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Performance Analysis of MUSIC, Root-MUSIC and ESPRIT DOA Estimation Algorithm

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Abstract—Direction of Arrival estimation refers to defining a mathematical function called a pseudospectrum that gives an indication of the angle a signal is impinging on the antenna array. This estimation is an efficient method of improving the quality of service in a communication system by focusing the reception and transmission only in the estimated direction thereby increasing fidelity with a provision to suppress interferers. This improvement is largely dependent on the performance of the algorithm employed in the estimation. Many DOA algorithms exists amongst which are MUSIC, Root-MUSIC and ESPRIT. In this paper, performance of these three algorithms is analyzed in terms of complexity, accuracy as assessed and characterized by the CRLB and memory requirements in various environments and array sizes. It is found that the three algorithms are high resolution and dependent on the operating environment and the array size.

Keywords—Direction of Arrival, Autocorrelation matrix, Eigenvalue decomposition, MUSIC, ESPRIT, CRLB.

I. INTRODUCTION

WIRELESS communication is one of the fastest growing fields in the engineering world. This has been necessitated by the advancement made in the research and design of communication equipment [1], [2]. It started with cellular communication, then the world-wide-web to the extent where these services are accessible wherever whenever. The Quality of Service (QoS) however, deteriorates with distance between the transmitting node and the receiver.

Estimating the direction of a transceiver is an effective method of improving the QoS between a node and a transceiver, by heightening and focusing the transmission only to the direction of the receiver and vice versa for the receiver [3]–[5]. This is achieved by the use antenna arrays with some added capability to estimate the Direction of Arrival (DOA) of all impinging signals.

The aforementioned improvement is to a large extent dependent on the performance of the employed algorithm. The performance of a DOA algorithm is in turn dependent on the size of the array, number of impinging signals, spacing between elements and the number of snapshots used in the estimation process.

Many DOA techniques exist: quadratic type e.g CAPON and those based on eigenvalue decomposition e.g. MUSIC, ESPRIT and their variants. Eigenvalue decomposition based algorithms involves manipulation of the signal autocorrelation matrix to yield noise and signal subspaces from which angles of arrival of impinging signals are extracted.

This paper therefore, seeks to analyze the performance of three DOA algorithms based on number of array elements and the Signal to Noise Ratio (SNR) to have an optimum choice of algorithm and design for a given environment.

MUSIC, Root-MUSIC and ESPRIT algorithms were developed and simulated in MATLAB software for signals operating at 2.4GHz for a ULA with inter-element spacing of $\frac{\lambda}{2}$. In Section II of this paper, the signal model of an N element uniform linear array receiving M signals from directions $\theta_1, \theta_2, \ldots, \theta_M$ is derived followed by the Cramer-Rao Lower Bound (CRLB) for DOA estimation in III. Sections IV-VI briefly describe the mathematical aspects behind the three algorithms. Simulation results are presented in Section VII followed by the discussion and conclusion in Sections VIII and IX respectively.

II. SIGNAL MODEL

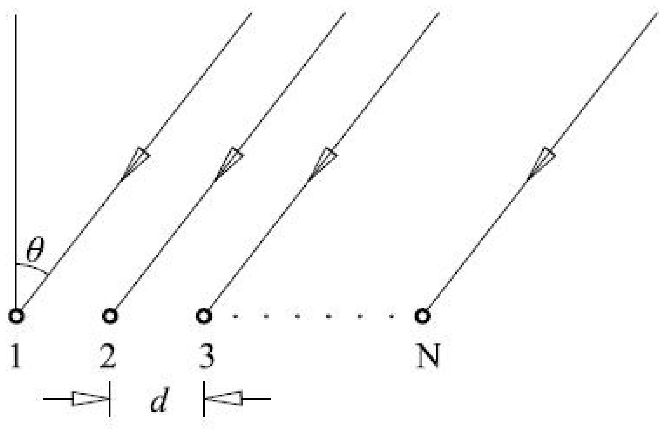

Fig. 1. Uniform linear array.

Consider a *uniform linear array* (ULA) having N identical elements, separated by a distance d as shown in Fig. 1. This array is receiving a far field signal impinging the array at an angle θ to the array axis. Taking element 1 as the reference, the line path from the source to the i^{th} element is shorter than

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that to the 1^{st} element. If the received signal at sensor 1 is $x_1(t) = s(t)$, it is delayed at sensor i by

$$\tau_i = \frac{(i-1)d \sin\theta}{c} \tag{1}$$

Then the received signal at sensor i is

$$\begin{aligned} x_i(t) = e^{-j\omega\tau_i} x_1(t) &= e^{-jkd(i-1)sin\theta} s(t) \\ &= e^{j(i-1)\psi} s(t) \end{aligned} \tag{2}$$

Putting received signals from all N elements together.

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ \vdots \\ x_N(t) \end{bmatrix} = \begin{bmatrix} 1 \\ e^{-j\psi} \\ \vdots \\ \vdots \\ e^{-j(N-1)\psi} \end{bmatrix} s(t) = \mathbf{a}(\theta) s(t) \tag{3}$$

where $\mathbf{a}(\theta)$is called the steering vector.

If there are M signal sources, received by the array arriving at angles $\theta_1, \theta_2, \ldots, \theta_M$, we get a signal model

$$\mathbf{x}(t) = \mathbf{A}s(t) \tag{4}$$

where the vector $\mathbf{A}$ is the array steering vector having the following vandermode structure.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ e^{-j2\psi_1} & e^{-j2\psi_2} & \ldots & e^{-j2\psi_M} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-j(N-1)\psi_1} & e^{-j(N-1)\psi_2} & \ldots & e^{-j(N-1)\psi_M} \end{bmatrix} \tag{5}$$

III. Cramer-Rao Lower Bound

Cramer-Rao Lower Bound provides an algorithm independent means of assessing and comparing the accuracy and performance of a DOA algorithm. The CRLB on the variance of direction estimation errors provides a useful characterization of the achievable accuracy of the DOA system. This is achieved by comparing the Mean Square Error(MSE) with the CRLB [6], [7].

The CRLB theorem states that for a length N vector of received signal x dependent on a set of parameters P, and corrupted by additive noise, the variance of an unbiased estimate of the p^{th} estimate is greater than the cramer-rao lower bound.

The CRLB of a DOA estimation problem is given by

$$var(\theta) \geq CRLB = \frac{6}{snr[N(N^2-1)(kd)^2 \sin^2\theta]} \tag{6}$$

IV. MUSIC

Multiple Signal Classification (MUSIC) is a popular high resolution algorithm based on eigenstructure technique. The main idea behind this DOA algorithm is that of performing eigenvalue decomposition on the correlation matrix [8], [9], separating it into two subspaces: signal subspace and the noise subspace. Since the signal subspace is spanned by the array steering vector of the received signals, this makes the steering vector orthogonal to the noise subspace. The product of the two: array steering vector and the noise subspace, therefore is a null for a particular Angle of Arrival (AOA).

For a uniform linear array with N elements and M signals $s_1(t)$,$s_2(t)$,$\ldots$,$s_M(t)$ arriving from directions θ_1,θ_2,$\ldots\theta_M$, and in the presence of noise $\mathbf{n}(t)$, the received signal $\mathbf{x}(t)$ is given by

$$\mathbf{x}(t) = \mathbf{A}s(t) + \mathbf{n}(t) \tag{7}$$

Defining an N×N autocorrelation matrix of the received signal $\mathbf{R}_{xx}$ as

$$\mathbf{R}_{xx} = \mathcal{E}\{\mathbf{x}(t)\mathbf{x}^H(t)\} = \mathbf{A}\mathbf{R}_{ss}\mathbf{A}^H + \sigma_0^2\mathbf{I} \tag{8}$$

where:

$$\mathbf{R}_{ss} = \mathcal{E}\{s(t)s^H(t)\} = diag\{\sigma_1^2, \ldots.., \sigma_M^2\} \tag{9}$$

$\mathbf{R}_{xx}$ has N eigenvalues $[\lambda_1, \lambda_2, \ldots, \lambda_N]$ and N associated eigenvectors making a subspace $\overline{E}$=$[\overline{e}_1, \overline{e}_2, \ldots, \overline{e}_N]$. Sorting the N eigenvalues from the smallest to the largest, the subspace $\overline{E}$ can be decomposed into two subspaces:

$$\begin{aligned} \overline{E} &= [\underbrace{\overline{e}_1, \ldots, \overline{e}_M}_{\mathrm{E}_N}, \underbrace{\overline{e}_{M+1}, \ldots, \overline{e}_N}_{\mathrm{E}_S}] && (10) \\ &= [\overline{E}_N \quad \overline{E}_S] && (11) \end{aligned}$$

$\overline{E}_N$ is the $N{\times}(N-M)$ noise subspace composed of the eigenvectors associated with the noise, whereas $\overline{E}_S$ is the $N \times M$ signal subspace composed of the eigenvectors associated with the arriving signal.

Due to the orthogonality of the noise subspace and the array steering vector at the angles of arrival θ_1,θ_2,$\ldots\theta_M$, the matrix product $a^H(\theta)E_N E_N^H a(\theta)$ is zero for this angles. The reciprocal of this matrix product creates sharp peaks at the angle of arrival. Thus the MUSIC pseudospectrum is given as

$$P(\theta) = \frac{1}{\mid a^H(\theta)E_N E_N^H a(\theta) \mid} \tag{12}$$

V. ROOT-MUSIC

This is a variant of MUSIC algorithm that employs more information than MUSIC [10]. Unlike MUSIC which involves plotting the pseudospectrum against the angles and searching for the peaks, ROOT-MUSIC involves finding the roots of a polynomial.

Starting with the pseudospectrum of MUSIC algorithm

$$P(\theta) = \frac{1}{\mid a^H(\theta)E_N E_N^H a(\theta) \mid} \tag{13}$$

defining $C = E_N E_N^H$, the denominator of equation 12 above can be rewritten as

$$P(\theta) = \frac{1}{\mid a^H(\theta)Ca(\theta) \mid} \tag{14}$$

the m^{th} element $a_m(\theta)$ of the array steering vector is defined as

$$a_m(\theta) = e^{-jkdm\sin\theta}, \qquad m = 0, 1, \ldots, N-1 \tag{15}$$

The denominator, thus can be rewritten as

$$\begin{aligned} a^H(\theta)Ca(\theta) &= \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} e^{-jkdm\sin\theta} C_{mn} e^{jkdn\sin\theta} \\ &= \sum_{\ell=-N+1}^{N-1} C_\ell e^{jkd\ell\sin\theta} \end{aligned} \tag{16}$$

where C_ℓ is the sum of the elements along the ℓ^{th} diagonal of C.

Letting $z = e^{-jkd\sin\theta}$, equation 16 above simplifies to

$$D(z) = \sum_{\ell=-N+1}^{N-1} C_\ell z^\ell \tag{17}$$

The roots of $D(z)$ that lie closest to the unit circle correspond to the poles of the MUSIC pseudospectrum. These $2(N-1)$ roots can be written as

$$z_i = \mid z_i \mid e^{jarg(z_i)}, \qquad i = 1, 2, \ldots, 2(N-1) \tag{18}$$

Choosing those roots inside the unit circle whose magnitude $\mid z_i \mid \simeq 1$, and comparing $e^{jarg(z_i)}$ to $e^{-jkd\sin\theta}$ gives

$$\theta_i = -sin^{-1}\left\{\frac{argz_i}{kd}\right\} \tag{19}$$

VI. ESPRIT

Estimation of Signal Parameters via Rotational Invariance Technique (ESPRIT) algorithm involves decomposing an N element array into two identical subarrays each with S element [8], [11]. The objective of ESPRIT algorithm is to estimate the angle of arrival by determining the rotation operator Φ.

The separation distance between the two subarrays is Δ (measured in wavelengths). Fig. 2 show a ten element linear array and possible subarray configurations for S=9, 7 and 5 elements respectively (Δ=1, 3 and 5). The 1^{st} element in the subarray is the first element in the first subarray whereas the $(\Delta+1)^{th}$ element of the original sensor is the first element in the second subarray.

Letting
N-Number of elements in the original array
S-Number of elements in each subarray
M-Number of signals hitting the subarray.

Also letting $x_1(t)$ and $x_2(t)$ be the received signal in the two subarrays, corrupted by additive white gaussian noise $n_1(t)$ and $n_2(t)$ respectively.

$$\begin{aligned} x_1(t) &= \mathbf{A}s(t) + n_1(t) \\ x_2(t) &= \mathbf{A}\Phi s(t) + n_2(t) \end{aligned} \tag{20}$$

where $x_1(t)$, $x_2(t)$, $n_1(t)$ and $n_2(t)$ are $M \times 1$ matrices. $\mathbf{A}$ is the $S \times M$ steering matrix and the variable Φ is $M \times M$ diagonal matrix called the rotation operator.

$$\Phi = diag\{e^{j\psi_1}, e^{j\psi_2}, \ldots, e^{j\psi_M}\} \tag{21}$$

where

$$\psi_i = -2k\Delta\sin\theta_i; \qquad 1 \le i \le M \tag{22}$$

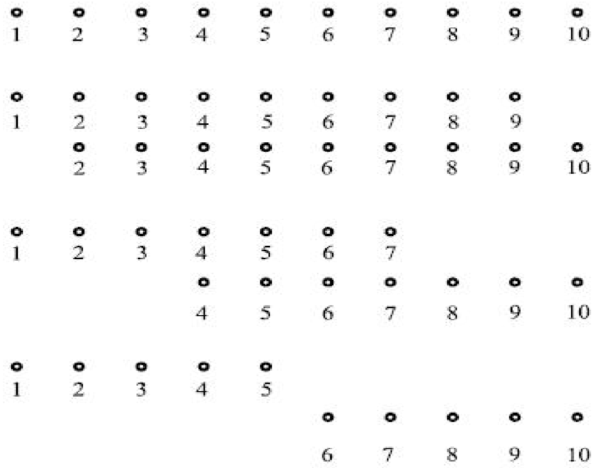


Fig. 2. ULA decomposition in ESPRIT algorithm.

From equation 20, correlation matrices $\mathbf{R}_{11}$ and $\mathbf{R}_{22}$ of the signals in the two subarrays can be estimated as

$$\begin{aligned} \mathbf{R}_{11} &= \mathcal{E}\{x_1(t)x_1^H(t)\} \\ \mathbf{R}_{22} &= \mathcal{E}\{x_2(t)x_2^H(t)\} \end{aligned} \tag{23}$$

Eigen-decomposing $\mathbf{R}_{11}$ and $\mathbf{R}_{22}$ result in two signal subspace E_1 and E_2 respectively. Defining a $2M \times 2M$ matrix C from the two subspaces such that

$$C = \begin{bmatrix} E_1^H \\ \\ E_2^H \end{bmatrix} \begin{bmatrix} E_1 & E_2 \end{bmatrix} = E_C \Lambda E_C^H \tag{24}$$

E_C is a $2M \times 2M$ matrix obtained by eigenvalue decomposition of C such that $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_{2M}$ and $\Lambda = diag\{\lambda_1\,\lambda_2\,\ldots\,\lambda_{2M}\}$.

Partitioning E_C into four $M \times M$ submatrices such that

$$E_C = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} \tag{25}$$

The rotation operator is estimated as

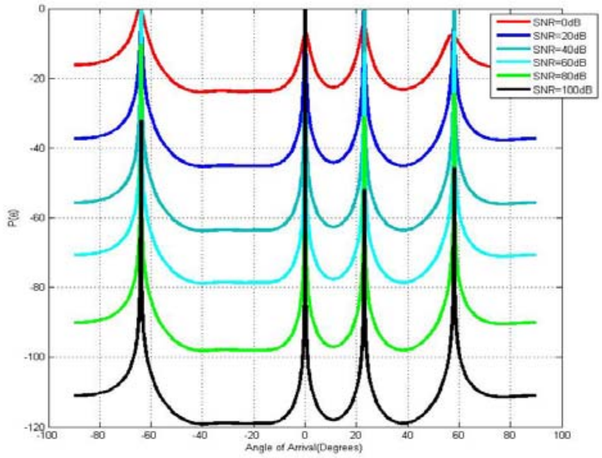
$$\Phi = -E_{12}E_{22}^{-1} \tag{26}$$

From M eigenvalues of Φ, angles of arrival can be estimated as

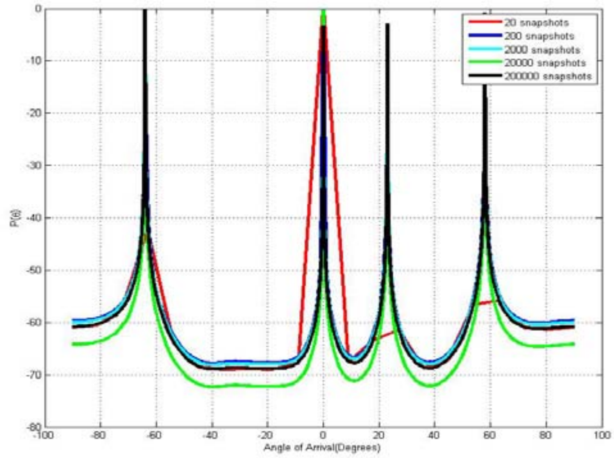
$$\theta_i = sin^{-1}\left\{\frac{arg(\lambda_i)}{k\Delta}\right\} \tag{27}$$

VII. SIMULATION RESULTS

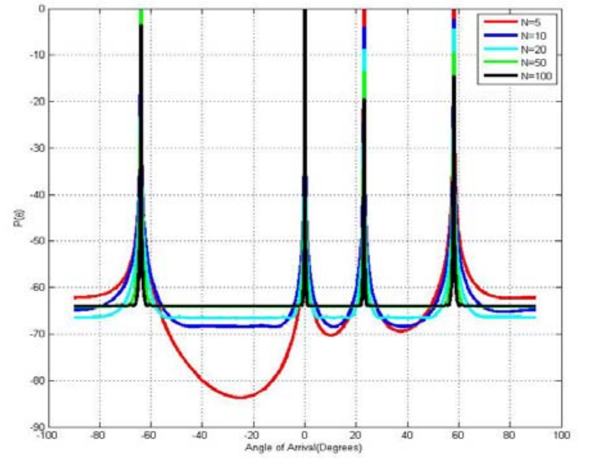
Simulations were done in MATLAB software for angles of arrival $\theta_1 = -64°$, $\theta_2 = 0°$, $\theta_1 = 23°$ and $\theta_1 = 58°$ respectively. The array size was held to 8 elements as the values of SNR were varied from 0-100dB in steps of 20dB. This was repeated holding SNR to 50dB and varying the array size from 5-100 elements.



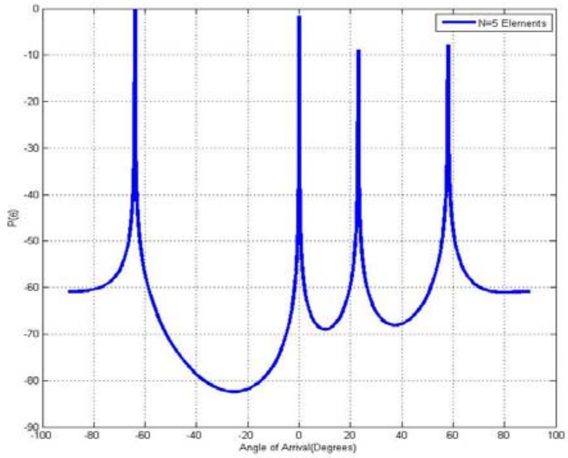
(a) MUSIC for varying SNR



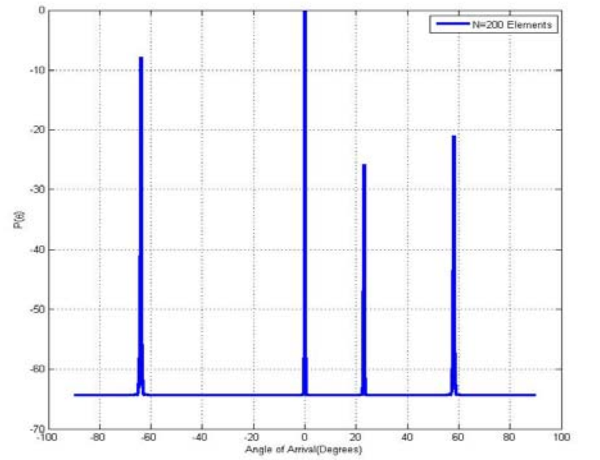
(b) MUSIC for varying snapshots



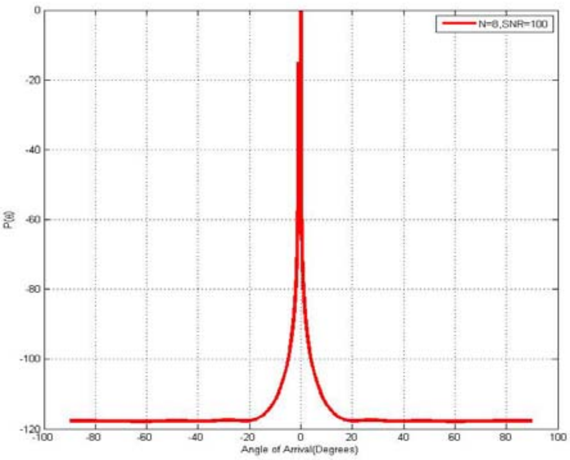
(c) MUSIC for varying array size



(d) MUSIC for a 5 element array



(e) MUSIC for a 200 element array

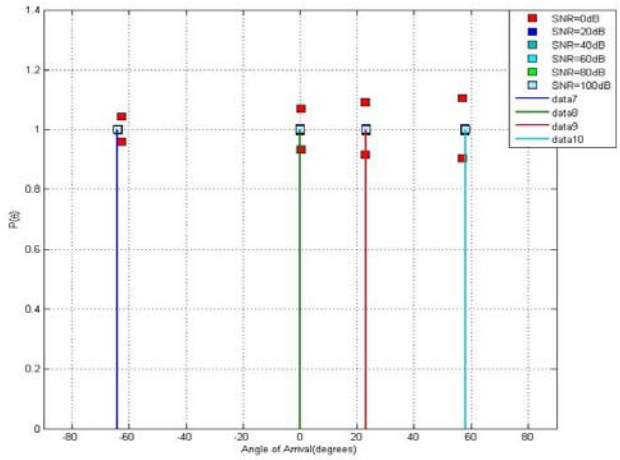


(f) MUSIC for two angles closely spaced

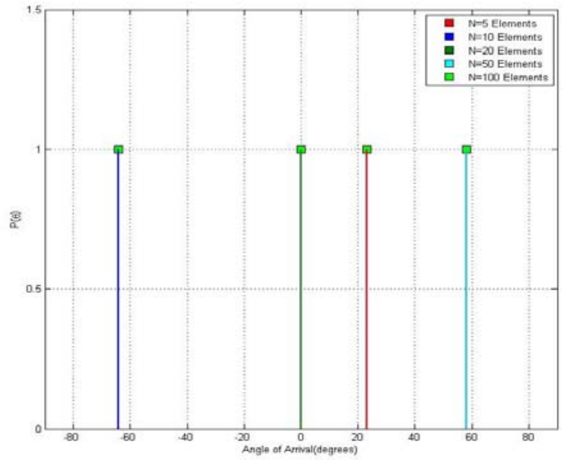
Fig. 3. Performance of MUSIC algorithm

VIII. Discussion

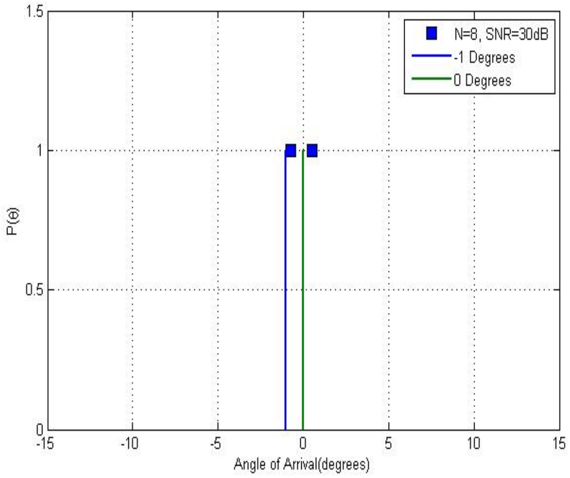
Fig. 3 shows the performance of MUSIC algorithm in various operating conditions. Fig. 3a shows the performance of this algorithms in an environment with varying SNR. For low values of SNR, 0dB, the spikes depicting the arrival of a



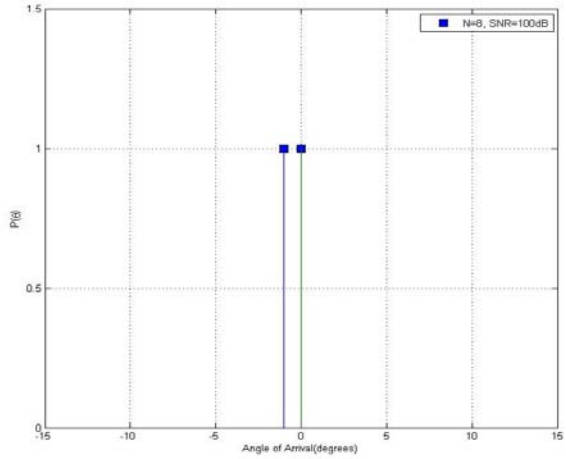
(a) Root-MUSIC for varying SNR



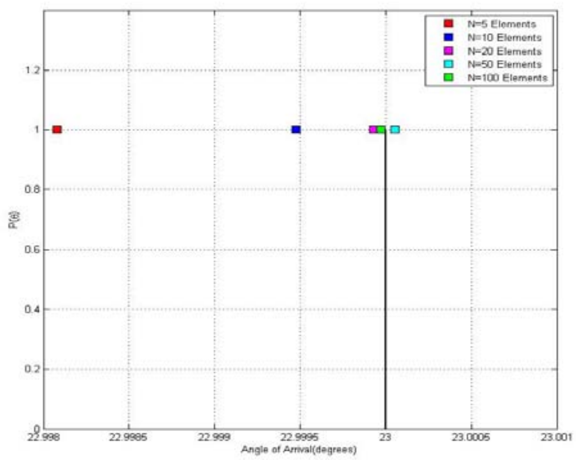
(b) Root-MUSIC for varying array size



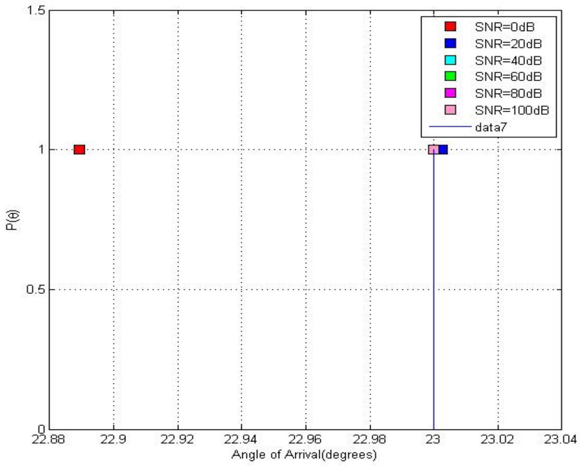
(c) Root-MUSIC for two closely spaced angles-30dB



(d) Root-MUSIC for two closely spaced angles-1000B



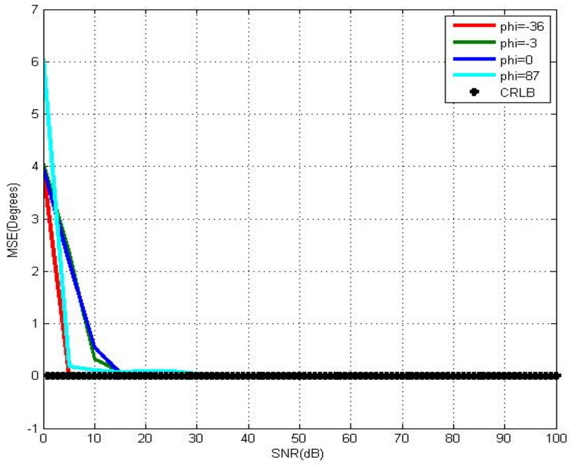
(e) Root-MUSIC at 23° for varying array size



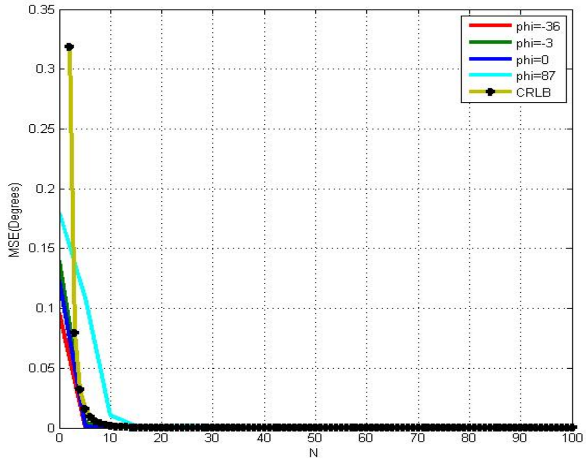
(f) Root-MUSIC at 23° for varying SNR

Fig. 4. Performance of Root-MUSIC

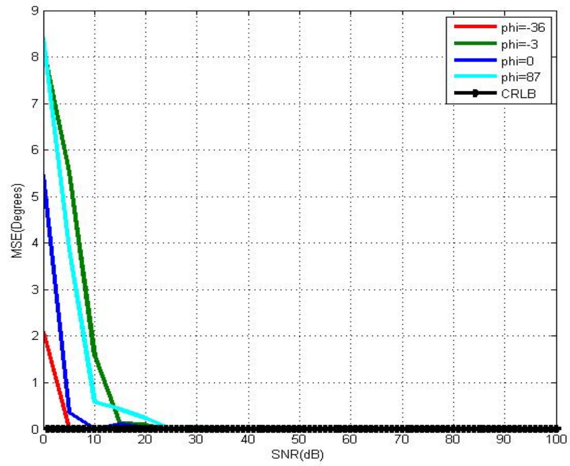
signal from certain direction are small and the response is almost flat. It is thus difficult to exactly extract the angles of arrival. As the values of SNR increase, however, the resolution of the algorithm is observed to improve considerably and the spikes become more definite. This is attributed to the fact that for low SNR the difference between the eigenvalues associated with the signal and those associated with the noise become smaller and the peaks



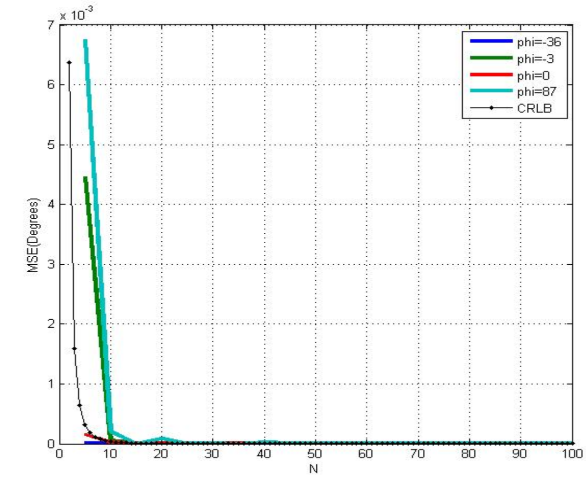
(a) MSE of MUSIC for varying SNR



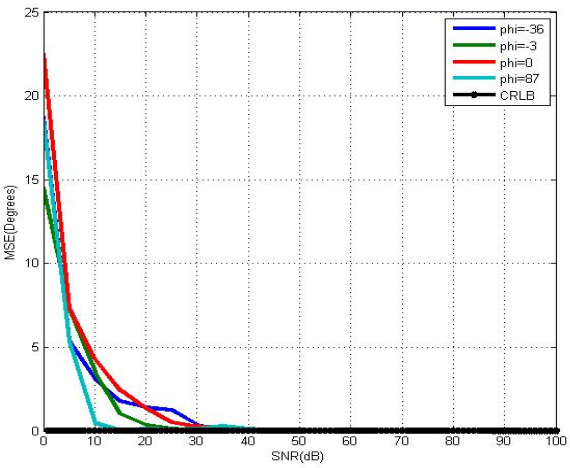
(b) MSE of MUSIC for varying array size



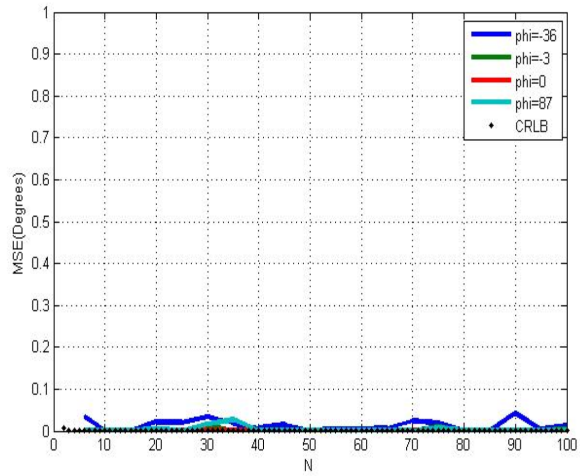
(c) MSE of Root-MUSIC for varying SNR



(d) MSE of Root-MUSIC for varying array size



(e) MSE of ESPRIT for varying SNR



(f) MSE of ESPRIT for varying array size

Fig. 5. MSE in various environments

TABLE I
ACCURACY OF ESPRIT FOR VARYING SNR

SNR	-64°	0°	23°	58°
0dB	-62.8826	1.3855	35.1460	57.0785
10dB	-65.4913	-0.0669	28.2430	57.5258
20dB	-64.3173	0.1507	24.2929	58.9783
30dB	-64.0421	0.0655	23.2738	58.3887
40dB	-64.0120	-0.0120	22.8796	58.0220
50dB	-64.0092	0.0001	23.0749	57.9863
60dB	-63.9981	0.0026	23.0057	57.9986
70dB	-64.0011	0.0004	22.9964	58.0001
80dB	-63.9997	0.0004	22.9997	58.0010
90dB	-63.9999	0	22.9994	58.0000
100dB	-64.0000	0	22.9997	57.9999

TABLE II
ACCURACY OF ESPRIT FOR VARYING ARRAY SIZE

N	-64°	0°	23°	58°
6	-64.0064	0.0007	22.6700	57.8722
10	-64.0052	-0.0068	23.0135	57.9998
20	-63.9983	0.0025	22.9432	58.0063
30	-63.9910	0.0006	22.9462	58.0328
40	-64.6071	0.0069	22.6469	56.5691
50	-63.9274	-0.0014	23.0007	57.9558
60	-64.0218	0.0118	23.4792	58.0016
70	-63.8838	0.0117	22.7634	57.3365
80	-62.9261	0.0001	22.0148	53.5281
90	-64.0007	0.0021	23.0129	57.9459
100	-63.9965	-0.0003	23.0709	58.0131

therefore become smaller with respect to the noise levels. With increase in SNR, the difference between the two sets of eigenvalues is substantial and the peaks are bigger with respect to the noise levels.

Fig. 3b depicts the response of the MUSIC algorithm to varying number of snapshots. It can be seen from the figure that for 20 snapshots the response has less pronounced spikes. The resolution is seen to improve with increase in the number of snapshots from 20 to 200,000. The number of snapshots affect the correlation between the received signals. For less snapshots, the received signals seem more correlated making it difficult to distinguish between them.

The response of MUSIC algorithm to the array size is as shown in Fig. 3c. From the figure, it can be seen that for 5 elements, the spikes are very definite and exactly correspond the the angles of arrival. As the array size increase, the resolution improves and the extraction of the angle of arrival becomes easier. An extract of the pseudospectrum for N=5 and N=200 are shown in Fig. 3d and 3e respectively.

MUSIC pseudospectrum for two closely spaced signals separated by 1° i.e. θ_1=-1° and θ_2=0° is captured in Fig. 3f for N=8 and SNR=100dB. From the figure, the two angles of arrival can be exactly extracted making MUSIC algorithm a high resolution DOA algorithm.

Fig. 4a shows the behaviour of root-MUSIC algorithm for varying values of SNR. From the figure, it can be seen that the accuracy is poor for 0dB SNR and improves as the SNR increases from 0dB to 100dB. The response of Root-MUSIC to variation in the array size is shown in Fig. 4b. It can be observed from the figure that the variation of the estimated angle of arrival is minimal with increase in the array size.

Fig. 4c and 4d presents two signals separated by 1°, i.e θ_1=-1° and θ_2=0° in different environments. From this, it is clear that Root-MUSIC is a high resolution DOA algorithm which can estimate two closely spaced angles of arrival as two angles and not one.

Fig. 4e and 4f depicts the accuracy of the Root-MUSIC for a signal impinging on the array specifically at 23° for varying values of array size and SNR. It can be observed that the performance of this algorithm is less dependent on the array size but largely on the operating environment but it anyhow improves with the improvement of either.

Tables I and II shows the response of the ESPRIT algorithm to varying values of SNR and array size. As in the other two algorithms, the resolution is very poor for low values of SNR and improves as the SNR values increase. Minimal variation in the accuracy of the algorithm with the array size is observed.

Fig. 5 summarizes the performance of the three algorithms in terms of the Mean Square Error (MSE) and in comparison with CRLB as the values of the SNR and the array sizes are varied. Four angles of arrival $\theta_1 = -36°$, $\theta_2 = -3°$, $\theta_1 = 0°$ and $\theta_1 = 87°$ (two closely spaced and one near endfire). From Fig. 5a-5e, it can be observed that the MSE diminishes with increase in both SNR and array size and the performance approaches the CRLB. However, variation of MSE with the array size is flat and approaching the CRLB behaviour for ESPRIT algorithm as captured in Fig. 5f.

IX. CONCLUSION

From the discussion above, where three DOA algorithms are developed in MATLAB and their response to various parameters presented, it can be concluded that the three algorithms are high resolution algorithms. This is derived from the precision with which the angles of arrival are estimated. The three algorithm are highly sensitive to the signal to noise ratio SNR where the resolution of the algorithms is found to improve with the increase in the SNR. Resolution is also found to considerably increase with the number of elements forming the antenna array.

MUSIC algorithms is suitable for an array with few array elements in an environment with high SNR with average snapshots whereas root-MUSIC requires relatively more elements than MUSIC in a high SNR environment again with average snapshots. ESPRIT algorithm on the other hand is less dependent on the array size and can perform relatively well than the other two in an environment having low SNR.

REFERENCES

[1] S. Kamboj and R. Dahiya, “Adaptive antenna array for satellite communication systems,” *Proceedings of the International Multiconference of Engineers and Computer Scientists*, vol. 2, March 2008,Hong Kong.

[2] A. Iozsa and A. Vesa, “Direction of arrival estimation for uniform sensor arrays,” *International Symposium on Electronics and Telecommunication(ISETC)*, no. 978-1-4244-8460-7/10, 2010.

[3] A. Iozsa and A. Vesa, “The esprit algorithm. variants and precision,” *International Symposium on Electronics and Telecommunication(ISETC)*, no. 978-1-4244-8460-7/10, 2010.

[4] A. A. Noori, S K. Gharghan, and A. A. Wahab, “Study of signal estimation parameters via rotational invariance technique by using ants colony optimization algorithm,” *Eng. & Tech. Journal*, vol. 29, no. 4, pp. 736–749, 2011.

[5] M. Barkar, R. M. Vani, and P. V. Hunagund, "Eigen structures based direction of arrival estimation algorithms for smart antennas systems," in *IJCSNS International Journal of Computer Science and Network Security*, vol. 9, pp. 96–100, Nov 2009.
[6] Engin Tuncer and Benjamin Friedlander, *Classical and Modern Direction-of-Arrival Estimation*, ch. 1, pp. 10–13. Elsevier Inc., 2009.
[7] Y. Wu, H. Liu, and H. C. So, "Fast and accurate direction-of-arrival estimation for a single source," *Progress In Electromagnetics Research C*, vol. 6, pp. 13–20, 2009.
[8] A. Vesa, "Direction of arrival estimation using music and root-music algorithm," in *18^{th} Telecommunication forum TELFOR*, November 2010.
[9] X. Gu and Y. H. Zhang, "Resolution threshold analysis of music algorithm in radar range imaging," *Progress In Electromagnetics Research B*, vol. 31, pp. 297–321, 2011.
[10] Z. Aliyazicioglu, H. K. Hwang, M. Grice, and A.Yakovlev, "Sensitivity analysis for direction of arrival estimation using a root-music algorithm," *Engineering Letters*, vol. 13, pp. 353–360, 2008.
[11] C. R. Dongarsane, A. N. Jadhav, and S. M. Hirikude, "Performance analysis of esprit algorithm for smart antenna system," *International Journal of Communication Networks and Security*, vol. 1, pp. 34–37, 2012.