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Permanence and Almost Periodic Solutions to an Epidemic Model with Delay and Feedback Control

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Abstract—This paper is concerned with an epidemic model with delay. By using the comparison theorem of the differential equation and constructing a suitable Lyapunov functional, Some sufficient conditions which guarantee the permeance and existence of a unique globally attractive positive almost periodic solution of the model are obtain. Finally, an example is employed to illustrate our result.

Keywords—Permanence, Almost periodic solution, Epidemic model, Delay, Feedback control.

I. Introduction

THE nonlinear differential equations

$$\dot{x}_i(t) = -a_i(t)x_i(t) + (c_i(t) - x_i(t))\sum_{j=1}^{n}\beta_{ij}(t) \times x_j(t-\tau_{ij}(t)),\ \ i = 1, 2, \ldots, n, \quad (1)$$

where $a_i(t), c_i(t), \beta_{ij}(t), \tau_{ij}(t) : R \to [0,\infty)$ are continuous functions for $i, j = 1, 2, \ldots, n$, have been used by [1-8] to describe the dynamics of an epidemic model. For example, Zhao et al. [9] considered the local exponential convergence of the solutions for model (1) with initial conditions:

$$0 \le x_i(s) = \varphi_i(s) < \tilde{c}_i, s \in [-\tau, 0].$$

where

$$\varphi_i \in C([-\tau, 0], R^n_+), \tau = \max_{1\le i,j\le n}\sup_{t\in R}\tau_{ij}(t) > 0,$$

$c_i = \inf_{t\in R}\tilde{c}_i(t), i = 1, 2, \ldots, n.$
Moreover, we assume that the delays are constants, then, the above epidemic model can be described to be of the following form

$$\dot{x}_i(t) = -a_i(t)x_i(t) + (c_i(t) - x_i(t))\sum_{j=1}^{n}\beta_{ij}(t)x_j(t-\tau),\ \ i = 1, 2, \ldots, n. \quad (2)$$

It is well- known that system (2) can be applied in the propagation of Gonorrhea and other epidemics (see [1-4]). The authors present some new sufficient conditions for all the solutions of system (2) with permitted initial conditions converging exponentially to zero.

In recent years, there have been extensive results on the problem of the convergence of the solutions for the epidemic model (1) with permitted initial conditions, in the literature. We refer the reader to [1-8] and the references cited therein. As well known, the exponential convergence is an important dynamic behavior since it characterizes the rate of convergence (See [10,11]). In 1993, Gopalsamy and Weng [12] introduce a models with feedback controls, in which the control variables satisfy certain differential equation. In the last decades, much work has been done on the ecosystem with feedback controls (see [13]-[18] and the references therein). In particular, Li and Liu [13], Lalli et al. [14], Liu and Xu [15] and Li [16] have studied delay equations with feedback controls.

Motivated by above, in this paper, we will study the following non-autonomous epidemic system with delay and feedback control

$$\begin{cases} \dot{x}_i(t) = -a_i(t)x_i(t) + (c_i(t) - x_i(t))\sum\limits_{j=1}^{n}\beta_{ij}(t) \\ \qquad \times x_j(t-\tau) - x_i(t)\sum\limits_{s=1}^{m}b_{is}(t)u_i(t-\sigma), \\ \dot{u}_i(t) = -\beta_i(t)u_i(t) + \sum\limits_{k=1}^{p}\alpha_{ik}(t)x_i(t-\eta), \end{cases} \quad (3)$$

where $a_i(t), c_i(t), \tau, \beta_{ij}(t), b_{is}(t), \beta_i, \alpha_{ik}(t) : R \to [0,+\infty)$ are continuous functions for $i, j = 1, 2, \ldots, n, s = 1, 2, \ldots, m, k = 1, 2, \ldots, p$ have been used by [1-8] to describe the dynamics of an epidemic model. Here, we formulate a frequency-dependent model consisting of n patches. The spatial arrangement of patches and rates of movement between patches are defined by a connection matrix. Suppose that $c_i(t)$ is the number of susceptible people (they don't develop the infectious disease, but will if in contact with infected people) in the ith patch without epidemic. $x_i(t)$ corresponds to the number of infected people in the ith patch at the time t. Assume that $\beta_{ij}(t)$ is the infection rate of the infected people in the jth patch infecting the susceptible people in the ith subarea at the time t. $a_i(t)$ is the recovery rate of the infectious people in the ith patch. $\tau \ge 0$ is the latent period of the virus in body, i.e. from the time infected people get the disease to the time they infect others. Suppose that the

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infected people will not die. Moreover, we do not consider people's immunity to this epidemic.

Let R and R^n denote the set of all real numbers and the n-dimensional real Euclidean space,respectively, R^n_+ denote the non-negative sonce of R^n. Let f be a continuous bounded function on R and we set

$$f^M = \sup_{t\in R} f(t), \quad f^l = \inf_{t\in R} f(t).$$

Throughout this paper we assume the coefficients of the almost periodic system (3) satisfy

$$\min\{a_i^l, c_i^l, \beta_{ij}^l, \tau, \sigma, \eta, b_{is}^l, \beta_i^l, \alpha_{ik}^l\} > 0,$$
$$\max\{a_i^M, c_i^M, \beta_{ij}^M, b_{is}^M, \beta_i^M, \alpha_{ik}^M\} < +\infty,$$

where $i, j = 1, 2, \ldots, n, s = 1, 2, \ldots, m, p = 1, 2, \ldots, p$.

The main purpose of this paper is to establish sufficient conditions for the existence of almost periodic solutions to system (3) by using the comparison theorem of the differential equation and constructing a suitable Lyapunov functional.

The organization of this paper is as follows. In next Section, we make some preparations. In Section three, by By using the comparison theorem of the differential equation and constructing a suitable Lyapunov functional, we establish sufficient conditions for the existence of almost periodic solutions to system (3). An illustrative example is given in Section four.

II. Preliminaries

Now let us state serval lemmas which will be useful in the proving of main result of this section.

Lemma 1: $R^n_+ = \{(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_n) | x_i > 0, u_i > 0, i = 1, 2, \ldots, n\}$ is positive invariant with respect to system (3).

Lemma 2: If $a > 0, b > 0$, and $\dot{x} \geq (\leq) x(b - ax^\alpha)$, where α is positive constant, then

$$\lim_{t\to\infty}\inf x(t) \geq \left(\frac{b}{a}\right)^{\frac{1}{\alpha}}, \quad \left(\lim_{t\to\infty}\sup x(t) \leq \left(\frac{b}{a}\right)^{\frac{1}{\alpha}}\right). \tag{4}$$

Lemma 3: If $a > 0, b > 0$, and $\dot{x} \geq (\leq) b - ax$, when $t \geq 0$ and $x(t) \geq 0$, we have

$$\lim_{t\to\infty}\inf x(t) \geq \frac{b}{a}, \quad \left(\lim_{t\to\infty}\sup x(t) \leq \frac{b}{a}\right). \tag{5}$$

Theorem 1: Let the following condition hold for the system (3)

$$(H): -a_i^M + c_i^l \sum_{j=1}^n \beta_{ij}^l e^{-\tau c_i^M} - \sum_{j=1}^n \beta_{ij}^M e^{-\tau c_i^l} - b_i^M Q_i > 0.$$

Then system (3) is permanent, i.e. any positive $(x_i(t), u_i(t))$ of the system (3) satisfies (when $i = 1, 2, \ldots, n$)

$$0 < m_i \leq \lim_{t\to\infty}\inf x_i(t) \leq \lim_{t\to\infty}\sup x_i(t) \leq M_i.$$
$$0 < q_i \leq \lim_{t\to\infty}\inf u_i(t) \leq \lim_{t\to\infty}\sup x_i(t) \leq Q_i.$$

Proof: Let $(x_i(t), u_i(t))^T$ be a positive solution of (3), from the first equation of system (3) it follows that

$$\dot{x}_i(t) \leq (c_i(t) - x_i(t)) \sum_{j=1}^n \beta_{ij}(t) x_j(t-\tau) \;\; \forall \;\; t \in R. \tag{6}$$

Hence, for any $\theta < 0$, integrating inequality (6) from $t+\theta$ to t, we obtain

$$x_i(t+\theta) \geq x_i(t) \exp\left(\int_t^{t+\theta} c_i(s)\mathrm{d}s\right). \tag{7}$$

So for any $t \in R$, from (7) and the first equation of system (3) we further obtain

$$\begin{aligned}\dot{x}_i(t) &\leq (c_i(t) - x_i(t)) \sum_{j=1}^n \beta_{ij}(t) x_j(t-\tau) \\ &\leq (c_i^M - x_i(t)) \sum_{j=1}^n \beta_{ij}(t) x_i(t) \\ &\quad \times \exp\big(-\tau c_i(t)\big)\end{aligned} \tag{8}$$

Since for any $t \in R$ and $s \in [-\tau, 0]$,

$$\int_t^{t+s} c_i(\theta)\mathrm{d}\theta \geq -\tau c_i^M,$$

we have

$$\begin{aligned}\dot{x}_i(t) &\leq \Bigg(\sum_{j=1}^n \beta_{ij}^M e^{-\tau c_i^l} c_i^M - \sum_{j=1}^n \beta_{ij}^l e^{-\tau c_i^M} \\ &\quad \times x_i(t)\Bigg) x_i(t).\end{aligned} \tag{9}$$

Applying lemma 2 to (9) leads to

$$\lim_{t\to\infty}\sup x_i(t) \leq \frac{c_i^M \sum_{j=1}^n \beta_{ij}^M \exp(-\tau c_i^l)}{\sum_{j=1}^n \beta_{ij}^l \exp(-\tau c_i^M)} := M_i$$
$$(i = 1, 2, \ldots, n). \tag{10}$$

From (10), for small enough positive constant $\epsilon_0 > 0$, there exist $T_i > 0$ enough large such that

$$x_i(t) \leq M_i + \epsilon_0 \;\; \forall \;\; t \geq T_i. \tag{11}$$

Then, from the second equation of system (3) and (10), we obtain for $t \geq T_i$,

$$\begin{aligned}\dot{u}_i(t) &\leq -\beta_i(t) u_i(t) + \sum_{k=1}^p \alpha_{ik}^M(t)(M_i + \epsilon_0) \\ &\leq -\beta_i^l u_i(t) + \sum_{k=1}^p \alpha_{ik}^M (M_i + \epsilon_0).\end{aligned} \tag{12}$$

Setting $\epsilon_0 \to 0$ and applying lemma 2 to (12), it follows that:

$$\dot{u}_i(t) \leq -\beta_i^l u_i(t) + \sum_{k=1}^p \alpha_{ik}^M M_i.$$

Since $u_i(t) > 0$ for all $t \in R$ holds, then $u_i(0) > 0$, so using Lemma 3 to above inequality we have

$$\lim_{t\to\infty} \inf u_i(t) \leq \frac{\sum\limits_{k=1}^{p} \alpha_{ik}^M M_i}{\beta_i^l} := Q_i. \quad (13)$$

form (13), for above small positive constant $\epsilon_1 > 0$, there exist $t > K_i$ such that

$$u_i(t) \leq Q_i + \epsilon_1 \quad \forall \ t \geq K_i. \quad (14)$$

From the first equation of system (3) and (11) and (14), we obtain that for $t \geq K_i$,

$$\begin{aligned} \dot{x}_i(t) \quad &\geq \quad -a_i^M x_i(t) + \Bigg(c_i^l \sum_{j=1}^{n} \beta_{ij}^l e^{-\tau c_i^M} x_i(t) - x_i(t) \\ &\times \sum_{j=1}^{n} \beta_{ij}^M e^{-\tau c_i^l} x_i(t) - \sum_{s=1}^{m} b_{is}^M (Q_i + \epsilon_1) x_i(t). \end{aligned}$$

Setting $\epsilon_1 \to 0$ in above inequality leads to

$$\begin{aligned} \dot{x}_i(t) \quad &\geq \quad x_i(t) \Bigg[- a_i^M + c_i^l \sum_{j=1}^{n} \beta_{ij}^l e^{-\tau c_i^M} - \sum_{j=1}^{n} \beta_{ij}^M \\ &\times e^{-\tau c_i^l} - \sum_{s=1}^{m} b_{is}^M Q_i - \sum_{j=1}^{n} \beta_{ij}^M e^{-\tau c_i^l} \\ &\times x_i(t) \Bigg], \end{aligned} \quad (15)$$

Then, by applying lemma 3 to (15), if follows that:

$$\begin{aligned} \lim_{t\to\infty} \inf x_i(t) \quad &\geq \quad \frac{-a_i^M + c_i^l \sum\limits_{j=1}^{n} \beta_{ij}^l e^{-\tau c_i^M}}{\sum\limits_{j=1}^{n} \beta_{ij}^M e^{-\tau c_i^l}} \\ &+ \quad \frac{-\sum\limits_{j=1}^{n} \beta_{ij}^M e^{-\tau c_i^l} - \sum\limits_{s=1}^{m} b_{is}^M Q_i}{\sum\limits_{j=1}^{n} \beta_{ij}^M e^{-\tau c_i^l}} \\ &:= m_i, i = 1, 2, \ldots, n, \end{aligned} \quad (16)$$

form (16), for above small positive constant $\epsilon_2 > 0$, there exist $K_{i+1} > T_i$ and $K_{i+2} > T_{i+1}$ such that

$$x_i(t) \geq m_i - \epsilon_2 \ \ \forall \ t > K_i (i = 1, 2, \ldots, n). \quad (17)$$

Hence, by applying (14) and (17) to the second equation of system (3), we have for $t \geq K_i$

$$\begin{aligned} u_i(t) \quad &\geq \quad -\beta_i(t) u_i(t) + \sum_{k=1}^{p} \alpha_{ik}(t)(m_i - \epsilon_2) \\ &\geq \quad -\beta_i^M u_i(t) + \sum_{k=1}^{p} \alpha_{ik}^l (m_i - \epsilon_2). \end{aligned}$$

Setting $\epsilon_2 \to 0$ in above inequality leads to

$$\begin{aligned} u_i(t) \quad &\geq \quad -\beta_i(t) u_i(t) + \sum_{k=1}^{p} \alpha_{ik}(t) m_i \\ &\geq \quad -\beta_i^M u_i(t) + \sum_{k=1}^{p} \alpha_{ik}^l m_i. \end{aligned} \quad (18)$$

Then applying Lemma 3 to (18), if follows that:

$$\lim_{t\to\infty} \inf x_i(t) \geq \frac{\sum\limits_{k=1}^{p} \alpha_{ik}^l m_i}{\beta_i^M} := q_i (i = 1, 2, \ldots, n). \quad (19)$$

Equations (10), (12),(15) and (18) show that under the assumption of the Theorem 1, system (3) is permanent. This ends the proof of the Theorem 1. ■

Next we will prove for $t \geq 0$, the above conclusions holds.

We denote by (S) the set of all solutions $z_i(t) = (x_i(t), u_i(t))^T$ of system (3) on R satisfying $m_i \leq x_i(t) \leq M_i, q_i \leq u_i(t) \leq Q_i (i = 1, 2, \ldots, n)$ for all $t \in R$.

Theorem 2: $(S) \neq \emptyset$.

Proof: From properties of almost periodic function, there exists a sequence $\{t_n\}, t_n \to \infty$ as $n \to \infty$, such that

$$\begin{aligned} &a_i(t + t_n) \to a_i(t), \ b_{is}(t + t_n) \to b_{is}(t), \\ &c_i(t + t_n) \to c_i(t), \ \beta_{ij}(t + t_n) \to \beta_{ij}(t), \\ &\alpha_{ik}(t + t_n) \to \alpha_{ik}(t), \ \beta_i(t + t_n) \to \beta_i(t), \\ &i, j = 1, 2, \ldots, n, \ s = 1, 2, \ldots, m, \ k = 1, 2, \ldots, p, \end{aligned}$$

as $n \to \infty$ uniformly on R. Let $z_i(t)$ be a solution of (1) satisfying $m_i \leq x_i(t) \leq M_i, q_i \leq u_i(t) \leq Q_i (i = 1, 2)$ for all $t \in R$. Clearly, the sequence $z_i(t + t_n)$ is uniformly bounded and equicontinuous on each bounded subset of R. Therefore by Ascoli's theorem we know that there exits a subsequence $z_i(t + t_k)$ which converges to a continuous function $P_i(t) = (p_i(t), g_i(t))^T (i = 1, 2, \ldots, n)$ as $k \to \infty$ uniformly on each bounded subset of R. Let $\bar{T} \in R$ be given. We may assume that $t_k + T_1 \geq T$ for all n. For all $t \geq 0$, we have

$$\begin{aligned} & x_i(t + t_k + \bar{T}) - x_i(t_k + \bar{T}) \\ = \quad & \int_{\bar{T}}^{t+\bar{T}} \Bigg[- a_i(s + t_k) x_i(s + t_k) \\ & + (c_i(s + t_k) - x_i(s + t_k)) \\ & \times \sum_{j=1}^{n} \beta_{ij}(s + t_k) x_j(s + t_k - \tau) \\ & \times x_i(s + t_k) - \sum_{s=1}^{m} b_{is}(s + t_k) \\ & \times u_i(s + t_k - \sigma) \Bigg] \mathrm{d}s, \end{aligned}$$

$$\begin{aligned} & u_i(t+t_k+\bar{T}) - u_i(t_k+\bar{T}) \\ = & \int_{\bar{T}}^{t+\bar{T}} \Big[\sum_{k=1}^{p} \alpha_{ik}(s+t_k) u_i(s+t_k-\eta) \\ & -\beta_i(s+t_k) u_i(s+t_k) \Big] \mathrm{d}s. \end{aligned}$$

Applying Lebesgue' dominated convergence theorem, and letting $n \to \infty$ in above equalities, we obtain

$$\begin{aligned} p_i(t+\bar{T}) - p_i(\bar{T}) = & \int_{\bar{T}}^{t+\bar{T}} \Big[- a_i(s) p_i(s) + (c_i(s) \\ & -p_i(s)) \sum_{j=1}^{n} \beta_{ij}(s) p_j(s-\tau) \\ & - \sum_{s=1}^{m} b_{is}(s) \times p_i(s) g_i(s-\sigma) \Big] \mathrm{d}s, \end{aligned}$$

$$\begin{aligned} g_i(t+\bar{T}) - g_i(\bar{T}) = & \int_{\bar{T}}^{t+\bar{T}} [-\beta_i(s) g_i(s) + \sum_{k=1}^{p} \alpha_{ik}(s) \\ & \times p_i(s-\eta)] \mathrm{d}s, \end{aligned}$$

for all $t \geq 0$. Since $\bar{T} \in R$ is arbitratily given, $P_i(t) = (p_i(t), g_i(t))^T$ is a solution of system (3) of R. It is clear that $m_i \leq p_i(t) \leq M_i (i = 1,2), q_i \leq g_i(t) \leq Q_i$, for all $t \in R$. Thus $P_i(t) \in (S)$.

This completes the proof. ■

III. Existence of a unique almost periodic solution

Now, we give the definition of the almost periodic function.

Definition 1: A function $f(t,x)$, where f is an m-vector, t is a real scalar and x is an n-vector, is said to be almost periodic in t uniformly with respect to $x \in X \subset R^n$, if $f(t,x)$ is continuous in $t \in R$ and $x \in X$, and if for any $\epsilon > 0$, it is possible to find a constnat $l(\epsilon) > 0$ such that in any interval of length $l(\epsilon)$ there exists a τ such that the inequality

$$\|f(t+\tau,x) - f(t,x)\| = \sum_{i=1}^{m} |f_i(t+\tau,x) - f_i(t,x)| < \epsilon$$

is satisfied for all $t \in R, x \in X$. The number τ called an ϵ-translation number of $f(t,x)$.

Definition 2: A function $f : R \to R$ is said to be asymptotically almost periodic function if there exists an almost-periodic function $q(t)$ and a continuous function $r(t)$ such that

$$f(t) = q(t) + r(t), \quad t \in R \ \ and \ \ r(t) \to 0 \ \ as \ \ t \to \infty.$$

We refer to [19,20] for the relevant definitions and the properties of almost periodic functions. In the followings, by constructing an suitable Lyapunov functional, we get the sufficient conditions for the existence of the globally attractive solution for systems (3).

Theorem 3: In addition to the conditions for Theorem 1, assume that (H) hold,

then for any two positive solutions $z_i(t) = (x_i(t), u_i(t))^T$ and $z_i^*(t) = (x_i^*(t), u_i^*(t))^T$ of system (3), we have

$$\lim_{t \to \infty} |z_i(t) - z_i^*(t)| = 0 \tag{20}$$

Proof: Let $z_i(t) = (x_i(t), u_i(t))^T$ and $z_i^*(t) = (x_i^*(t), u_i^*(t))^T$ be any two positive solutions of system (3). From conditions (H), it follows that there exits an enough small $\varphi > 0$ such that

$$A(\varphi) = -a_i^M - d_i^M (Q_i + \varepsilon) + c_i^l \exp(-\tau r_i^M) > \varphi, \tag{21}$$

It follows (6), (9),(14), (16) and (19) that for above $\varepsilon > 0$, there exists $T > 0$ such that

$$\begin{aligned} & m_i - \varepsilon \leq x_i(t) \leq M_i(t) + \varepsilon, \quad q_i - \varepsilon \leq u_i(t) \leq Q_i(t) \\ & +\varepsilon, \ i = 1,2. \end{aligned} \tag{22}$$

Let

$$V_1(t) = |\ln x_i(t) - \ln x_i^*(t)|. \tag{23}$$

Calculating the upper right derivatives of $V_1(t)$ along the solution of (3), by using (22) it follows that

$$\begin{aligned} D^+ V_1(t) = & \operatorname{sgn}(x_i(t) - x_i^*(t))[(\ln x_i(t))' - (\ln x_i^*(t))'] \\ = & \operatorname{sgn}(x_i(t) - x_i^*(t)) \Big[- \sum_{j=1}^{n} \beta_{ij}(t)(x_j(t-\tau) \\ & -x_j^*(t-\tau)) - d_i x_i(t) u_i(t-\tau) \\ & +d_i(t) x_i^*(t) u_i^*(t-\tau) \Big] \\ \leq & \operatorname{sgn}(x_i(t) - x_i^*(t)) \Big[- \sum_{j=1}^{n} \beta_{ij}(t) |(x_j(t-\tau) \\ & -x_j^*(t-\tau))| + d_i(t) x_i^*(t) |(u_i^*(t-\tau) \\ & -u_i(t-\tau))| + d_i(t) u_i(t-\tau) |x_i^*(t) - x_i(t)| \Big] \\ = & - \sum_{j=1}^{n} \beta_{ij}(t) |(x_j(t-\tau) - x_j^*(t-\tau))| \\ & -d_i(t) x_i^*(t) |(u_i^*(t-\tau) - u_i(t-\tau))| \\ & -d_i(t) u_i(t-\tau) |x_i(t) - x_i^*(t)|. \end{aligned}$$

Let

$$V_2(t) = |\ln u_i(t) - \ln u_i^*(t)|.$$

Calculating the upper right derivatives of $V_2(t)$ along the solution of (3), by using (22) it follows that

$$\begin{aligned} D^+V_2(t) &= \operatorname{sgn}(u_i(t)-u_i^*(t))[(\ln u_i(t))' - (\ln u_i^*(t))'] \\ &= \operatorname{sgn}(u_i(t)-u_i^*(t))\alpha_i(t)\Big[\frac{x_i(t-\tau)}{u_i(t)} \\ &\quad -\frac{x_i^*(t-\tau)}{u_i^*(t)}\Big] \\ &= -\frac{\alpha_i(t)x_i(t-\tau)}{u_i(t)u_i^*(t)}|u_i(t)-u_i^*(t)| + \frac{\alpha_i(t)}{u_i^*(t)} \\ &\quad \times|x_i(t-\tau)-x_i^*(t-\tau)|. \end{aligned}$$

Now let us define

$$V(t) = V_1(t) + V_2(t). \tag{24}$$

Therefore, for $t > T$, it follows from above analysis that

$$\begin{aligned} D^+V(t) &\le -\sum_{j=1}^n \beta_{ij}(t)|(x_j(t-\tau)-x_j^*(t-\tau))| \\ &\quad -d_i(t)x_i^*(t)|(u_i^*(t-\tau)-u_i(t-\tau))| \\ &\quad -d_i(t)u_i(t-\tau)|x_i(t)-x_i^*(t)| \\ &\quad -\frac{\alpha_i(t)x_i(t-\tau)}{u_i(t)u_i^*(t)}|u_i(t)-u_i^*(t)| \\ &\quad +\frac{\alpha_i(t)}{u_i^*(t)}|x_i(t-\tau)-x_i^*(t-\tau)| \\ &\le -\sum_{j=1}^n \beta_{ij}^l|(x_j(t-\tau)-x_j^*(t-\tau))| \\ &\quad -d_i^l(m_i-\varepsilon)|(u_i^*(t-\tau)-u_i(t-\tau))| \\ &\quad -d_i^l(q_i-\varepsilon)|x_i(t)-x_i^*(t)| - \frac{\alpha_i^l(m_i-\varepsilon)}{(Q_i-\varepsilon)^2}|u_i(t) \\ &\quad -u_i^*(t)| + \frac{\alpha_i^M}{(q_i-\varepsilon)}|x_i(t-\tau)-x_i^*(t-\tau)| \\ &\le -\beta_{ii}^l|(x_i(t-\tau)-x_i^*(t-\tau))| - d_i^l(m_i-\varepsilon) \\ &\quad \times|(u_i^*(t-\tau)-u_i(t-\tau))| - d_i^l(q_i-\varepsilon)|x_i(t) \\ &\quad -x_i^*(t)| - \frac{\alpha_i^l(m_i-\varepsilon)}{(Q_i-\varepsilon)^2}|u_i(t)-u_i^*(t)| \\ &\quad +\frac{\alpha_i^M}{(q_i-\varepsilon)}|x_i(t-\tau)-x_i^*(t-\tau)| \\ &= -\Big(\beta_{ii}^l - \frac{\alpha_i^M}{q_i-\varepsilon}\Big)|(x_j(t-\tau)-x_j^*(t-\tau))| \\ &\quad -d_i^l(m_i-\varepsilon)|(u_i^*(t-\tau)-u_i(t-\tau)) \\ &\quad \times| - d_i^l(q_i-\varepsilon)|x_i(t)-x_i^*(t)| - \frac{\alpha_i^l(m_i-\varepsilon)}{(Q_i-\varepsilon)^2} \\ &\quad \times|u_i(t)-u_i^*(t)|. \end{aligned}$$

From (20), we know that there must be a positive constant ε such that

$$\begin{aligned} &D^+V(t) \\ \le\ & -\varepsilon|(x_i(t-\tau)-x_i^*(t-\tau))| - \varepsilon|(u_i^*(t-\tau) \\ & -u_i(t-\tau))| - \varepsilon|x_i(t)-x_i^*(t)| - \varepsilon|u_i(t)-u_i^*(t)|. \end{aligned}$$

Integration the above inequality on internal $[T, t]$, it follows that for $t > T$

$$\begin{aligned} & V(t) + \varepsilon\int_T^t |x_i(s)-x_i^*(s)|\mathrm{d}s + \varepsilon\int_T^t |x_i(s-\tau) \\ & -x_i^*(s-\tau)|\mathrm{d}s + \varepsilon\int_T^t |u_i(s)-u_i^*(s)|\mathrm{d}s \\ & +\varepsilon\int_T^t |u_i(s-\tau)-u_i^*(s-\tau)|\mathrm{d}s \le V(T) < +\infty. \end{aligned}$$

Threrfore,

$$\begin{aligned} \lim_{t\to\infty}\sup\int_T^t |x_i(s)-x_i^*(s)|\mathrm{d}s \le \frac{V(T)}{\varepsilon} < +\infty, \\ \lim_{t\to\infty}\sup\int_T^t |u_i(s)-u_i^*(s)|\mathrm{d}s \le \frac{V(T)}{\varepsilon} < +\infty. \end{aligned}$$

From the above inequalities, one could easily deduce that

$$\lim_{t\to+\infty}|x_i(t)-x_i^*(t)| = 0, \quad \lim_{t\to+\infty}|u_i(t)-u_i^*(t)| = 0.$$

The completes the proof. ■

Theorem 4: Suppose all conditions of Theorem 1 hold, then there exits a unique almost periodic solution of system (3).

Proof: From Theorem 1, there exits a bounded positive solution

$$z_i(t) = (w_i(t), v_i(t))^T, \quad t \ge 0.$$

Suppose that $z_i(t) = (w_i(t), v_i(t))^T$ is a solution of (3), then there exits a sequence $\{t_k^{'}\}, \{t_k^{'}\} \to \infty$ as $k \to \infty$, such that $(w_i(t+t_k^{'}), v_i(t+t_k^{'}))^T$ is a solution of the following system:

$$\begin{cases} \dot{x}_i(t) = -a_i(t+t_k^{'})x_i(t) + (c_i(t+t_k^{'}) - x_i(t)) \\ \qquad\quad \times \sum_{j=1}^n \beta_{ij}(t+t_k^{'})x_j(t-\tau) \\ \qquad\quad -b_i(t+t_k^{'})x_i(t)u_i(t-\tau), \\ \dot{u}_i(t) = -\beta_i(t+t_k^{'})u_i(t) + \alpha_i(t+t_k^{'})x_i(t-\tau). \end{cases} \tag{25}$$

From above discussion and Theorem 1, we have that not only $\{z_i(t+t_k^{'})\}(i = 1, 2, \ldots, n)$ but also $\{\dot{z}_i(t+t_k^{'})\}(i = 1, 2, \ldots, n)$ are uniformly bounded, thus $\{z_i(t+t_k^{'})\}(i = 1, 2, \ldots, n)$ are uniformly bounded and equi-continuous. By Ascoli's theorem there exists a uniformly convergent subsequence $\{z_i(t+t_k)\} \subseteq \{z_i(t+t_k^{'})\}(i = 1, 2, \ldots, n)$ such that for any $\varepsilon > 0$, there exists a $k(\varepsilon) > 0$ with the property that if $m, k > k(\varepsilon)$ then

$$|z_i(t+t_m) - z_i(t+t_k)| < \varepsilon \quad (i = 1, 2, \ldots, n). \tag{26}$$

It shows that $\{z_i(t+t_k^{'})\}(i=1,2,\dots,n)$ are systmptotically almost periodic solutions, then $\{z_i(t+t_k^{'})\}(i=1,2,\dots,n)$ are the sum of an almost periodic function $q_i(t+t_k)(i=1,2,,\dots,n)$ and a continuous function $P_i(t+t_k)(i=1,2,\dots,n)$ defined on R, such that

$$z_i(t+t_k) = P_i(t+t_k) + q_{ij}(t+t_k) \quad \forall\ t \in R, \quad j=1,2, \tag{27}$$

where

$$\lim_{k\to\infty} P_i(t+t_k) = 0, \ \lim_{k\to\infty} q_{ij}(t+t_k) = q_{ij}(t),$$

$q_{ij}(t)$ is an almost periodic function. It means that $\lim_{k\to\infty} z_i(t+t_k) = q_{ij}(t), (i=1,2,\dots,n, \ \ j=1,2)$.

On the other hand

$$\begin{aligned} &\lim_{k\to+\infty} \dot{z}_i(t+t_k) \\ =& \lim_{k\to+\infty}\lim_{h\to 0} \frac{z_i(t+t_k+h)-z_i(t+t_k)}{h} \\ =& \lim_{h\to 0}\lim_{k\to+\infty} \frac{z_i(t+t_k+h)-z_i(t+t_k)}{h} \\ =& \lim_{h\to 0} \frac{q_i(t+h)-q_i(t)}{h}. \end{aligned} \tag{28}$$

So the limit $q_{1i}(t), q_{2i}(t)(i=1,2,\dots,n)$ exist.

Now we will prove that $(q_{1i}(t), q_{2i}(t))^T$ is an almost periodic solution of system (3). From properties of almost periodic function, there exists a sequence $\{t_n\}, t_n \to \infty$ as $n\to\infty$, such that

$$\begin{aligned} &a_i(t+t_n)\to a_i(t),\ b_i(t+t_n)\to b_i(t),\\ &c_i(t+t_n)\to c_i(t), \quad \beta_{ij}(t+t_n)\to\beta_{ij}(t),\\ &\alpha_i(t+t_n)\to\alpha_i(t),\ \beta_i(t+t_n)\to\beta_i(t),\ i=1,2,\dots,n, \end{aligned}$$

as $n\to\infty$ uniformly on R. It is easy to show that $z_i(t+t_n)\to z_i(t)$ as $n\to+\infty(i=1,2,\dots,n)$, then we have

$$\begin{aligned} \dot{q}_{1i}(t) &= \lim_{n\to+\infty} \dot{u}_i(t+t_n) \\ &= \lim_{n\to+\infty}\Big[-a_i(t+t_n)u_i(t) + (c_i(t+t_n) \\ &\quad -u_i(t))\sum_{j=1}^{n}\beta_{ij}(t+t_n)u_j(t-\tau)\Big] \\ &\quad -b_i(t+t_n)u_i(t)v_i(t+t_n-\tau) \\ &= q_{1i}(t)\Big[-a_i(t)q_{1i}(t) + (c_i(t)-q_{1i}(t))\sum_{j=1}^{n}\beta_{ij}(t) \\ &\quad \times q_{1j}(t-\tau) - b_i(t)q_{1i}(t)q_{2i}(t-\tau)\Big], \end{aligned}$$

$$\begin{aligned} \dot{q}_{2i}(t) &= \lim_{n\to+\infty} \dot{v}_i(t+t_n) \\ &= \lim_{n\to+\infty}[-\beta_i(t+t_n)u_i(t) + \alpha_i(t+t_n)v_i(t-\tau)] \\ &= -\beta_i(t)q_{2i}(t) + \alpha_i(t)q_{1i}(t-\tau). \end{aligned}$$

This prove that $(q_{1i}(t), q_{2i}(t))^T$ satisfied system (3) and $(q_{1i}(t), q_{2i}(t))^T$ is a positive periodic solution, by Theorem 3, it follows that there exits a unique positive almost periodic solution of system (3). The proof is completed. ∎

IV. An example

Now, we will give an example to show the feasibility of Theorem 3.

Example 1: Consider the following epidemic model:

$$\begin{cases} \dot{x}_1(t) = -\frac{19+\sin t}{1000}x_1(t) + (20-x_1(t))\Big[\frac{\sin^2(\sqrt{3}t)}{1000} \\ \qquad \times x_1(t-\frac{1}{41}) + \frac{\sin^2(\sqrt{3}t)}{1000}x_2(t-\frac{1}{41})\Big] \\ \qquad -\frac{\cos^2(t)+1}{1000}x_1(t)u_1(t-\frac{1}{41}), \\ \dot{x}_2(t) = -\frac{1+\cos t}{1000}x_2(t) + (10-x_2(t))\Big[\frac{\cos^2(\sqrt{3}t)}{1000} \\ \qquad \times x_1(t-\frac{1}{41}) + \frac{\cos^2(\sqrt{3}t)}{1000}x_2(t-\frac{1}{41})\Big] \\ \qquad -\frac{\cos^2(t)+1}{1000}x_2(t)u_2(t-\frac{1}{41}), \\ \dot{u}_1(t) = -(1+\cos^2(t))u_1(t) + (1+\sin^2(t)) \\ \qquad \times x_1(t-\frac{1}{41}), \\ \dot{u}_1(t) = -(2+\sin^2(t))u_1(t) + (2+\cos^2(t)) \\ \qquad \times x_1(t-\frac{1}{41}), \end{cases} \tag{29}$$

In this case, we have

$$M_1^M = \frac{c_1^M\sum\limits_{j=1}^{2}\beta_{1j}^M\exp(-\tau c_1^l)}{\sum\limits_{j=1}^{2}\beta_{1j}^l\exp(-\tau c_1^M)} = 20,$$

$$M_2^M = \frac{c_2^M\sum\limits_{j=1}^{2}\beta_{2j}^M\exp(-\tau c_2^l)}{\sum\limits_{j=1}^{2}\beta_{2j}^l\exp(-\tau c_2^M)} = 19,$$

$$Q_1 = \frac{\alpha_1^M M_1}{\beta_1^l} = \frac{2}{5}, \quad Q_2 = \frac{\alpha_2^M M_1}{\beta_2^l} = \frac{19}{1000},$$

$$a_1^M = \frac{1}{50},\ a_2^M = \frac{1}{500},\ b_1^M = b_2^M = \frac{1}{500}.$$

From above, we have

$$\begin{aligned} &-a_1^M + c_1^l\sum_{j=1}^{2}\beta_{1j}^l\exp(-\tau c_1^M) - \sum_{j=1}^{2}\beta_{1j}^M\exp(-\tau c_1^l) \\ &-b_1^M Q_1 \approx 0.00104817 > 0, \\ &-a_2^M + c_2^l\sum_{j=1}^{2}\beta_{2j}^l\exp(-\tau c_2^M) - \sum_{j=1}^{2}\beta_{2j}^M\exp(-\tau c_2^l) \\ &-b_2^M Q_2 \approx 0.0100184 > 0. \end{aligned}$$

Hence, all conditions of Theorem 3 are satisfied. By Theorem 3, system (29) has at one positive almost ω-periodic solutions.

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