World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:5, No:8, 2011

Sinc-Galerkin Method for the Solution of Problems in Calculus of Variations

M. Zarebnia, N. Aliniya

***Abstract*—**In this paper, a numerical solution based on sinc functions is used for finding the solution of boundary value problems which arise from the problems of calculus of variations. This approximation reduce the problems to an explicit system of algebraic equations. Some numerical examples are also given to illustrate the accuracy and applicability of the presented method.

***Keywords*—**Calculus of variation; Sinc functions; Galerkin; Numerical method

I. Introduction

Minimization principles form one of the most wide-ranging means of formulating mathematical models governing the equilibrium configurations of physical systems. Moreover, many popular numerical integration schemes such as the powerful finite element method are also founded upon a minimization paradigm. Classical solutions to minimization problems in the calculus of variations are prescribed by boundary value problems involving certain types of differential equations, known as the associated Euler–Lagrange equations. The mathematical techniques that have been developed to handle such optimization problems are fundamental in many areas of mathematics, physics, engineering, and other applications.

The history of the calculus of variations is tightly interwoven with the history of mathematics. The field has drawn the attention of a remarkable range of mathematical luminaries, beginning with Newton, then initiated as a subject in its own right by the Bernoulli family. The first major developments appeared in the work of Euler, Lagrange and Laplace. In the nineteenth century, Hamilton, Dirichlet and Hilbert are but a few of the outstanding contributors. In modern times, the calculus of variations has continued to occupy center stage, witnessing major theoretical advances, along with wide-ranging applications in physics, engineering and all branches of mathematics.

Several numerical methods for approximating the solution of problems in the calculus of variations are known. Galerkin method is used for solving variational problems in [1]. The Ritz method [2], usually based on the subspaces of kinematically admissible complete functions, is the most commonly used approach in direct methods of solving variational problems. Chen and Hsiao [3] introduced the Walsh series method to variational problems. Due to the nature of the Walsh functions, the solution obtained was piecewise constant. Some orthogonal polynomials are applied on variational problems to find the continuous solutions for these problems [4-6]. A simple algorithm for solving variational problems via Bernstein orthonormal polynomials of degree six is proposed by Dixit et al. [7]. Razzaghi et al. [8] applied a direct method for solving variational problems using Legendre wavelets. Chebyshev finite difference method has been employed for solving some problems in calculus of variations in [9].

In this paper, we solve variational problems using sinc-Galerkin method. First, in Section 2, we will give some preliminary definitions and theorems in [10,11] that are employed to derive the formulations and analysis of the sinc-Galerkin method in Section 3. Also in this section, we report our numerical results and demonstrate the efficiency and accuracy of the proposed numerical scheme by considering some numerical examples.

M. Zarebnia is with the Department of Mathematics, University of Mohaghegh Ardabili, P. O. Box. 179, Ardabil, Iran, e-mail: zarebnia@uma.ac.ir,
N. Aliniya is with the Department of Mathematics, University of Mohaghegh Ardabili, P. O. Box. 179, Ardabil, Iran, e-mail: aliniya_n@yahoo.com

II. Statement of the Problem

The genaral form of a variational problem is finding extremum of the

$$J[u_1(t),u_2(t),\ldots,u_n(t)]=\int_a^b G(t,u_1(t),u_2(t),\ldots,u_n(t),u_1'(t),u_2'(t),\ldots,u_n'(t))dt. \quad (1)$$

To find the extreme value of J, the boundary conditions of the admissible curves are known in the following form:

$$u_i(a)=\gamma_i, \qquad i=1,2,\ldots,n, \quad (2)$$

$$u_i(b)=\delta_i, \qquad i=1,2,\ldots,n. \quad (3)$$

The necessary condition for $u_i(t),\ \ i=1,2,\ldots,n$, to extremize $J[u_1(t),u_2(t),\ldots,u_n(t)]$ is to satisfy the Euler-Lagrange equations that is obtained by applying the well known procedure in the calculus of variation [2],

$$\frac{\partial G}{\partial u_i}-\frac{d}{dt}\left(\frac{\partial G}{\partial u_i'}\right)=0, \qquad i=1,2,\ldots,n \quad (4)$$

subject to the boundary conditions given by Eqs. (2)-(3).

In this paper, we consider the special form of the variational problem(1) as

$$J[u(t)]=\int_a^b G(t,u(t),u'(t))dt, \quad (5)$$

with boundary conditions

$$u(a)=\gamma, \qquad u(b)=\delta, \quad (6)$$

and

$$J[u_1(t),u_2(t)] = \int_a^b G(t,u_1(t),u_2(t),u_1'(t),u_2'(t))dt \quad (7)$$

subject to boundary conditions

$$u_1(a) = \gamma_1, \qquad u_1(b) = \delta_1, \quad (8)$$

$$u_2(a) = \gamma_2, \qquad u_2(b) = \delta_2. \quad (9)$$

Thus, for solving the variational problems (5), we consider the second order differential equation

$$\frac{\partial G}{\partial u} - \frac{d}{dt}\left(\frac{\partial G}{\partial u'}\right) = 0, \quad (10)$$

with the boundary condition (6). And also, for solving the variational problems (7), we find the solution of the system of second-order differential equations

$$\frac{\partial G}{\partial u_i} - \frac{d}{dt}\left(\frac{\partial G}{\partial u_i'}\right) = 0, \qquad i = 1,2, \quad (11)$$

with the boundary conditions (8)-(9). Therefore, by applying sinc method for the Euler-Lagrange equations (10) and (11) we can obtain an approximate solution to the variational problems (5) and (7).

III. SINC-GALERKIN METHOD

In this section, we will review sinc function properties, sinc quadrature rule, and the sinc method. These are discussed thoroughly in [10] and [11]. For solving variational equation (5) and (7), these properties will be used extensively in section3.

The sinc function. The sinc function is defined on the whole real line, $-\infty < z < \infty$, by

$$Sinc(z) = \begin{cases} \dfrac{\sin(\pi z)}{\pi z}, & z \neq 0; \\ 1, & z = 0. \end{cases} \quad (12)$$

For any $h > 0$, the translated sinc functions with evenly spaced nodes are given as

$$S(j,h)(z) = Sinc\left(\frac{z - jh}{h}\right), \qquad j = 0,\pm1,\pm2,... \quad (13)$$

They are based in the infinite strip D_d in the complex plane

$$D_d = \{w = u + iv : |v| < d \leq \frac{\pi}{2}\}. \quad (14)$$

To construct approximation on the interval $\Gamma = [a,b]$ we consider the conformal map

$$\phi(z) = \ln\left(\frac{z-a}{b-z}\right). \quad (15)$$

The map ϕ carries the eye-shaped region

$$D_E = \{z = x + iy : \left|\arg\left(\frac{z-a}{b-z}\right)\right| < d \leq \frac{\pi}{2}\}. \quad (16)$$

For the sinc method, the basis functions on the $\Gamma = [a,b]$ for $z \in D_d$ are derived from the composite translated sinc functions

$$S_j(z) = S(j,h)o\phi(z) = Sinc\left(\frac{\phi(z) - jh}{h}\right). \quad (17)$$

The function

$$z = \phi^{-1}(w) = \frac{a + be^w}{1 + e^w}, \quad (18)$$

is an inverse mapping of $w = \phi(z)$. We define the range of ϕ^{-1} on the real line as

$\Gamma = \{\psi(u) = \phi^{-1}(u) \in D : -\infty < u < \infty\}$.

The sinc grid points $z_k \in \Gamma$ in D_d will be denoted by x_k because they are real. For the evenly spaced nodes $\{kh\}_{k=-\infty}^{\infty}$ on the real line, the image which corresponds to these nodes is denoted by

$$x_k = \phi^{-1}(kh) = \frac{a + be^{kh}}{1 + e^{kh}}, \; k = \pm1,\pm2,.... \quad (19)$$

*Sinc interpolation and quadrature rules***.** For further explanation of the procedure, the important class of functions is denoted by $B(D_E)$. The properties of functions in $B(D_E)$ and detailed discussions are given in [10] and [11]. We recall the following definitions and theorems for our purpose.

Definition 1. Let $B(D_E)$ denote the family of functions F which are analytic in D_E and satisfy

$$\int_{\psi(u+L)} |F(z)dz| \to 0, \;\; u \to \pm\infty \quad (20)$$

where $L = \{iv : |v| < d \leq \frac{\pi}{2}\}$, and on the boundary of D_E (denoted by ∂D_E) satisfy

$$N(F) \equiv \int_{\partial D_E} |F(z)dz| < \infty, \quad (21)$$

Theorem 1. If $\phi' \in B(D_E)$ then for all $x \in \Gamma$

$$\left|F(x)-\sum_{k=-\infty}^{+\infty}F(x_k)S(k,h)o\phi(x)\right|\le\frac{N(F\phi')}{2\pi d\sinh(\pi d/h)}\le\frac{2N(F\phi')}{\pi d}e^{-\pi d/h}. \qquad (22)$$

Moreover, if $|F(x)|\le C_1e^{-\alpha|\phi(x)|}$, $x\in\Gamma$, for some positive constants C_1 and α, and let $h=\sqrt{2\pi d/\alpha N}$ then

$$F(x)=\sum_{k=-N}^{N}F(x_k)S(k,h)o\phi(x)+O(\exp(-(2\pi d\alpha N)^{1/2})). \qquad (23)$$

Theorem 2. Let $F\in B(D_E)$ and ϕ be a conformal map with constants α and C_2 so that

$$\left|\frac{F(z)}{\phi'(z)}\right|\le C_2e^{-\alpha|\phi(z)|},\quad z\in\Gamma, \qquad (24)$$

then the sinc trapezoidal quadrature rule is

$$\int_\Gamma F(z)dz=h\sum_{j=-N}^{N}\frac{F(z_j)}{\phi'(z_j)}+O(\exp(-\alpha Nh))+O(\exp(-2\pi d/h)). \qquad (25)$$

Hence, by selecting

$$h=(\frac{2\pi d}{\alpha N})^{1/2}, \qquad (26)$$

the exponential order of the sinc trapezoidal quadrature rule in (25) is $O(\exp(-2\pi d\alpha N)^{1/2})$. By applying the Theorem 2, we conclude the following Corollary that is special case of (25).

Corollary. Let $F\in B(D_E)$, and let h be selected by (26), then

$$\int_\Gamma F(z)S(k,h)o\phi(z)dz=h\frac{F(z_k)}{\phi'(z_k)}+O(\exp(-2\pi d\alpha N)^{1/2}). \qquad (27)$$

Theorem 3. Let ϕ be a conformal one-to-one map of the simply connected domain D_d onto D_E. then

$$\delta_{jk}^{(0)}=[S(j,h)o\phi(x)]|_{x=x_k}=\begin{cases}1, & k=j;\\ 0, & k\ne j,\end{cases} \qquad (28)$$

$$\delta_{jk}^{(1)}=h\frac{d}{d\phi}[S(j,h)o\phi(x)]|_{x=x_k}=\begin{cases}0, & k=j;\\ \dfrac{(-1)^{(k-j)}}{(k-j)}, & k\ne j,\end{cases} \qquad (29)$$

and

$$\delta_{jk}^{(2)}=h^2\frac{d^2}{d\phi^2}[S(j,h)o\phi(x)]|_{x=x_k}=\begin{cases}\dfrac{-\pi^2}{3}, & k=j;\\ \dfrac{-2(-1)^{(k-j)}}{(k-j)^2}, & k\ne j.\end{cases} \qquad (30)$$

In order to illustrate the performance of the sinc-Galerkin method, we present some examples.

Example 1. We first consider the following variational problem with the exact solution $u(t)=e^{3t}$ in [9]:

$$\min J=\int_0^1(u(x)+u'(x)-4e^{3x})dx, \qquad (31)$$

subject to boundary conditions

$$u(0)=1,\qquad u(1)=e^3. \qquad (32)$$

Considering the Eq. (31), the Euler-Lagrange equation of this problem can be written in the following form:

$$u''(x)-u(x)-8e^{3x}=0. \qquad (33)$$

The solution of the second-order differential equation (33) with boundary conditions (32) is approximated by the sinc method. For our purpose, first we convert the non-homogeneous boundary condition (32) to homogeneous boundary condition by considering the following transformation

$$y(x)=u(x)-\frac{b-x}{b-a}u(a)-\frac{x-a}{b-a}u(b). \qquad (34)$$

Using the above change of variable yields the following boundary value problem

$$y''(x)-y(x)=(1-x)+xe^3+8e^{3x}, \qquad (35)$$

with boundary conditions

$$y(0)=0,\qquad y(1)=0. \qquad (36)$$

We consider the boundary value problem (35)-(36) in general form as follows:

$$y''(x)+\upsilon(x)y(x)=\sigma(x),\quad y(a)=y(b)=0 \qquad (37)$$

We suppose that the boundary value problem (37) has a unique solution $y \in B(D_E)$. In this case consider sinc pproximation by the formula

$$y(x) \approx y_N(x) = \sum_{j=-N}^{N} w_j S(j,h) o \phi(x), \qquad (38)$$

Our purpose is applying the Galerkin method based on sinc function. Therefore, consider inner product for arbitrary function f and g in the following form

$$< f, g >= \int_\Gamma w(x) f(x) g(x) dx, \qquad (39)$$

where

$$w(x) = \frac{1}{[\phi'(x)]^{1/2}}. \qquad (40)$$

We apply the inner product (39) for $S(k,h)o\phi(x)$ and equation (37) as follows :

$$< y''(x), S(k,h)o\phi(x) > + < \upsilon(x) y(x), S(k,h)o\phi(x) > - < \sigma(x), S(k,h)o\phi(x) >= 0 \qquad (41)$$

Multiplying the both side of equation (41) in h and considering the Equations (38) and (39) we obtain :

$$h \int_\Gamma \frac{S(k,h)o\phi(x)}{[\phi'(x)]^{1/2}} y_N''(x) dx + h \int_\Gamma \frac{S(k,h)o\phi(x)}{[\phi'(x)]^{1/2}} \upsilon(x) y_N(x) dx - h \int_\Gamma \frac{S(k,h)o\phi(x)}{[\phi'(x)]^{1/2}} \sigma(x) dx = 0 \qquad (42)$$

Now, we apply part by part integration for the first term of Eq.(42) and then we get

$$h \int_\Gamma \frac{S(k,h)o\phi(x)}{[\phi'(x)]^{1/2}} y_N''(x) dx = h \int_\Gamma \left(\frac{S(k,h)o\phi(x)}{[\phi'(x)]^{1/2}}\right)'' y_N(x) dx. \qquad (43)$$

By using Eqs.(27) and (38) we can write

$$h \int_\Gamma \left(\frac{S(k,h)o\phi(x)}{[\phi'(x)]^{1/2}}\right)'' y_N(x) dx = h^2 \sum_{j=-N}^{N} \left[\left(\frac{S(k,h)o\phi(x_j)}{[\phi'(x_j)]^{1/2}}\right)'' . \frac{1}{\phi'(x_j)}\right] w_j. \qquad (44)$$

Having used the relations (28)-(30) we have

$$[S(j,h)o\phi(x)]|_{x=x_k} = \phi' \frac{d}{d\phi}[S(j,h)o\phi(x)]|_{x=x_k} = \phi'(x_k) h^{-1} \delta_{jk}^{1}, \qquad (45)$$

$$[S(j,h)o\phi(x)]''|_{x=x_k} = [\phi' \frac{d}{d\phi}[S(j,h)o\phi(x)]]'|_{x=x_k} = \phi''(x_k) h^{-1} \delta_{jk}^{(1)} + [\phi'(x_k)]^2 h^{-2} \delta_{jk}^{(2)}. \qquad (46)$$

From the above relations and Eq.(44) we obtain

$$h \int_\Gamma \frac{S(j,h)o\phi(x)}{[\phi'(x)]^{1/2}} y_N''(x) dx = \sum_{j=-N}^{N} \{\delta_{k,j}^{(2)} [\phi'(z_j)]^{1/2} + h \delta_{k,j}^{(1)} \left(\frac{\phi''(z_j)}{[\phi'(z_j)]^{3/2}} + 2([\phi'(z_j)]^{-1/2})'\right) + h^2 \delta_{k,j}^{(0)} \left(\frac{[\phi'(z_j)^{-1/2}]''}{\phi'(z_j)}\right)\} w_j. \qquad (47)$$

Now, consider second term of Eq.(42). By applying the Eqs.(27) and (38) we can obtain

$$h \int_\Gamma \frac{S(k,h)o\phi(x)}{[\phi'(x)]^{1/2}} \upsilon(x) y_N(x) dx = \sum_{j=-N}^{N} h^2 \delta_{k,j}^{(0)} \frac{\upsilon(z_j)}{[\phi'(z_j)]^{3/2}}. \qquad (48)$$

Similarly, for the third term of Eq.(42) we get

$$h \int_\Gamma \frac{S(k,h)o\phi(x)}{[\phi'(x)]^{1/2}} \sigma(x) dx = h^2 \frac{\sigma(z_k)}{[\phi'(z_k)]^{3/2}}. \qquad (49)$$

Now, the Galerkin result is obtained by using substituting Eqs. (47)–(49) in Eq. (42), in matrix form as follows:

$$BW = P, \qquad (50)$$

where

$$B = \left(I^2 + h^2 D\left(\frac{1}{(\phi')^{3/2}} \left(\frac{1}{\sqrt{\phi'}}\right)'' + \frac{\upsilon}{[\phi']^2}\right)\right).D\left([\phi'^{\frac{1}{2}}]\right),$$

$$P = h^2 D\left(\frac{1}{[\phi']^{3/2}}\right) S, \qquad S = (\sigma(z_{-N}), \ldots, \sigma(z_N))^T,$$

$$W = (w_{-N}, w_{-N+1}, \ldots, w_{N-1}, w_N)^T.$$

Corresponding to a given function u, $D(u)$ defined a diagonal matrix and $I^2 = [\delta_{k,j}^{(2)}]$. The above linear system containing $(2N+1)$ equations with $(2N+1)$ unknown coefficients $\{w_j\}_{j=-N}^{N}$. Solving this linear system, we can obtain the approximate solution as follows:

$$y(x) \approx y_N(x) = \sum_{j=-N}^{N} w_j S(j,h) o \phi(x).$$

The errors are reported on the set of uniform grid points

$$S = \{a = x_0, \ldots, x_i, \ldots, x_n = b\},$$
$$x_i = x_0 + ih, \quad i = 0,1,2,\ldots,n, \quad h = \frac{b-a}{n}. \qquad (51)$$

The maximum error on the uniform grid points S is

$$\left\| E_y(h) \right\|_\infty = \max_{0 \le j \le n} \left| y(x_j) - y_N(x_j) \right|, \qquad (52)$$

where $y(x_j)$ is the exact solution of the given example, and $y_N(x_j)$ is the computed solution by the sinc method. The examples have been solved by the presented method with different values of N and α, $0 < \alpha \le 1$. Examples 1 and 2 are solved for $d = \frac{\pi}{4}$ and $\alpha = \frac{1}{2}$ and also example 3 is solved for $d = \frac{\pi}{4}$ and $\alpha = 1$. The maximum absolute errors in numerical solution of the Example 1 are tabulated in Table I. These results show the efficiency and applicability of the presented method. The plot of exact solution and the solution of sinc-Galerkin method for $N = 3$ and $N = 10$ have been displayed in Figure 1.

TABLE I
RESULTS FOR EXAMPLE 1

n	h	$\left\| E_y(h) \right\|_\infty$
5	1.4049629	8.64656×10^{-2}
10	0.9934588	6.47961×10^{-3}
20	0.7024815	1.39879×10^{-4}
30	0.5735737	6.10976×10^{-6}
40	0.4967294	4.30248×10^{-7}
50	0.4442883	6.92302×10^{-8}

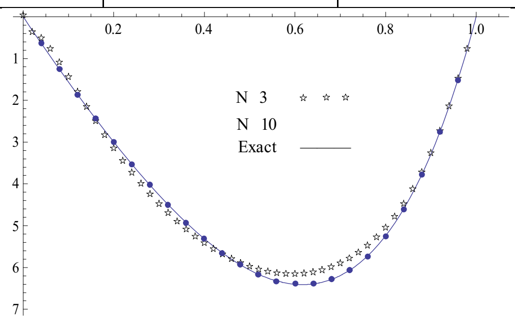

Fig. 1 Exact and approximate solutions for Example 1,(N = 3, 10)

Example 2. In this example, consider the following problem of finding the extremals of the functional[8]:

$$J[u_1(t), u_2(t)] = \int_0^{\frac{\pi}{2}} (u_1'^2(t) + u_2'^2(t) + 2u_1(t)u_2(t))dt, \qquad (53)$$

with boundary conditions

$$u_1(0) = 0, \qquad u_1(\frac{\pi}{2}) = 1, \qquad (54)$$

$$u_2(0) = 0 \qquad u_2(\frac{\pi}{2}) = -1 \qquad (55)$$

which has the exact solution given by $(u_1(t), u_2(t)) = (\sin(t), -\sin(t))$. For this problem, the corresponding Euler-Lagrange equations are

$$\begin{cases} u_1''(t) - u_2(t) = 0, \\ u_2''(t) - u_1(t) = 0, \end{cases} \qquad (56)$$

with boundary conditions (54) and (55). In a similar manner, the Eqs. (54)-(56) produce a linear system that contains $2 \times (2N+1)$ equations with $2 \times (2N+1)$ unknown coefficients. Solving this linear system, we can obtain the approximate solution of the system of second-order boundary value problems (54)-(56).We solved Example 2 for different values of N. The maximum of absolute errors on the uniform grid points (51) are tabulated in Table II . The plot of exact solution and the solution of sinc-Galerkin method for $N = 3$ and $N = 10$ have been displayed in Figures 2 and 3.

TABLE II
RESULTS FOR EXAMPLE 2.

n	h	$\left\| E_{y_1}(h) \right\|_\infty$	$\left\| E_{y_2}(h) \right\|_\infty$
5	1.4049629	3.01744×10^{-3}	3.01744×10^{-3}
10	0.9934588	2.7295×10^{-4}	2.7295×10^{-4}
20	0.7024815	8.69675×10^{-6}	8.69675×10^{-6}
30	0.5735737	5.65263×10^{-7}	5.65263×10^{-7}
40	0.4967294	5.47391×10^{-8}	5.47391×10^{-8}
50	0.4442883	6.93422×10^{-9}	6.93422×10^{-9}

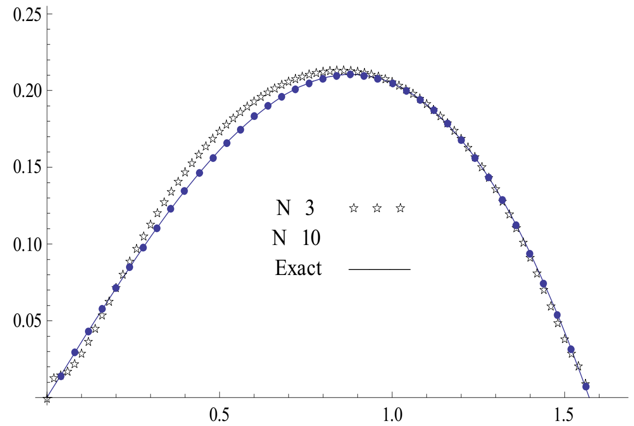

Fig. 2 Exact and approximate solutions for Example 2,(y_1)

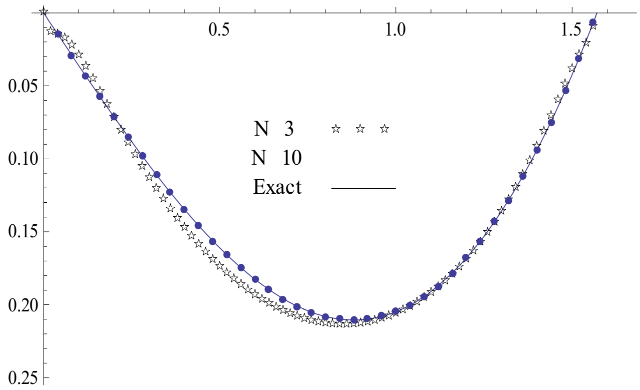

Fig. 3 Exact and approximate solutions for Example 2,(y_2)

Example 3. In this example, we consider the following variational problem

$$\min J = \int_0^1 \frac{1+u^2(x)}{u^2(x)} dx,$$

that satisfies the conditions

$$u(0)=0, \qquad u(1)=0.5$$

THE EXACT SOLUTION OF THIS PROBLEM IS

$$u(x)=\sinh(0.4812118250x).$$

In this case the Euler–Lagrange equation is written as:

$$u''+u''u^2-uu'^2=0.$$

The numerical results for different values of N are tabulated in table III. The plot of this case for exact and obtained solutions is shown in Figure 4.

TABLE III
RESULTS FOR EXAMPLE 3

n	h	$\|E_y(h)\|_\infty$
5	0.9934588	1.1512×10^{-4}
10	0.7024814	1.82294×10^{-5}
20	0.4967294	1.18951×10^{-6}
30	0.4055779	1.38133×10^{-7}
40	0.3512407	2.21156×10^{-8}
50	0.3141593	4.40759×10^{-9}

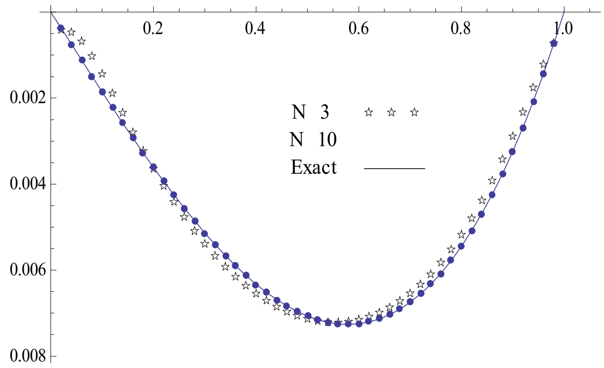

Fig. 4 Exact and approximate solutions for example 3

IV. CONCLUSION

In this paper sinc-Galerkin method employed for finding the extremum of a functional over the specified domain. The main purpose is to find the solution of boundary value problems which arise from the variational problems. The numerical examples show that the accuracy improve with increasing the number of sinc grid points.

REFERENCES

[1] L. Elsgolts, Differential Equations and Calculus of Variations, Mir, Moscow, 1977 (translated from the Russian by G. Yankovsky).
[2] I.M. Gelfand, S.V. Fomin, Calculus of Variations, Prentice-Hall, Englewood Cliffs, NJ, 1963.
[3] C.F. Chen, C.H. Hsiao, A walsh series direct method for solving variational problems, J. Franklin Inst.vol. 300, pp. 265-280, 1975.
[4] R.Y. Chang, M.L.Wang, Shifted Legendre direct method for variational problems, J. Optim. Theory Appl.vol. 39, pp. 299-306, 1983.
[5] I.R. Horng, J.H. Chou, Shifted Chebyshev direct method for solving variational problems, Internat. J. Systems Sci. vol. 16, pp. 855-861,1985.
[6] C. Hwang, Y.P. Shih, Laguerre series direct method for variational problems, J. Optim. Theory Appl. Vol. 39, no. 1, pp. 143-149, 1983.
[7] S. Dixit, V.K. Singh, A.K. Singh, O.P. Singh, Bernstei Direct Method for Solving Variational Problems, International Mathematical Forum,vol. 5, 2351-2370, 2010.
[8] M. Razzaghi, S. Yousefi, Legendre wavelets direct method for variational problems, Mathematics and Computers in Simulation, vol. 53, pp. 185-192, 2000.
[9] A. Saadatmandi, M. Dehghan, The numerical solution of problems in calculus of variation using Chebyshev finite difference method, Physics Letters A, vol. 372, pp. 4037- 4040, 2008.
[10] F. Stenger, Numerical Methods Based on Sinc and Analytic Functions, Springer-Verlag, New York, 1993. [11] J. Lund, K. Bowers, Sinc Methods for Quadrature and Differential Equations, SIAM, Philadelphia, PA , 1992.