	-1.94	-1.33	0.66	0.95	1.23	1.59
250	-3.28	-2.63	-2.08	-1.48	0.79	1.09	1.38	1.75
500	-3.32	-2.69	-2.15	-1.55	0.87	1.18	1.47	1.85

The probability shown at the head of the column is the area in the left-hand tail.

III. Monte Carlo Experiments

In this section, we describe the results of several Monte Carlo experiments carried out to evaluate the performance of the unit root tests, $\hat{\kappa}_{ols}, \hat{\kappa}_r, \hat{\tau}_{ols}$ and $\hat{\tau}_r$. The first-order autoregressive process in (1) with parameters (μ, σ_e) = (0, 1) is generated under the null hypothesis H_0 by setting $y_0 \sim N\left(0, 1/(1-\rho^2)\right)$ and the initial one hundred observations are generated and discarded in order to eliminate the effect of the initial value. The scope of the simulations is set under the autoregressive parameter values ρ = 0.7, 0.8, 0.85, 0.9, 0.93, 0.95, 0.97, 0.98, 0.99 and 1.00; the sample sizes n = 25, 50, 100 and 250. The random variables e_t are generated from a normal distribution with a mean of zero and variance of one. Five hundred thousand time series were simulated by using R statistical software [24]. The significance levels α for the unit root tests are equal to 0.05 and 0.10. The simulations compared the empirical probability of Type I error and power of the unit root tests. Simulation results are summarized in Tables III and IV.

TABLE III
PROBABILITIES OF TYPE I ERROR OF THE $\hat{\kappa}_{ols}, \hat{\kappa}_r, \hat{\tau}_{ols}$ AND $\hat{\tau}_r$

α	n	$\hat{\kappa}_{ols}$	$\hat{\kappa}_r$	$\hat{\tau}_{ols}$	$\hat{\tau}_r$
0.05	25	0.0511	0.0498	0.0489	0.0490
	50	0.0508	0.0487	0.0492	0.0498
	100	0.0512	0.0498	0.0514	0.0508
	250	0.0519	0.0505	0.0501	0.0491
0.10	25	0.1044	0.0999	0.1001	0.0996
	50	0.1025	0.1013	0.1009	0.0973
	100	0.1004	0.0978	0.1010	0.1013
	250	0.1001	0.0982	0.1008	0.1000

We begin with the results for the probability of Type I error of the unit root test (Table III). Bradley's [25] criterion was considered. This criterion is that if the empirical probability of Type I error of any unit root test is within the interval 0.5α and $1.5\alpha,$ then that unit root test can control the probability of Type I error. For the significance level $\alpha = 0.05,$ the empirical probability of Type I error should be between 0.025 and 0.075. It can be seen from Table III that all unit root tests can control the probability of Type I error for all sample sizes and all levels of significance. The empirical probability of Type I error for all unit root tests gets closer to the significance level with increasing sample sizes n. This is intuitive in nature because as n increases it is possible to estimate the autoregressive coefficients more accurately. Table IV shows that the empirical power of the $\hat{\kappa}_r$ test is higher than the $\hat{\kappa}_{ols}$ test. Furthermore, The $\hat{\tau}_r$ test provides the higher empirical power. Apart from that, the empirical power of the tests tends to increase as the sample size gets larger. On the other hand, the empirical power of the tests decreases when ρ approaches unity as the AR(1) process becomes less distinguishable from random walks.

IV. CONCLUSIONS

This paper proposes new unit root tests based on the robust estimator of Guo [23]. The proposed testing for a unit root and the unit root test based on the ordinary least squares (OLS) estimator were studied and compared by examining the empirical probabilities of Type I error and powers of the tests. The tables of critical values for testing of the unit root are created by setting $\rho = 1.$ Based on simulation studies, all unit root tests can control the probability of Type I error for all situations. The empirical power of the unit root tests based on the robust estimator ($\hat{\kappa}_r$ and $\hat{\tau}_r$) are higher than the unit root tests based on the OLS estimator ($\hat{\kappa}_{ols}$ and $\hat{\tau}_{ols}$).

TABLE IV
EMPIRICAL POWERS OF THE $\hat{\kappa}_{ols}, \hat{\kappa}_r, \hat{\tau}_{ols}$ AND $\hat{\tau}_r$

n	ρ	$\alpha = 0.05$				$\alpha = 0.10$			
		$\hat{\kappa}_{ols}$	$\hat{\kappa}_r$	$\hat{\tau}_{ols}$	$\hat{\tau}_r$	$\hat{\kappa}_{ols}$	$\hat{\kappa}_r$	$\hat{\tau}_{ols}$	$\hat{\tau}_r$
25	0.70	0.312	0.736	0.235	0.583	0.501	0.852	0.404	0.772
	0.80	0.174	0.572	0.136	0.435	0.312	0.726	0.254	0.640
	0.85	0.130	0.464	0.106	0.353	0.240	0.637	0.199	0.556
	0.90	0.093	0.339	0.080	0.267	0.174	0.514	0.154	0.449
	0.93	0.076	0.260	0.069	0.213	0.146	0.423	0.136	0.373
	0.95	0.065	0.203	0.063	0.174	0.131	0.352	0.125	0.316
	0.97	0.059	0.149	0.059	0.133	0.117	0.268	0.117	0.252
	0.98	0.055	0.114	0.058	0.108	0.112	0.217	0.112	0.208
	0.99	0.052	0.078	0.054	0.077	0.106	0.154	0.111	0.154
50	0.70	0.807	0.918	0.676	0.775	0.928	0.954	0.845	0.895
	0.80	0.472	0.784	0.354	0.599	0.676	0.873	0.546	0.777
	0.85	0.304	0.659	0.222	0.484	0.485	0.788	0.375	0.683
	0.90	0.176	0.491	0.131	0.354	0.309	0.651	0.240	0.553
	0.93	0.120	0.360	0.096	0.266	0.221	0.525	0.180	0.447
	0.95	0.092	0.270	0.078	0.208	0.180	0.423	0.151	0.371
	0.97	0.071	0.177	0.063	0.149	0.141	0.306	0.125	0.281
	0.98	0.065	0.134	0.060	0.120	0.125	0.240	0.117	0.229
	0.99	0.058	0.086	0.055	0.085	0.113	0.165	0.111	0.167
100	0.70	1.000	0.993	0.997	0.945	1.000	0.997	1.000	0.982
	0.80	0.954	0.955	0.885	0.824	0.992	0.978	0.967	0.925
	0.85	0.783	0.887	0.642	0.706	0.921	0.942	0.829	0.851
	0.90	0.459	0.731	0.342	0.530	0.664	0.848	0.530	0.718
	0.93	0.271	0.566	0.197	0.399	0.444	0.725	0.343	0.591
	0.95	0.174	0.422	0.130	0.300	0.310	0.597	0.242	0.480
	0.97	0.107	0.255	0.085	0.199	0.202	0.427	0.165	0.356
	0.98	0.080	0.176	0.069	0.150	0.158	0.317	0.136	0.277
	0.99	0.063	0.106	0.058	0.100	0.126	0.203	0.116	0.192
250	0.70	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	0.80	1.000	1.000	1.000	0.991	1.000	1.000	1.000	0.998
	0.85	1.000	0.997	1.000	0.964	1.000	0.999	1.000	0.988
	0.90	0.994	0.973	0.974	0.860	1.000	0.988	0.996	0.945
	0.93	0.884	0.899	0.767	0.718	0.968	0.951	0.909	0.860
	0.95	0.627	0.773	0.481	0.568	0.812	0.874	0.688	0.745
	0.97	0.300	0.522	0.214	0.375	0.477	0.691	0.366	0.561
	0.98	0.171	0.346	0.126	0.260	0.307	0.525	0.237	0.430
	0.99	0.091	0.170	0.074	0.150	0.175	0.306	0.148	0.277

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