On the Positive Definite Solutions of Nonlinear Matrix Equation
Tian Baoguan, Liang Chunya, Chen Nan

Abstract—In this paper, the nonlinear matrix equation is investigated. Based on the fixed-point theory, the boundary and the existence of the solution with the case $r > -\delta_i$ are discussed. An algorithm that avoids matrix inversion with the case $-1 < \delta_i < 0$ is proposed.

Keywords—Nonlinear matrix equation, Positive definite solution, The maximal-minimal solution, Iterative method, Free-inversion.

I. INTRODUCTION

In this paper, we consider the nonlinear matrix equation

$$X + \sum_{i=1}^{n} A_i X^k A_i = I$$

where $r, m$ are positive integers, $I$ is an $n \times n$ identity matrix, $A_i(i=1, 2, \cdots, m)$ are $n \times n$ nonsingular complex matrices and $A_i'$ is the conjugate transpose of $A_i$. As nonlinear matrix equations have applications in control theory, dynamic programming ladder networks and other fields, similar kinds of nonlinear matrix equations have been widely studied [1]-[4]. The case when $m=1$, $\delta_i=-1$, is one of the important study fields of the numerical algebra [5], [6]. To solve such matrix equations, numerical solutions are usually taken into consideration.

Different iterative methods including a kind of inversion-free method have been put forward and improved [7]-[9]. A. M. Sarhan et al. have studied the extremal positive definite solutions of (1) when $-1 < \delta_i < 0$ holds [10]. They have given a necessary condition and a sufficient condition for the existence of the solution and also, several algorithms are derived to compute the extremal (maximal or minimal) positive definite solutions.

In this paper, we first continue to discuss (1) with $r > -\delta_i (\delta_i < 0)$ and then we have several results for the definite solution of the equation. We also give an iterative algorithm that avoids matrix inversion, and according to this method we obtain the maximal solution when $-1 < \delta_i < 0$. Finally, we use some numerical examples to illustrate our algorithm.

The notations used in this paper are summarized as follows. $A \geq 0 (A > 0)$ means that matrix $A$ is Hermitian positive semi-definite (definite); the maximal and minimal eigenvalues of $A$ are denoted by $\lambda_{\text{max}}(A)$ and $\lambda_{\text{min}}(A)$, respectively; $\| \cdot \|$ and $\| \cdot \|_F$ represent the spectral norm and the Frobenius norm of $A$, respectively. For matrix $A = (a_{ij})$, $\text{vec}(A)$ is a vector defined by $\text{vec}(A) = (a_{11}, a_{12}, \cdots, a_{mn})'$; $A \otimes B = (a_{ij}B)$ is a Kronecker product.

Lemma 1 If $A \geq 0$ (or $A > 0$), then $A' > B^r$ (or $A \geq B^r$) for all $a \in (0,1]$, and $A'' > B^r$ (or $0 < A'' \leq B^r$) for all $a \in [-1,0]$ [11].

Lemma 2 If $C$ and $P$ are Hermitian matrices of the same order with $P > 0$, then $\|P - Q\| \leq a\|P - R\|$. Here $\| \cdot \|$ stands for one kind of matrix norm [11].

II. THE MAIN RESULTS

Now, we consider the following nonlinear matrix equation

$$X + \sum_{i=1}^{n} A_i X^k A_i = I, \quad 0 < t_i < 1$$

Theorem 1 If $r > -\delta_i (\delta_i < 0, i=1, 2, \cdots, m)$, then (1) is equivalent to (2).

Proof. Consider $X' + \sum_{i=1}^{n} A_i X^k A_i = I$. Let $Y = X'$, then $X' = Y^{1/r}$, $X^k = Y^{k/r}$. The equation can be rewritten as $Y + \sum_{i=1}^{n} A_i Y^{k/r} A_i = I$. Since $r > -\delta_i$, $i=1, 2, \cdots, m$, we have $1 < \delta_i / r$, that is $0 < -\delta_i / r < 1$. Replace $\delta_i / r$ with $-t_i$, then $Y + \sum_{i=1}^{n} A_i Y^{k} A_i = I, \quad 0 < t_i < 1$. So (1) is equivalent to (2). As for $r > -\delta_i$, the solution of (1) can be obtained by (2). In the following theorems, we will discuss the solution of (2).

Theorem 2 If (2) has a HPD solution, then we have $X \in (T_i, T_j)$, where

$$T_i = \text{max}(M_i, M_j), \quad T_j = \sum_{i=1}^{m} A_i A_i' + \sum_{i=1}^{m} A_i A_i'$$

Supported by the Natural Science Foundation of Shandong Province, China (No. ZR2010AL018)
Tian Baoguan, Liang Chun Yan, and Chen Nan are with Mathematics and Physics College of Qingdao University of Science and Technology, China (e-mail: tianbaoguangqd@163.com, 953361830@qq.com, 601933395@qq.com, respectively).

International Scholarly and Scientific Research & Innovation 8(3) 2014 580 ISNI:0000000091950263
Proof. If (2) has a HPD solution $X$, then $X = I - \sum_{i=1}^{n} A_i X_i A_i < I$, i.e., $X < I$. Moreover, $X = I - \sum_{i=1}^{n} A_i X_i A_i < I - \sum_{i=1}^{n} A_i A_i$. On the other hand, $\sum_{i=1}^{n} A_i X_i A_i < I$ implies $A_i X_i A_i < I$, hence $X > (A_i A_i)^{\delta}$ and $m X = \sum_{i=1}^{n} (A_i A_i)^{\delta}$, i.e., $X > \frac{1}{m} \sum_{i=1}^{n} (A_i A_i)^{\delta} = M_i$. Further, $X = I - \sum_{i=1}^{n} A_i X_i A_i < I - \sum_{i=1}^{n} A_i M_i A_i = M_i$. Let $T_i = \max(M_i, M_j)$, $T_i = I - \sum_{i=1}^{n} A_i A_i$, then we have $X \in \{T_i, T_j\}$.

Lemma 4 Let $X$ and $Y$ be positive definite matrices, satisfying $X \geq aI$, $Y \geq aI$ where $a$ is a positive number. Then $\|Y - X\| \leq a\|Y - X\|$ where $0 < a < 1 [10]$. 

Theorem 3 If $M_i + \sum_{i=1}^{n} A_i M_i A_i < I$, then (2) has solutions in $\Omega = [T_i, T_j]$. And, if $a I \leq T_i \leq X \leq T_j$ (where $a$ is the minimum eigenvalue of $T_i$) and $\beta = \frac{\sum_{i=1}^{n} \|A_i A_i\|^{\delta}}{\sum_{i=1}^{n} \|A_i A_i\|^{\delta}} < 1$, then (2) has a unique HPD solution.

Proof. We consider the map $f(X) = I - \sum_{i=1}^{n} A_i X_i A_i$ and $X \in \Omega = [T_i, T_j]$. Since $X < I$ and $X \geq T_i \geq M_i$, $f(X) = I - \sum_{i=1}^{n} A_i X_i A_i < I - \sum_{i=1}^{n} A_i A_i = T_i$, and $f(X) = I - \sum_{i=1}^{n} A_i X_i A_i < I - \sum_{i=1}^{n} A_i M_i A_i$ $= M_i$. Since $M_i + \sum_{i=1}^{n} A_i M_i A_i < I$, we have $f(X) \geq I - \sum_{i=1}^{n} A_i M_i A_i$ $\geq M_i$. Hence, $f(\Omega) \subseteq \Omega$, then by Brouwer’s fixed point theory, (2) has solutions in $\Omega$. For arbitrary $X, Y \in \Omega$, we have $X \geq aI$, $Y \geq aI$ and $f(X) - f(Y) = (I - \sum_{i=1}^{n} A_i X_i A_i) - (I - \sum_{i=1}^{n} A_i Y_i A_i) = \sum_{i=1}^{n} (A_i Y_i - A_i X_i)$. Then,

$$f(X) - f(Y) \geq I \sum_{i=1}^{n} \|A_i Y_i - A_i X_i\| \geq \sum_{i=1}^{n} \|A_i Y_i - A_i X_i\|.$$ 

By Lemma 4, we derive

$$f(X) - f(Y) \geq \sum_{i=1}^{n} \|A_i Y_i - A_i X_i\| \geq \|X - Y\|.$$ 

Since $\beta < 1$, by Banach’s fixed point theory, (2) has an unique solution in $\Omega$.

Algorithm 1 Consider the following algorithm $X_0 = aI$

$$X_{k+1} = I - \sum_{i=1}^{n} A_i X_k A_i, k = 0, 1, 2, \cdots$$

Theorem 4 Suppose that $A_i (i = 1, 2, \cdots)$ are nonsingular complex matrices, and we consider the sequence of positive definite matrices $\{X_i\}$ derived from Algorithm 1. If (2) has a PHD solution $X$ and $\alpha > 1$, $\sum_{i=1}^{n} \alpha^{\delta} A_i A_i > (1 - \alpha) I$ (or $0 < \alpha < 1$, $\sum_{i=1}^{n} \alpha^{\delta} A_i A_i < (1 - \alpha) I$), then $\{X_i\}$ is monotonic decreasing (or increasing) and converges to the maximal solution (or the minimal solution).

Proof. Compare to the proof of Theorem 1 and Lemma 4 in [10], and replace $\delta$ with $-\delta$, $r = 1$, we can easily get the result.

III. AN ITERATIVE METHOD THAT AVOIDS MATRIX INVERSION FOR $-1 < \delta < 0$

We consider the following nonlinear matrix equation

$$X^r + \sum_{i=1}^{n} A_i X_i A_i = I, \quad (-1 < \delta < 0, i = 1, 2, \cdots, m)$$

Let $\eta_i = -\delta, i = 1, 2, \cdots, m$. Since $-1 < \delta < 0$, then $0 < \eta_i < 1$.

Algorithm 2 Consider the iterative algorithm $Y_0 = I$

$$Y_{k+1} = (I - \sum_{i=1}^{n} A_i^Y A_i)^{\nu}$$

Theorem 5 Let $A_i (i = 1, 2, \cdots, m)$ be nonsingular complex matrices and satisfy the condition $\sum_{i=1}^{n} A_i A_i \leq I$. And if (4) has a positive definite solution $X$, then the sequence of positive definite matrices $\{X_m\}$ derived from Algorithm 2 is monotonic decreasing and converges to the maximal solution $X_m$.

Proof. Let $X$ be a HPD solution of (4). Then $X \leq I$ and hence $X = (I - \sum_{i=1}^{n} A_i X_i A_i)^{\nu} \leq (I - \sum_{i=1}^{n} A_i A_i)^{\nu}$ that is, $X \leq (I - \sum_{i=1}^{n} A_i A_i)^{\nu}$. Therefore, by algorithm 2, we have $X_0 = (I - \sum_{i=1}^{n} A_i A_i)^{\nu} \geq X_i, Y_i \leq X^{r+1}$. By Lemma 2 and Lemma 3, we have $Y_i = 2 Y_i - Y_i X_i Y_i \leq X^{r+1}$ and $Y_i = Y_i - Y_i X_i Y_i = Y_i (Y_i - X_i) Y_i = Y_i - Y_i \geq 0$, that is, $Y_i \leq X^{r+1}$ and $Y_i \geq Y_i$. It follows from Lemma 1 that $X_i = (I - \sum_{i=1}^{n} A_i A_i)^{\nu} \geq (I - \sum_{i=1}^{n} A_i A_i)^{\nu} = X$ and $X_i = (I - \sum_{i=1}^{n} A_i A_i)^{\nu} \leq (I - \sum_{i=1}^{n} A_i A_i)^{\nu} = X_i$, that is, $X_i \geq X_i \geq X$ and $Y_i \leq Y_i \leq X^{r+1}$. Assume that $X_{k+1} \geq X$ and $Y_{k+1} \leq X^{r+1}, k = 2, 3, \cdots$, we have $Y_{k+1} = 2 Y_{k+1} - Y_{k+1} X_{k+1} \leq X_{k+1}^{r+1}$ and $X_{k+1} = (I - \sum_{i=1}^{n} A_i A_i)^{\nu} \geq (I - \sum_{i=1}^{n} A_i A_i)^{\nu} = X_i$.
Further, since \( Y_i \leq X_{i+1} \leq X^\dagger_1 \), i.e., \( Y_i \geq X_i \), we have
\[
Y_{i+1} = Y_i - Y_iX_iY_i = (Y_i^\dagger - X_i^\dagger)Y_i \geq 0 \quad \text{and} \quad X_{i+1} = (I - \sum_{k=1}^{i} A_k^\dagger A_k)^{\dagger} \leq X_i^\dagger.
\]
By induction, we derive the following results \( X_0 \geq X_1 \geq \cdots \geq X_i \geq X \) and \( Y_i \leq Y_{i-1} \leq \cdots \leq Y_1 \leq X^{-\dagger} \).

Appropriately the limits of \( \{X_i\} \) and \( \{Y_i\} \) exist. Taking limit in the Algorithm 2 leads to
\[
\lim_{i \to \infty} Y_i = (\lim_{i \to \infty} X_i)^{\dagger} = \lim_{i \to \infty} (\lim_{i \to \infty} X_i)^{\dagger} = \sum_{k=1}^{\infty} A_k^\dagger (\lim_{i \to \infty} X_i)^{\dagger} A_k = I \quad \text{i.e.,} \quad (\lim_{i \to \infty} X_i)^{\dagger} + \sum_{i=1}^{\infty} A_i^\dagger (\lim_{i \to \infty} X_i)^{\dagger} A_i = I.
\]
So, we get \( \lim_{i \to \infty} X_i \) is a HPD solution of (4). As \( X_i \geq X, k = 0,1, \ldots \) holds for any HPD solution of (4), we derive \( \lim_{i \to \infty} X_i = X \) where \( X \) stands for the maximal solution.

**Theorem 6** After \( k \) iterative steps of Algorithm 2, if
\[
\lVert I - X_i X_i \rVert < \epsilon, \text{ then } X_i + \sum_{k=1}^{\infty} A_k^\dagger A_k \leq \epsilon \quad \text{where} \quad p = \left\lVert I - \sum_{i=1}^{\infty} \eta_i \right\rVert.
\]

**Proof.** Since \( I = X_{i+1} + \sum_{i=1}^{\infty} A_k^\dagger A_k \), then
\[
X_i^\dagger + \sum_{i=1}^{\infty} A_k^\dagger A_k - I = X_i^\dagger - X_{i+1} + \sum_{i=1}^{\infty} A_k^\dagger (X_i^\dagger - Y_i^\dagger) A_k
\]
\[
= \sum_{i=1}^{\infty} A_k^\dagger (Y_i^\dagger - Y_i^\dagger) A_k + \sum_{i=1}^{\infty} A_k^\dagger (X_i^\dagger - Y_i^\dagger) A_k.
\]
Use of Lemma 3, we have
\[
\left\lVert X_i^\dagger + \sum_{i=1}^{\infty} A_k^\dagger A_k - I \right\rVert \leq \sum_{i=1}^{\infty} \left\lVert A_k^\dagger (X_i^\dagger - Y_i^\dagger) A_k \right\rVert
\]
\[
\leq \sum_{i=1}^{\infty} \left\lVert A_k^\dagger \right\rVert \left\lVert X_i^\dagger - Y_i^\dagger \right\rVert
\]
\[
\leq \left\lVert X_i^\dagger - Y_i^\dagger \right\rVert \sum_{i=1}^{\infty} \left\lVert A_k^\dagger \right\rVert \eta_i.
\]
Since \( X_i \geq X \geq T_i \), which implies \( \lVert X_i^\dagger \rVert \leq \lVert T_i^\dagger \rVert \) and
\[
\left\lVert X_i^\dagger - Y_i \right\rVert = \left\lVert X_i^\dagger (I - X_i Y_i) \right\rVert \leq \left\lVert X_i^\dagger \right\rVert \left\lVert I - X_i Y_i \right\rVert \leq \left\lVert X_i \right\rVert \left\lVert I - X_i Y_i \right\rVert \leq \left\lVert X_i \right\rVert \left\lVert I - X_i Y_i \right\rVert .
\]
Then,
\[
\left\lVert X_i^\dagger + \sum_{i=1}^{\infty} A_k^\dagger A_k - I \right\rVert \leq \left\lVert I - X_i X_i \right\rVert \sum_{i=1}^{\infty} \left\lVert A_k^\dagger \right\rVert \eta_i < \epsilon.
\]

**References**


