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An Iterative Method for Quaternionic Linear Equations

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Abstract—By the real representation of the quaternionic matrix, an iterative method for quaternionic linear equations $Ax = b$ is proposed. Then the convergence conditions are obtained. At last, a numerical example is given to illustrate the efficiency of this method.

Keywords—Quaternionic linear equations, Real representation, Iterative algorithm.

I. Introduction

IN quaternionic quantum mechanics and some other applications of quaternions[1], [2], [3], the problem of solutions of quaternionic linear equations is often encountered. Because of noncommutativity of quaternions, solving quaternionic linear equations is difficult. In papers[4], [5], [6], [7], by means of a complex representation and a companion vector, the authors have studied quatemionic linear equations and presented a Cramer rule for quaternionic linear equations and an algebraic algorithm for the least squares problem, respectively, in quaternionic quantum theory. In the paper[8], by using the complex representation of quaternion matrices, and the Moore-Penrose generalized inverse, the authors derive the expressions of the least squares solution with the least norm, the least squares pure imaginary solution with the least norm, and the least squares real solution with the least norm for the quaternion matrix equation $AX = B$, respectively.

In the paper [9] and [10], by means of a real representation of the quaternionic matrix, we gave an iterative algorithms for the least squares problem in quaternionic quantum theory, and the relation between the positive (semi)definite solutions of quaternionic matrix equations and those of corresponding real matrix equations, respectively.

In this paper, we will pay attention to quaternionic linear equations $Ax = b$ by means of the real representation, and propose an iterative method, which is more suitable in the large-scale systems.

Let $\mathbf{R}$ denote the real number field, $\mathbf{Q} = \mathbf{R}\oplus\mathbf{R}i\oplus\mathbf{R}j\oplus\mathbf{R}k$ the quaternion field, where

$$i^2 = j^2 = k^2 = -1, ij = -ji = k.$$

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Let $A_l \in \mathbf{R}^{m\times n}(l = 1, 2, , 3, 4)$. The real representation matrix is defined[7] in the form

$$A^R \equiv \begin{pmatrix} A_1 & -A_2 & -A_3 & -A_4 \\ A_2 & A_1 & -A_4 & A_3 \\ A_3 & A_4 & A_1 & -A_2 \\ A_4 & -A_3 & A_2 & A_1 \end{pmatrix} \in \mathbf{R}^{4m\times 4n}. \quad (1)$$

The real matrix A^R is uniquely determined by quaternion matrix

$$A = A_1 + A_2 i + A_3 j + A_4 k \in \mathbf{Q}^{m\times n},$$

and it is said to be a real representation matrix of quaternion matrix A.

Then it is easy to verify the following properties.

Proposition 1.[10] *Let* $A, B \in \mathbf{Q}^{m\times n}, C \in \mathbf{Q}^{n\times s}, \alpha \in \mathbf{R}$. *Then*
$(A+B)^R = A^R + B^R, (\alpha A)^R = \alpha A^R, (AC)^R = A^R C^R.$

II. Main results

In this section, we will give an iterative method for

$$Ax = b, \quad (2)$$

where $A = A_1 + A_2 i + A_3 j + A_4 k \in \mathbf{Q}^{n\times n}$,$b = b_1 + b_2 i + b_3 j + b_4 k \in \mathbf{Q}^n$, and $x = x_1 + x_2 i + x_3 j + x_4 k \in \mathbf{Q}^n$. A and A_1 are nonsingular. Then, we will discuss the convergence conditions for this iterative method.

The real representation equation of (2) is

$$A^R x^R = b^R, \quad (3)$$

that is,

$$\begin{pmatrix} A_1 & -A_2 & -A_3 & -A_4 \\ A_2 & A_1 & -A_4 & A_3 \\ A_3 & A_4 & A_1 & -A_2 \\ A_4 & -A_3 & A_2 & A_1 \end{pmatrix}\begin{pmatrix} x_1 & -x_2 & -x_3 & -x_4 \\ x_2 & x_1 & -x_4 & x_3 \\ x_3 & x_4 & x_1 & -x_2 \\ x_4 & -x_3 & x_2 & x_1 \end{pmatrix} = \begin{pmatrix} b_1 & -b_2 & -b_3 & -b_4 \\ b_2 & b_1 & -b_4 & b_3 \\ b_3 & b_4 & b_1 & -b_2 \\ b_4 & -b_3 & b_2 & b_1 \end{pmatrix},$$

which may be written as

$$\begin{pmatrix} A_1 & & & \\ & A_1 & & \\ & & A_1 & \\ & & & A_1 \end{pmatrix}\begin{pmatrix} x_1 & -x_2 & -x_3 & -x_4 \\ x_2 & x_1 & -x_4 & x_3 \\ x_3 & x_4 & x_1 & -x_2 \\ x_4 & -x_3 & x_2 & x_1 \end{pmatrix} = \begin{pmatrix} 0 & -A_2 & -A_3 & -A_4 \\ A_2 & 0 & -A_4 & A_3 \\ A_3 & A_4 & 0 & -A_2 \\ A_4 & -A_3 & A_2 & 0 \end{pmatrix}\begin{pmatrix} b_1 & -b_2 & -b_3 & -b_4 \\ b_2 & b_1 & -b_4 & b_3 \\ b_3 & b_4 & b_1 & -b_2 \\ b_4 & -b_3 & b_2 & b_1 \end{pmatrix}.$$

It follows from Proposition 1 that the above formula is equivalent to

$$\begin{cases} A_1x_1 = A_2x_2 + A_3x_3 + A_4x_4 + b_1 \\ A_1x_2 = -A_2x_1 + A_4x_3 - A_3x_4 + b_2 \\ A_1x_3 = -A_3x_1 - A_4x_2 + A_2x_4 + b_3 \\ A_1x_4 = -A_4x_1 + A_3x_2 - A_2x_3 + b_4 \end{cases}.$$

So we can construct the iterative algorithm as follows:

Algorithm 1. the algorithm for $Ax = b$

(I). Initialization. Given arbitrary $x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, x_4^{(0)}$;
(II). Iteration. For $l = 1, 2, \cdots$

$$\begin{cases} x_1^{(l+1)} = A_1^{-1}(A_2x_2^{(l)} + A_3x_3^{(l)} + A_4x_4^{(l)} + b_1) \\ x_2^{(l+1)} = -A_1^{-1}(A_2x_1^{(l)} + A_4x_3^{(l)} - A_3x_4^{(l)} + b_2) \\ x_3^{(l+1)} = -A_1^{-1}(A_3x_1^{(l)} - A_4x_2^{(l)} + A_2x_4^{(l)} + b_3) \\ x_4^{(l+1)} = -A_1^{-1}(A_4x_1^{(l)} + A_3x_2^{(l)} - A_2x_3^{(l)} + b_4) \end{cases},$$

(III). check convergence.

Theorem 1. If $\|A_1^{-1}(A_l)\| < 1, l = 2, 3, 4$, then the sequence $\{(x_1^{(l)}x_2^{(l)}x_3^{(l)}x_4^{(l)})\}$, generated by Algorithm 1, converges to $(x_1^{(*)}x_2^{(*)}x_3^{(*)}x_4^{(*)})$, where

$$x = x_1^{(*)} + x_2^{(*)}i + x_3^{(*)}j + x_4^{(*)}k$$

is the solution of (2).

Proof. we can easily obtain
$x_1^{(l+1)} - x_1^{(*)}$
$= A_1^{-1}\left(A_2(x_2^{(l)} - x_2^{(*)}) + A_3(x_3^{(l)} - x_3^{(*)}) + A_4(x_4^{(l)} - x_4^{(*)})\right)$
$= \sum_{i=1}^{4} f_i(A_1^{-1}A_2, A_1^{-1}A_3, A_1^{-1}A_4)(x_i^{(0)} - x_i^{(*)})$,
where $f_i(x, y, z)$ is a $(l+1)$-order homogeneous polynomial on x, y, z.

If $\|A_1^{-1}(A_l)\| < 1, l = 2, 3, 4$, it is obvious that

$$x_1^{(l)} \to x_1^{(*)}.$$

Similarly, $x_i^{(l)} \to x_i^{(*)}, i = 2, 3, 4.\square$

Since the iterative matrix of Algorithm 1 is

$$B = \begin{pmatrix} 0 & -A_1^{-1}A_2 & -A_1^{-1}A_3 & -A_1^{-1}A_4 \\ A_1^{-1}A_2 & 0 & -A_1^{-1}A_4 & A_1^{-1}A_3 \\ A_1^{-1}A_3 & A_1^{-1}A_4 & 0 & -A_1^{-1}A_2 \\ A_1^{-1}A_4 & -A_1^{-1}A_3 & A_1^{-1}A_2 & 0 \end{pmatrix}, \quad (4)$$

we can get the following result.

Theorem 2. If and only if $\rho(B) < 1$, the sequence $\{(x_1^{(l)}x_2^{(l)}x_3^{(l)}x_4^{(l)})\}$, generated by Algorithm 1, converges to $(x_1^{(*)}x_2^{(*)}x_3^{(*)}x_4^{(*)})$, where $x = x_1^{(*)} + x_2^{(*)}i + x_3^{(*)}j + x_4^{(*)}k$ is the solution of (2).

Because $\rho(B) \leq \|B\|_\infty$, we have

Theorem 3. If $\|A_1^{-1}(A_2A_3A_4)\|_\infty < 1$, the sequence $\{(x_1^{(l)}x_2^{(l)}x_3^{(l)}x_4^{(l)})\}$, generated by Algorithm 1, converges to $(x_1^{(*)}x_2^{(*)}x_3^{(*)}x_4^{(*)})$, where $x = x_1^{(*)} + x_2^{(*)}i + x_3^{(*)}j + x_4^{(*)}k$ is the solution of (2).

III. Numerical Example

In this section, we present a numerical example to illustrate the efficiency of our algorithm.

Given $A = A_1 + A_2i + A_3j + A_4k, x = x_1 + x_2i + x_3j + x_4k$ with

$$A_1 = \begin{pmatrix} 9 & 12 & -37 & 6 \\ -8 & 0 & 19 & -7 \\ 17 & 43 & -19 & 0 \\ 78 & -98 & 0 & 12 \end{pmatrix} A_2 = \begin{pmatrix} 10 & 2 & -9 & 8 \\ 7 & 0 & 19 & -7 \\ 1 & -4 & 9 & 21 \\ 7 & 0 & 4 & -1 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 8 & 0 & 36 \\ -3 & 0 & 9 & -9 \\ 1 & 0 & 9 & 12 \\ -7 & 13 & 0 & 7 \end{pmatrix} A_4 = \begin{pmatrix} 17 & 0 & -17 & 3 \\ 0 & 8 & 0 & 0 \\ 1 & 0 & 9 & 19 \\ 0 & 10 & 1 & -12 \end{pmatrix},$$

$x_1 = (1 \quad 1 \quad 3 \quad -4)', x_2 = (0 \quad -7 \quad 8 \quad 11)', x_3 = (-8 \quad 14 \quad 20 \quad 3)', x_4 = (32 \quad 14 \quad 0 \quad -17)', A_1 = 10A_1$. $b = b_1 + b_2i + b_3j + b_4k \equiv Ax$.

A and A_1 are nonsingular, $Ax = b$ has a unique solution x.

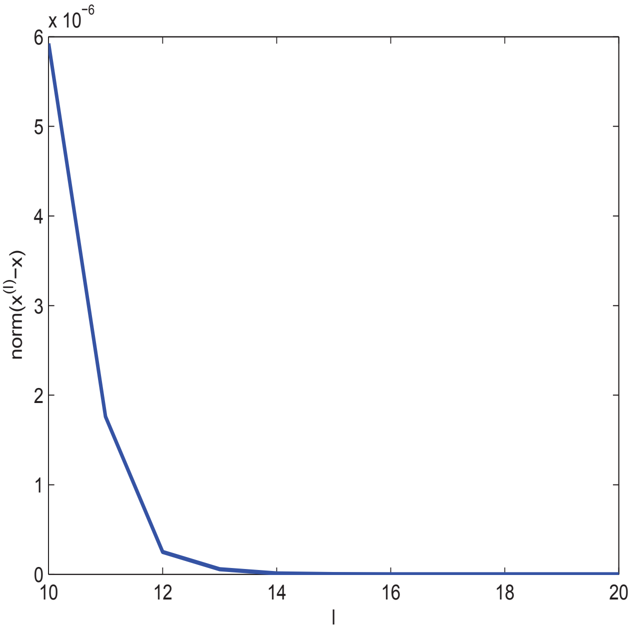

Fig. 1. The error of the computed solutions by Algorithm 1

Fig.1 depicts the relation of the k-step approximate

$$x^{(l)} = x_1^{(l)} + x_2^{(l)}i + x_3^{(l)}j + x_4^{(l)}k$$

and the true solution x. From Fig.1, we can see that Algorithm 1 is efficient for this example.

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